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On Fejér and Hermite-Hadamard type Inequalities involving h-Convex Functions and Applications

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Abstract.: In this paper, we use a new weighted identity to establish new integral inequalities of Fejér and Hermite-Hadamard type involving h-convex and quasi-convex functions. Several applications of our findings for random variables, different types of means of positive real numbers and approximation of some integrals are given.

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1. Introduction

Let us recall the following two definitions which are well recognized throughout the literature.

Definition 1.1. [20] *Let $h : [0, 1] \to (-\infty, +\infty)$ denotes a non-negative and non-zero function. A function $\lambda : \mathcal{I}_{\mathcal{R}} \subset (-\infty, +\infty) \to (-\infty, +\infty)$, where $\mathcal{I}_{\mathcal{R}}$ is an interval, is* *said to be h-convex if the inequality*

$$\lambda\left(sx+\left(1-s\right)y\right)\leq h\left(s\right)\lambda\left(x\right)+h\left(1-s\right)\lambda\left(y\right)$$

holds for all $x, y \in \mathcal{I}_{\mathcal{R}}$ and $s \in [0, 1]$.

Definition 1.2. [17] *A function $\lambda : \mathcal{I}_{\mathcal{R}} \subset (-\infty, +\infty) \rightarrow (-\infty, +\infty)$, where $\mathcal{I}_{\mathcal{R}}$ is an interval, is said to be quasi convex if the inequality*

$$\lambda\left(sx+\left(1-s\right)y\right)\leq \max\left\{\lambda\left(x\right),\lambda\left(y\right)\right\}$$

holds for all x, $y \in \mathcal{I}_{\mathcal{R}}$ and $s \in [0, 1]$.

In [20], Varsanec introduced the concept of the so-called h-convex function. It generalizes the concept of convex, Godunova–Levin and s-convex functions. The interested reader is referred to [18] and [19] for further properties of h-convex functions.

One of the most well known inequalities in the literature of mathematical inequalities is the Hermite-Hadamard inequality, which is stated as follows:

Let d, e be some real numbers with $d < e$, then the inequality

$$\lambda\left(\frac{d+e}{2}\right)\leq\frac{1}{e-d}\int_{d}^{e}\lambda\left(t\right)dt\leq\frac{\lambda\left(d\right)+\lambda\left(e\right)}{2}$$

holds, where $\lambda : [d, e] \rightarrow (-\infty, +\infty)$ is a convex function. Recently, several authors established a number of inequalities to estimate the difference between $\frac{\lambda(d)+\lambda(e)}{2}$ and $\frac{1}{e-d}\int_{d}^{e}\lambda\left(t\right)dt$, see [1]-[19] and the list of references cited there.

In [2], Hwang obtained the following weighted inequalities:

Theorem 1.3. [2] *Let $\lambda : \mathcal{I}_{\mathcal{R}} \subseteq (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ be a differentiable mapping on $\mathcal{I}_{\mathcal{R}}^{\circ}$, where $\mathcal{I}_{\mathcal{R}}$ is an interval, and $\xi : [d, e] \rightarrow [0, \infty)$ be a continuous mapping which is symmetric about $\frac{d+e}{2}$, where d, $e \in \mathcal{I}_{\mathcal{R}}^{\circ}$ with $d < e$. If $\lambda' \in L_1([d, e])$ and $|\lambda'|$ is convex on $[d, e]$, then*

$$\left|\frac{\lambda\left(d\right)+\lambda\left(e\right)}{2}\int\limits_{d}^{e}\xi\left(t\right)dt-\int\limits_{d}^{e}\xi\left(t\right)\lambda\left(t\right)dt\right|$$
$$\leq\frac{\left(e-d\right)}{2}\left[\frac{\left|\lambda'\left(d\right)\right|+\left|\lambda'\left(e\right)\right|}{2}\right]\int\limits_{0}^{1}\int\limits_{\eta(s)}^{\mu(s)}\xi\left(t\right)dtds,\quad(1.\ 1)$$

where

$$\eta(s)=\frac{1+s}{2}d+\frac{1-s}{2}e\text{ and }\mu(s)=\frac{1-s}{2}d+\frac{1+s}{2}e, s\in\left[0,1\right].$$

Theorem 1.4. [2] *Let $\lambda : \mathcal{I}_{\mathcal{R}} \subseteq (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ be a differentiable mapping on $\mathcal{I}_{\mathcal{R}}^{\circ}$, where $\mathcal{I}_{\mathcal{R}}$ is an interval, and $\xi : [d, e] \rightarrow [0, \infty)$ be a continuous mapping which is symmetric about $\frac{d+e}{2}$, where d, $e \in \mathcal{I}_{\mathcal{R}}^{\circ}$ with $d < e$. If $\lambda' \in L_1([d, e])$ and $|\lambda'|^{\hat{q}}$ is convex* *on* $[d,e]$ *for* $\hat{q} \geq 1$, *then*

$$\left| \frac{\lambda(d)+\lambda(e)}{2} \int_d^e \xi(t)\,dt - \int_d^e \xi(t)\lambda(t)\,dt \right| \leq \frac{(e-d)}{2} \left[\frac{|\lambda'(d)|^{\hat{q}} + |\lambda'(e)|^{\hat{q}}}{2} \right]^{\frac{1}{\hat{q}}} \int_0^1 \int_{\eta(s)}^{\mu(s)} \xi(t)\,dtds, \quad (1.\ 2)$$

where

$$\eta(s) = \frac{1+s}{2}d + \frac{1-s}{2}e \text{ and } \mu(s) = \frac{1-s}{2}d + \frac{1+s}{2}e, s \in [0,1].$$

In this paper, we generalize the inequalities obtained by Hwang in [2] for h-convex and quasi convex functions. Then we discuss some applications of these results to random variables, special means of positive real numbers and approximation of some integrals.

