Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(6)(2020) pp. 45-63

Opial-Type Inequalities Involving Radial Fractional Derivative Operators

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Received: 10 April, 2020 / Accepted: 07 May, 2020 / Published online: 01 June, 2020

Abstract. The main purpose of this paper is to give the Opial-type inequalities using radial fractional derivative operators, such as Riemann-Liouville, Caputo radial fractional derivative operators with related extreme cases. Moreover, counter part of the main results is also provided.

AMS (MOS) Subject Classification Codes: 26D15; 26D10; 26A33
Key Words: Inequalities; kernels; fractional derivatives.

1. INTRODUCTION

In 1960, Opial [10] presented an interesting inequality which is called Opial's inequality afterwards. Opial-type inequalities have importance in applications in the theory of ordinary differential equations and boundary value problems. A large number of papers have appeared in literature, for example, ([5], [8], [11]).

In 1968, D. Willett [11], considered the Opial-type inequality which involves a higher order derivative. Later on Das [4], found a more general result but in the last twenty years a large number of papers have been appeared in the literature which deals with the simple proves. In [12], Yang has established some interesting generalization of the Opial's inequality by using the Mallows method [9]. Agarwal, Alzer and Pang ([1]-[3]) studied the variety of Opial-type inequalities including ordinary derivatives and their application in differential equation and difference equation. Most of the writers dealt with them in diverse directions and for numerous cases, and still it halt a most active area of research. Opial's inequality has been generalized to frequent dissimilar situations and setting. Recently Iqbal et al [7], gave a variety of Opial-type inequalities but particularly our interest is to give the results for Riemann-Liouville radial fractional derivative operator, Caputo radial fractional derivative operators.

The Opial's inequality is stated in next theorem.

Theorem 1.1. *Let $a > 0, \Phi \in C^1[0,a]$ with $\Phi(a) = \Phi(0) = 0$ and $\Phi(\xi) > 0$ on $(0,a)$, then*

$$\int_0^a |\Phi(\xi)\Phi'(\xi)|d\xi \leq \frac{a}{4}\int_0^a (\Phi'(\xi))^2 d\xi.$$

the constant $\frac{a}{4}$ is the best possible.

The space of all functions on $[a,b]$ which have continuous derivatives up to order n is denoted by $C^n[a,b]$, and the space of all absolutely continuous functions on $[a,b]$ is denoted by $AC[a,b]$. By $AC^n[a,b]$ we denote the space of all functions $\Phi \in C^{n-1}[a,b]$ with $\Phi^{(n-1)} \in AC[a,b]$.

The space of all Lebesgue measurable functions Φ for which $|\Phi^p|$ is Lebesgue integrable on $[a,b]$ is denoted by $L_p[a,b], 1 \leq p < \infty$, and the set of all measurable and essentially bounded functions on $[a,b]$ denoted by $L_\infty[a,b]$. Clearly, $L_\infty[a,b] \subset