Throughout this paper, $(-\infty,+\infty)$ denotes the set of all real numbers, $\mathcal{I}_{\mathcal{R}} \subset (-\infty,+\infty)$ denotes an interval and $h : [0,1] \to (-\infty,+\infty)$ denotes a non-negative and non-zero function. In addition, for any positive integer ν and $d,e \in \mathcal{I}_{\mathcal{R}}$ with $d<e$, the functions $\phi_{\nu,d,e}, \psi_{\nu,d,e} : [0,\nu] \to (-\infty,+\infty)$ are defined as:

$$\phi_{\nu,d,e}(s) = \left(\frac{s+\nu}{2\nu}\right)e + \left(\frac{\nu-s}{2\nu}\right)d$$

and

$$\psi_{\nu,d,e}(s) = \left(\frac{s+\nu}{2\nu}\right)d + \left(\frac{\nu-s}{2\nu}\right)e.$$

Moreover, the discrete power mean inequality is given as follows:

$$x^r + y^r \leq 2^{1-r}(x+y)^r, \quad (1.\ 3)$$

where $x>0$, $y>0$ and $r<1$.

2. Main Results

We start this section with the following Lemma which will be used repeatedly in the sequel.

Lemma 2.1. *Let* $\lambda : \mathcal{I}_{\mathcal{R}} \to (-\infty,+\infty)$ *be a differentiable mapping on* $\mathcal{I}_{\mathcal{R}}^{\circ}$ *and* $d, e \in \mathcal{I}_{\mathcal{R}}^{\circ}$ *with* $d<e$. *Suppose that* $\xi : [d,e] \to [0,\infty)$ *is a continuous mapping which is symmetric about* $\frac{d+e}{2}$. *If* $\lambda' \in L_1([d,e])$, *then the equality*

$$\frac{\lambda(d)+\lambda(e)}{2} \int_d^e \xi(t)\,dt - \int_d^e \xi(t)\lambda(t)\,dt = \left(\frac{e-d}{4\nu}\right) \left[\int_0^\nu \left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt \right) [\lambda'(\psi_{\nu,d,e}(s)) - \lambda'(\phi_{\nu,d,e}(s))]\,ds \right] \quad (2.\ 4)$$

holds for any positive integer ν.

Proof. Using integration by parts, we get

$$\mathcal{I}_1 = \int_0^{\nu} \left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt \right) \lambda'(\phi_{\nu,d,e}(s))\,ds$$
$$= \frac{2\nu}{e-d} \left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt \right) \lambda(\phi_{\nu,d,e}(s)) \Bigg|_0^{\nu}$$
$$+ \int_0^{\nu} \left[\xi(\psi_{\nu,d,e}(s)) + \xi(\phi_{\nu,d,e}(s))\right] \lambda(\phi_{\nu,d,e}(s))\,ds.$$

Since ξ is symmetric with respect to $\frac{d+e}{2}$, we have

$$\xi(\psi_{\nu,d,e}(s)) = \xi(\phi_{\nu,d,e}(s)).$$

Hence

$$\mathcal{I}_1 = -\frac{2\nu}{e-d} \left(\int_d^e \xi(t)\,dt \right) \lambda(e) + 2\int_0^{\nu} \xi(\phi_{\nu,d,e}(s))\,\lambda(\phi_{\nu,d,e}(s))\,ds.$$

Setting $\phi_{\nu,d,e}(s) = t$, we have

$$\mathcal{I}_1 = -\frac{2\nu}{e-d} \left(\int_d^e \xi(t)\,dt \right) \lambda(e) + \frac{4\nu}{e-d} \int_{\frac{d+e}{2}}^{e} \xi(t)\,\lambda(t)\,dt. \tag{2. 5}$$

Similarly,

$$\mathcal{I}_2 = \int_0^{\nu} \left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt \right) \lambda'(\psi_{\nu,d,e}(s))\,ds$$
$$= \frac{2\nu}{e-d} \left(\int_d^e \xi(t)\,dt \right) \lambda(d) - \frac{4\nu}{e-d} \int_d^{\frac{d+e}{2}} \xi(t)\,\lambda(t)\,dt. \tag{2. 6}$$

Subtracting (2. 5) from (2. 6) and multiplying the resulting equality by $\frac{e-d}{4\nu}$, we obtain Identity 2. 4 . □

Remark 2.2. *In Lemma 2.1*

(1) *If* $\nu = 1$ *then*

$$\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt - \int_d^e \xi(t)\lambda(t)\,dt = \left(\frac{e-d}{4}\right)\left[\int_0^1 \left(\int_{\phi_{1,d,e}(s)}^{\psi_{1,d,e}(s)} \xi(t)\,dt\right)\left[\lambda'(\psi_{1,d,e}(s)) - \lambda'(\phi_{1,d,e}(s))\right] ds\right]. \quad (2.\ 7)$$

(2) *If* $\xi(t) = \frac{1}{e-d}$ *for all* $t \in [d,e]$*, then*

$$\frac{\lambda(d)+\lambda(e)}{2} - \frac{1}{e-d}\int_d^e \lambda(t)\,dt = \left(\frac{d-e}{4\nu^2}\right)\int_0^\nu s\left[\lambda'(\psi_{\nu,d,e}(s)) - \lambda'(\phi_{\nu,d,e}(s))\right] ds \quad (2.\ 8)$$

for any integer $\nu \geq 1$.

(3) *If* $\xi(t) = \frac{1}{e-d}$ *for all* $t \in [d,e]$ *and* $\nu = 1$*, then*

$$\frac{\lambda(d)+\lambda(e)}{2} - \frac{1}{e-d}\int_d^e \lambda(t)\,dt = \left(\frac{d-e}{4}\right)\int_0^1 s\left[\lambda'(\psi_{1,d,e}(s)) - \lambda'(\phi_{1,d,e}(s))\right] ds. \quad (2.\ 9)$$

Theorem 2.3. *Let* $\lambda : \mathcal{I}_{\mathcal{R}} \to (-\infty, +\infty)$ *be a differentiable mapping on* $\mathcal{I}_{\mathcal{R}}^\circ$ *and* $d, e \in \mathcal{I}_{\mathcal{R}}^\circ$ *with* $d < e$*. Suppose that* $\xi : [d,e] \to [0,\infty)$ *is a continuous mapping which is symmetric about* $\frac{d+e}{2}$*. If* $\lambda' \in L_1([d,e])$ *and* $|\lambda'|^{\hat{q}}$ *is* h*-convex on* $[d,e]$*, where* $\hat{q} \geq 1$*, then*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt - \int_d^e \xi(t)\lambda(t)\,dt\right| \leq \left(\frac{e-d}{2\nu}\right)\left[\frac{|\lambda'(d)|^{\hat{q}} + |\lambda'(e)|^{\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}} \times \left(\int_0^\nu \left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt\right)\left(h\left(\frac{\nu+s}{2\nu}\right) + h\left(\frac{\nu-s}{2\nu}\right)\right) ds\right)^{\frac{1}{\hat{q}}} \quad (2.\ 10)$$

holds for any positive integer ν.

Proof. Using Lemma 2.1 and applying the power-mean integral inequality, we get

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right|\le\left(\frac{e-d}{4\nu}\right)$$
$$\times\left(\int_0^\nu\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)}\xi(t)\,dtds\right)^{1-\frac{1}{\hat{q}}}\left\{\left(\int_0^\nu\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)}\xi(t)\,dt\right)\left|\lambda'(\psi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}\right.$$
$$\left.+\left(\int_0^\nu\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)}\xi(t)\,dt\right)\left|\lambda'(\phi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}\right\}. \quad (2.\ 11)$$

Using Inequality (1. 3) and the h-convexity of $|\lambda'|^{\hat{q}}$ on $[d,e]$, we get

$$\left(\int_0^\nu\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)}\xi(t)\,dt\right)\left|\lambda'(\psi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}$$
$$+\left(\int_0^\nu\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)}\xi(t)\,dt\right)\left|\lambda'(\phi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}$$
$$\le 2^{1-\frac{1}{\hat{q}}}\left(\int_0^\nu\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)}\xi(t)\,dt\right)\left[\left|\lambda'(\psi_{\nu,d,e}(s))\right|^{\hat{q}}ds+\left|\lambda'(\phi_{\nu,d,e}(s))\right|^{\hat{q}}\right]ds\right)^{\frac{1}{\hat{q}}}$$
$$=2^{1-\frac{1}{\hat{q}}}\left(\left|\lambda'(d)\right|^{\hat{q}}+\left|\lambda'(e)\right|^{\hat{q}}\right)^{\frac{1}{\hat{q}}}$$
$$\times\left(\int_0^\nu\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)}\xi(t)\,dt\right)\left(h\left(\frac{\nu+s}{2\nu}\right)+h\left(\frac{\nu-s}{2\nu}\right)\right)ds\right)^{\frac{1}{\hat{q}}}. \quad (2.\ 12)$$

Combining (2. 12) and (2. 11), we obtain (2. 10). □

Corollary 2.4. *Under the same conditions as in Theorem 2.3 and if $\nu=1$, then*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right|\le\left(\frac{e-d}{2}\right)\left[\frac{\left|\lambda'(d)\right|^{\hat{q}}+\left|\lambda'(e)\right|^{\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}}$$
$$\times\left(\int_0^1\left(\int_{\phi_{1,d,e}(s)}^{\psi_{1,d,e}(s)}\xi(t)\,dt\right)\left(h\left(\frac{1+s}{2}\right)+h\left(\frac{1-s}{2}\right)\right)ds\right)^{\frac{1}{\hat{q}}}. \quad (2.\ 13)$$

Corollary 2.5. *Under the same conditions as in Theorem 2.3, if $h(t) = t$, $t \in [0,1]$, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt - \int_d^e \xi(t)\lambda(t)\,dt\right| \le \left(\frac{e-d}{2\nu}\right)\left[\frac{|\lambda'(d)|^{\hat{q}}+|\lambda'(e)|^{\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}}\left(\int_0^{\nu}\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dtds\right)^{\frac{1}{\hat{q}}} \quad (2.\ 14)$$

holds for any positive integer ν.

Corollary 2.6. *According to the conditions of Theorem 2.3, if $h(t) = t^u$, $t \in [0,1]$, where $u > 0$, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt - \int_d^e \xi(t)\lambda(t)\,dt\right| \le \left(\frac{e-d}{2\nu}\right)\left[\frac{|\lambda'(d)|^{\hat{q}}+|\lambda'(e)|^{\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}}$$
$$\times\left(\int_0^{\nu}\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt\right)\left(\left(\frac{\nu+s}{2\nu}\right)^u+\left(\frac{\nu-s}{2\nu}\right)^u\right)ds\right)^{\frac{1}{\hat{q}}} \quad (2.\ 15)$$

holds for any positive integer ν.

Example 2.7. *Let $\hat{q} \ge 1$, $v \ge \hat{q}+1$, $0 < u < v$, $0 < d \le 1$ and $\lambda(t) = \frac{\hat{q}}{v}t^{\frac{v}{\hat{q}}}$ for $t > 0$. Then $|\lambda'(t)|^{\hat{q}} = t^{v-\hat{q}}$ is h-convex on $[d,e]$, where $h(t) = t^u, t \in [0,1]$. If $\xi : [d,e] \to [0,\infty)$ is a continuous mapping which is symmetric about $\frac{d+e}{2}$, then Theorem 2.3 implies that*

$$\left|\left(\frac{d^{\frac{v}{\hat{q}}}+e^{\frac{v}{\hat{q}}}}{2}\right)\int_d^e \xi(t)\,dt - \int_d^e t^{\frac{v}{\hat{q}}}\xi(t)\,dt\right| \le \left(\frac{e-d}{2\hat{q}}\right)\left[\frac{d^{\nu-\hat{q}}+e^{\nu-\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}}$$
$$\times\left(\int_0^{\nu}\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt\right)\left(\left(\frac{\nu+s}{2\nu}\right)^u+\left(\frac{\nu-s}{2\nu}\right)^u\right)ds\right)^{\frac{1}{\hat{q}}} \quad (2.\ 16)$$

for any positive integer ν.

Corollary 2.8. *Let the assumptions of Theorem 2.3 be met and $\hat{q} = 1$, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt - \int_d^e \xi(t)\lambda(t)\,dt\right| \le \left(\frac{e-d}{2\nu}\right)\left[\frac{|\lambda'(d)|+|\lambda'(e)|}{2}\right]$$
$$\times\int_0^{\nu}\left(\int_{\phi_{\nu,d,e}(s)}^{\psi_{\nu,d,e}(s)} \xi(t)\,dt\right)\left(h\left(\frac{\nu+s}{2\nu}\right)+h\left(\frac{\nu-s}{2\nu}\right)\right)ds \quad (2.\ 17)$$

holds for any positive integer ν.

Corollary 2.9. *If the assumptions of Theorem 2.3 are satisfied and if $\hat{q}=\nu=1$, then we have*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right| \leq \left(\frac{e-d}{2}\right)\left[\frac{|\lambda'(d)|+|\lambda'(e)|}{2}\right] \times\int_0^1\left(\int_{\phi_{1,d,e}(s)}^{\psi_{1,d,e}(s)}\xi(t)\,dt\right)\left(h\left(\frac{1+s}{2}\right)+h\left(\frac{1-s}{2}\right)\right)ds. \quad (2.\ 18)$$

Theorem 2.10. *Let $\lambda:\mathcal{I}_{\mathcal{R}}\to(-\infty,+\infty)$ be a differentiable mapping on $\mathcal{I}_{\mathcal{R}}^{\circ}$ and d, $e\in e^{\circ}$ with $d<e$. Suppose that $\xi:[d,e]\to[0,\infty)$ is a continuous mapping which is symmetric about $\frac{d+e}{2}$. If $\lambda'\in L_1([d,e])$ and $|\lambda'|^{\hat{q}}$ is h-convex on $[d,e]$ for $\hat{q}>1$, then*

$$\begin{aligned}&\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right|\\ &\leq\left(\frac{e-d}{2^{2-\frac{1}{\hat{q}}}\nu^{1-\frac{1}{\hat{q}}}}\right)\left[\int_0^{\nu}\left(\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dt\right)^{\frac{\hat{q}}{\hat{q}-1}}ds\right]^{1-\frac{1}{\hat{q}}}\\ &\times\left[\left(|\lambda'(d)|^{\hat{q}}\int_{\frac{1}{2}}^{1}h(s)\,ds+|\lambda'(e)|^{\hat{q}}\int_0^{\frac{1}{2}}h(s)\,ds\right)^{\frac{1}{\hat{q}}}\right.\\ &\left.+\left(|\lambda'(e)|^{\hat{q}}\int_{\frac{1}{2}}^{1}h(s)\,ds+|\lambda'(d)|^{\hat{q}}\int_0^{\frac{1}{2}}h(s)\,ds\right)^{\frac{1}{\hat{q}}}\right] \quad (2.\ 19)\end{aligned}$$

holds for any positive integer ν.

Proof. Using Lemma 2.1 and Hölder's Inequality, we get

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right|$$
$$\leq\left(\frac{e-d}{4\nu}\right)\left[\int_0^\nu\left(\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dt\right)^{\frac{\hat{q}}{\hat{q}-1}}ds\right]^{1-\frac{1}{\hat{q}}}$$
$$\times\left\{\left(\int_0^\nu\left|\lambda'(\psi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}+\left(\int_0^\nu\left|\lambda'(\phi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}\right\}. \quad (2.\ 20)$$

By the h-convexity of $|\lambda'|^{\hat{q}}$ on $[d,e]$, we have

$$\left(\int_0^\nu\left|\lambda'(\psi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}+\left(\int_0^\nu\left|\lambda'(\phi_{\nu,d,e}(s))\right|^{\hat{q}}ds\right)^{\frac{1}{\hat{q}}}$$
$$\leq\left(\int_0^\nu\left[\left|\lambda'(d)\right|^{\hat{q}}h\left(\frac{\nu+s}{2\nu}\right)+h\left(\frac{\nu-s}{2\nu}\right)\left|\lambda'(e)\right|^{\hat{q}}\right]ds\right)^{\frac{1}{\hat{q}}}$$
$$+\left(\int_0^\nu\left[\left|\lambda'(e)\right|^{\hat{q}}h\left(\frac{\nu+s}{2\nu}\right)+h\left(\frac{\nu-s}{2\nu}\right)\left|\lambda'(d)\right|^{\hat{q}}\right]ds\right)^{\frac{1}{\hat{q}}}$$
$$=(2\nu)^{\frac{1}{\hat{q}}}\left[\left(\left|\lambda'(d)\right|^{\hat{q}}\int_{\frac{1}{2}}^1 h(s)\,ds+\left|\lambda'(e)\right|^{\hat{q}}\int_0^{\frac{1}{2}} h(s)\,ds\right)^{\frac{1}{\hat{q}}}\right.$$
$$\left.+\left(\left|\lambda'(e)\right|^{\hat{q}}\int_{\frac{1}{2}}^1 h(s)\,ds+\left|\lambda'(d)\right|^{\hat{q}}\int_0^{\frac{1}{2}} h(s)\,ds\right)^{\frac{1}{\hat{q}}}\right]. \quad (2.\ 21)$$

Combining (2. 21) and (2. 20), we obtain inequality 2. 19 . □

Corollary 2.11. *According to the suppositions of Theorem 2.10, if h is symmetric about $\frac{1}{2}$, then*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right| \le \left(\frac{e-d}{2\nu^{1-\frac{1}{\hat{q}}}}\right)\left(\int_0^1 h(s)\,ds\right)^{\frac{1}{\hat{q}}}\left[\int_0^\nu\left(\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dt\right)^{\frac{\hat{q}}{\hat{q}-1}}ds\right]^{1-\frac{1}{\hat{q}}} \times\left[|\lambda'(d)|^{\hat{q}}+|\lambda'(e)|^{\hat{q}}\right]^{\frac{1}{\hat{q}}} \quad (2.\ 22)$$

holds for any positive integer ν.

Corollary 2.12. *Suppose that the assumptions of Theorem 2.10 are satisfied. If $\nu=1$ and $h(t)=t, t\in[0,1]$, then*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right| \le \left(\frac{e-d}{2^{2-\frac{1}{\hat{q}}}}\right)\left[\int_0^1\left(\int_{\psi_{1,d,e}(s)}^{\phi_{1,d,e}(s)}\xi(t)\,dt\right)^{\frac{\hat{q}}{\hat{q}-1}}ds\right]^{1-\frac{1}{\hat{q}}} \times\left[\left(\frac{3|\lambda'(d)|^{\hat{q}}+|\lambda'(e)|^{\hat{q}}}{8}\right)^{\frac{1}{\hat{q}}}+\left(\frac{3|\lambda'(e)|^{\hat{q}}+|\lambda'(d)|^{\hat{q}}}{8}\right)^{\frac{1}{\hat{q}}}\right]. \quad (2.\ 23)$$

Theorem 2.13. *Suppose that the assumptions of Theorem 2.3 are satisfied. If the mapping $|\lambda'|$ is quasi-convex on $[d,e]$, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right| \le \left(\frac{e-d}{4\nu}\right)\left[\max\left\{|\lambda'(d)|,\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right\} +\max\left\{|\lambda'(e)|,\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right\}\right]\int_0^\nu\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dtds \quad (2.\ 24)$$

holds for any positive integer ν.

Proof. Using Identity (2. 4), we get

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right|$$
$$\leq\left(\frac{e-d}{4\nu}\right)\left[\int_0^{\nu}\left(\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dt\right)\left[\left|\lambda'\left(\psi_{\nu,d,e}(s)\right)\right|+\left|\lambda'\left(\phi_{\nu,d,e}(s)\right)\right|\right]ds\right]. \quad (2.\ 25)$$

By the quasi-convexity of $|\lambda'|$ on $[d,e]$, we have

$$\left|\lambda'\left(\phi_{\nu,d,e}(s)\right)\right|\leq\max\left\{\left|\lambda'(e)\right|,\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right\} \quad (2.\ 26)$$

and

$$\left|\lambda'\left(\psi_{\nu,d,e}(s)\right)\right|\leq\max\left\{\left|\lambda'(d)\right|,\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right\}, \quad (2.\ 27)$$

for all $s\in[0,\nu]$. Combining the inequalities in (2. 25), (2. 26) and (2. 27), we obtain Inequality (2. 24). □

Corollary 2.14. (1) *If $|\lambda'|$ is non-decreasing in Theorem 2.13, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right|$$
$$\leq\left(\frac{e-d}{4\nu}\right)\left[\left(\left|\lambda'(e)\right|+\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right)\right]\int_0^{\nu}\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dtds \quad (2.\ 28)$$

holds for any positive integer ν.

(2) *If $|\lambda'|$ is non-increasing in Theorem 2.13, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right|$$
$$\leq\left(\frac{e-d}{4\nu}\right)\left[\left(\left|\lambda'(d)\right|+\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right)\right]\int_0^{\nu}\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dtds \quad (2.\ 29)$$

holds for any positive integer ν.

Theorem 2.15. *Consider the same assumptions as in Theorem 2.3. If the mapping $|\lambda'|^{\hat{q}}$ is quasi-convex on $[d,e]$ for $\hat{q}\geq 1$, then*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right| \leq \left(\frac{e-d}{4\nu}\right)\left[\left(\max\left\{|\lambda'(d)|^{\hat{q}},\left|\lambda'\left(\frac{d+e}{2}\right)\right|^{\hat{q}}\right\}\right)^{\frac{1}{\hat{q}}} + \left(\max\left\{|\lambda'(e)|^{\hat{q}},\left|\lambda'\left(\frac{d+e}{2}\right)\right|^{\hat{q}}\right\}\right)^{\frac{1}{\hat{q}}}\right]\int_0^\nu\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dtds \quad (2.\ 30)$$

for any positive integer ν.

Proof. Using Identity (2. 4) and the power-mean integral inequality, we get

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\lambda(t)\,dt\right| \leq \left(\frac{e-d}{4\nu}\right)\left(\int_0^\nu\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dtds\right)^{1-\frac{1}{\hat{q}}} \times\left\{\left(\int_0^\nu\left(\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dt\right)|\lambda'(\psi_{\nu,d,e}(s))|^{\hat{q}}\,ds\right)^{\frac{1}{\hat{q}}} + \left(\int_0^\nu\left(\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dt\right)|\lambda'(\phi_{\nu,d,e}(s))|^{\hat{q}}\,ds\right)^{\frac{1}{\hat{q}}}\right\}. \quad (2.\ 31)$$

By the quasi-convexity of $|\lambda'|^{\hat{q}}$ on $[d,e]$, we have

$$|\lambda'(\psi_{\nu,d,e}(s))|^{\hat{q}}\leq\max\left\{|\lambda'(d)|^{\hat{q}},\left|\lambda'\left(\frac{d+e}{2}\right)\right|^{\hat{q}}\right\} \quad (2.\ 32)$$

and

$$|\lambda'(\phi_{\nu,d,e}(s))|^{\hat{q}}\leq\max\left\{|\lambda'(e)|^{\hat{q}},\left|\lambda'\left(\frac{d+e}{2}\right)\right|^{\hat{q}}\right\} \quad (2.\ 33)$$

for all $s\in[0,\nu]$. Combining the inequalities (2. 31), (2. 32) and (2. 33), we obtain Inequality 2. 30 . □

Corollary 2.16. (1) *If $|\lambda'|$ is non-decreasing in Theorem 2.15, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\,\lambda(t)\,dt\right| \leq \left(\frac{e-d}{4\nu}\right)\left[\left|\lambda'(e)\right|+\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right]\int_0^{\nu}\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dtds \quad (2.\ 34)$$

holds for any positive integer ν.

(2) *If $|\lambda'|$ is non-increasing in Theorem 2.15, then the inequality*

$$\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e \xi(t)\,dt-\int_d^e \xi(t)\,\lambda(t)\,dt\right| \leq \left(\frac{e-d}{4\nu}\right)\left[\left(\left|\lambda'(d)\right|+\left|\lambda'\left(\frac{d+e}{2}\right)\right|\right)\right]\int_0^{\nu}\int_{\psi_{\nu,d,e}(s)}^{\phi_{\nu,d,e}(s)}\xi(t)\,dtds \quad (2.\ 35)$$

holds for any positive integer ν.

3. Some Applications

Our first application will be to random variables. Let χ be a random variable taking its values in the finite interval $[d,e]$, where $d<e$,with a probability density function $\xi:[d,e]\rightarrow[0,1]$ which is symmetric about $\frac{d+e}{2}$. The s-moment is defined as:

$$E_s(\chi)=\int_d^e t^s\xi(t)\,dt.$$

Proposition 3.1. *Let χ be a random variable taking its values in the finite interval $[d,e]$, where $d<e$, with a probability density function $\xi:[d,e]\rightarrow[0,1]$ which is symmetric about $\frac{d+e}{2}$, then*

$$\left|E_{\frac{s}{\hat{q}}}(\chi)-\frac{\left(d^{\frac{s}{\hat{q}}}+e^{\frac{s}{\hat{q}}}\right)}{2}\right| \leq \frac{s}{\hat{q}}\left(\frac{e-d}{2^{2+\frac{d}{\hat{q}}}(u+1)^{\frac{1}{\hat{q}}}}\right) \times\left[\left(d^{s-\hat{q}}\left(2^{u+1}-1\right)+e^{s-\hat{q}}\right)^{\frac{1}{\hat{q}}}+\left(e^{s-\hat{q}}\left(2^{u+1}-1\right)+d^{s-\hat{q}}\right)^{\frac{1}{\hat{q}}}\right] \quad (3.\ 36)$$

for each $t\in[d,e]$ and $\hat{q}>1$.

Proof. Let $\lambda(t)=\frac{\hat{q}}{s}t^{\frac{s}{\hat{q}}},t\in[d,e]$, where $s\geq\hat{q}+2$. Then $|\lambda'(t)|^{\hat{q}}=t^{s-\hat{q}}$ is h-convex for $h(t)=t^u,t\in(0,1)$, where $u\leq 1$. Applying Theorem 2.10 for $\nu=1$, we have

$$\begin{aligned}
&\left|\frac{\lambda(d)+\lambda(e)}{2}\int_d^e\xi(t)\,dt-\int_d^e\xi(t)\lambda(t)\,dt\right| \\
&\leq\left(\frac{e-d}{2^{2-\frac{1}{\hat{q}}}}\right)\left[\int_0^1\left(\int_{\psi_{1,d,e}(t)}^{\phi_{1,d,e}(t)}\xi(t)\,dt\right)^{\frac{\hat{q}}{\hat{q}-1}}dt\right]^{1-\frac{1}{\hat{q}}} \\
&\times\left[\left(|\lambda'(d)|^{\hat{q}}\int_{\frac{1}{2}}^1h(t)\,dt+|\lambda'(e)|^{\hat{q}}\int_0^{\frac{1}{2}}h(t)\,dt\right)^{\frac{1}{\hat{q}}}\right. \\
&\left.+\left(|\lambda'(e)|^{\hat{q}}\int_{\frac{1}{2}}^1h(t)\,dt+|\lambda'(d)|^{\hat{q}}\int_0^{\frac{1}{2}}h(t)\,dt\right)^{\frac{1}{\hat{q}}}\right] \qquad (3.\ 37)
\end{aligned}$$

But

$$\frac{\lambda(d)+\lambda(e)}{2}=\frac{\hat{q}\left(d^{\frac{s}{\hat{q}}}+e^{\frac{s}{\hat{q}}}\right)}{2s},\int_d^e\xi(t)\,dt=1,$$

$$\int_d^e\xi(t)\lambda(t)\,dt=\frac{\hat{q}}{s}E_{\frac{s}{\hat{q}}}(\chi),\int_0^{\frac{1}{2}}h(t)\,dt=\frac{1}{(u+1)2^{u+1}},$$

$$\int_{\frac{1}{2}}^1h(t)\,dt=\frac{1}{u+1}\left(1-\frac{1}{2^{u+1}}\right)$$

and

$$\int_0^1\left(\int_{\psi_{1,d,e}(t)}^{\phi_{1,d,e}(t)}\xi(t)\,dt\right)^{\frac{\hat{q}}{\hat{q}-1}}dt\leq 1,$$

which implies that

$$\left|\frac{\hat{q}\left(d^{\frac{s}{\hat{q}}}+e^{\frac{s}{\hat{q}}}\right)}{2s}-\frac{\hat{q}}{s}E_{\frac{s}{\hat{q}}}(\chi)\right| \leq \left(\frac{e-d}{2^{2-\frac{1}{\hat{q}}}}\right)\left[\left(\frac{d^{s-\hat{q}}}{u+1}\left(1-\frac{1}{2^{u+1}}\right)+\frac{e^{s-\hat{q}}}{(u+1)\,2^{u+1}}\right)^{\frac{1}{\hat{q}}} + \left(\frac{e^{s-\hat{q}}}{u+1}\left(1-\frac{1}{2^{u+1}}\right)+\frac{d^{s-\hat{q}}}{(u+1)\,2^{u+1}}\right)^{\frac{1}{\hat{q}}}\right] \tag{3. 38}$$

$$\left|E_{\frac{s}{\hat{q}}}(\chi)-\frac{\left(d^{\frac{s}{\hat{q}}}+e^{\frac{s}{\hat{q}}}\right)}{2}\right| \leq \frac{s}{\hat{q}}\left(\frac{e-d}{2^{2+\frac{d}{\hat{q}}}(u+1)^{\frac{1}{\hat{q}}}}\right) \times\left[\left(d^{s-\hat{q}}\left(2^{u+1}-1\right)+e^{s-\hat{q}}\right)^{\frac{1}{\hat{q}}}+\left(e^{s-\hat{q}}\left(2^{u+1}-1\right)+d^{s-\hat{q}}\right)^{\frac{1}{\hat{q}}}\right]. \tag{3. 39}$$

□

For our second application, recall that for positive numbers $d>0$ and $e>0$, the arithmetic mean $A(d,e)$, the geometric mean $G(d,e)$ and the generalized logarithmic mean $L_r(d,e)$ of d and e are defined as:

$$A(d,e)=\frac{d+e}{2},$$

$$G(d,e)=\sqrt{de}$$

and

$$L_r(d,e)=\begin{cases}\left[\frac{e^{r+1}-d^{r+1}}{(r+1)(e-d)}\right]^{\frac{1}{r}}, & r\neq -1,0,\\ \frac{e-d}{\ln e-\ln d}, & r=-1,\\ \frac{1}{e}\left(\frac{e^e}{d^d}\right)^{\frac{1}{e-d}}, & r=0.\end{cases}$$

Let

$$\lambda(t)=\frac{\hat{q}t^{1+\frac{1}{\hat{q}}}}{\hat{q}+1} \text{ for } t>0,\ \hat{q}\geq 1. \tag{3. 40}$$

Then, obviously $|\lambda'(t)|^{\hat{q}}=t$ is h-convex on $[d,e]$ for $h(t)=t, t\in[0,1]$.

Moreover, the function

$$\xi(t)=\left(t-\frac{d+e}{2}\right)^2, \tag{3. 41}$$

where $d,e>0$ and $t\in[d,e]$, is symmetric respect to $\frac{d+e}{2}$ on $[d,e]$.

Proposition 3.2. *If $e > d > 0$ and $\hat{q} > 1$, then*

$$\left| \frac{\hat{q}(e-d)^2 A\left(d^{1+\frac{1}{\hat{q}}}, e^{1+\frac{1}{\hat{q}}}\right)}{6(\hat{q}+1)} - \frac{\hat{q}\left(4\hat{q}^2+3\hat{q}+1\right) L_{3+\frac{1}{\hat{q}}}^{3+\frac{1}{\hat{q}}}(d,e)}{2(\hat{q}+1)(2\hat{q}+1)(3\hat{q}+1)} \right.$$
$$\left. + \frac{\hat{q}G^2(d,e) L_{1+\frac{1}{\hat{q}}}^{1+\frac{1}{\hat{q}}}(d,e)}{(3\hat{q}+1)} - \frac{\hat{q}G^4(d,e) L_{\frac{1}{\hat{q}}-1}^{\frac{1}{\hat{q}}-1}(d,e)}{2(\hat{q}+1)(2\hat{q}+1)} \right| \leq \frac{\nu^{\frac{1}{\hat{q}}}(e-d)^3}{48}\left(\frac{d+e}{2}\right)^{\frac{1}{\hat{q}}}. \tag{3. 42}$$

For our last application, recall that a partition P of a finite interval $[d,e]$, $d < e$, is a finite sequence of numbers $d = d_0 < d_1 < \cdots < d_n = e$.

Proposition 3.3. *Let $\lambda : \mathcal{I}_{\mathcal{R}} \subseteq (-\infty, +\infty) \to (-\infty, +\infty)$ be a differentiable mapping on $\mathcal{I}_{\mathcal{R}}^{\circ}$ and $d, e \in \mathcal{I}_{\mathcal{R}}^{\circ}$ with $d < e$. Suppose that $\xi : [d,e] \to [0,\infty)$ is a continuous mapping which is symmetric about $\frac{d+e}{2}$. Let $P : d = d_0 < d_1 < \cdots < d_n = e$ be a partition of $[d,e]$. If $\lambda' \in L_1([d,e])$ and $|\lambda'|^{\hat{q}}$ is h-convex on $[d,e]$, $\hat{q} \geq 1$, then*

$$\int_d^e \xi(t)\lambda(t)\,dt = A(\lambda,\xi,P) + E(\lambda,\xi,P),$$

where

$$A(\lambda,\xi,P) = \sum_{j=0}^{n-1} \frac{\lambda(d_j)+\lambda(d_{j+1})}{2} W(d_j,d_{j+1}),$$

$$W(d_j,d_{j+1}) = \int_{d_j}^{d_{j+1}} \xi(t)\,dt,$$

and

$$|E(\lambda,\xi,P)| \leq \sum_{j=0}^{n-1} \left(\frac{d_{j+1}-d_j}{2\nu}\right) \left[\frac{|\lambda'(d_j)|^{\hat{q}} + |\lambda'(d_{j+1})|^{\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}}$$
$$\times \left(\int_0^{\nu} \left(\int_{\psi_{\nu,d_j,d_{j+1}}(s)}^{\phi_{\nu,d_j,d_{j+1}}(s)} \xi(t)\,dt \right) \left(h\left(\frac{\nu+s}{2\nu}\right) + h\left(\frac{\nu-s}{2\nu}\right) \right) ds \right)^{\frac{1}{\hat{q}}}, \tag{3. 43}$$

where ν is a positive integer.

Proof. For each $j=0,1,\cdots,n-1$, applying Theorem 2.3 over the interval $[d_j,d_{j+1}]$, we have

$$\left|\frac{\lambda(d_j)+\lambda(d_{j+1})}{2}\int_{d_j}^{d_{j+1}}\xi(t)\,dt-\int_{d_j}^{d_{j+1}}\xi(t)\lambda(t)\,dt\right|$$
$$\leq\left(\frac{d_{j+1}-d_j}{2\nu}\right)\left[\frac{|\lambda'(d_j)|^{\hat{q}}+|\lambda'(d_{j+1})|^{\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}}$$
$$\times\left(\int_0^{\nu}\left(\int_{\psi_{\nu,d_j,d_{j+1}}(s)}^{\phi_{\nu,d_j,d_{j+1}}(s)}\xi(t)\,dt\right)\left(h\left(\frac{\nu+s}{2\nu}\right)+h\left(\frac{\nu-s}{2\nu}\right)\right)ds\right)^{\frac{1}{\hat{q}}}. \quad (3.\ 44)$$

Note that

$$E(\lambda,\xi,P)=\int_d^e\xi(t)\lambda(t)\,dt-A(\lambda,\xi,P)$$
$$=\int_d^e\xi(t)\lambda(t)\,dt-\sum_{j=0}^{n-1}\frac{\lambda(d_j)+\lambda(d_{j+1})}{2}W(d_j,d_{j+1})$$
$$=\sum_{j=0}^{n-1}\left[\int_{d_i}^{d_{j+1}}\xi(t)\lambda(t)\,dt-\frac{\lambda(d_j)+\lambda(d_{j+1})}{2}\int_{d_j}^{d_{j+1}}\xi(t)\,dt\right].$$

Using the triangle inequality and inequality 3. 44 , we get that

$$|E(\lambda,\xi,P)|\leq\sum_{j=0}^{n-1}\left(\frac{d_{j+1}-d_j}{2\nu}\right)\left[\frac{|\lambda'(d_j)|^{\hat{q}}+|\lambda'(d_{j+1})|^{\hat{q}}}{2}\right]^{\frac{1}{\hat{q}}}$$
$$\times\left(\int_0^{\nu}\left(\int_{\psi_{\nu,d_j,d_{j+1}}(s)}^{\phi_{\nu,d_j,d_{j+1}}(s)}\xi(t)\,dt\right)\left(h\left(\frac{\nu+s}{2\nu}\right)+h\left(\frac{\nu-s}{2\nu}\right)\right)ds\right)^{\frac{1}{\hat{q}}}.$$

□

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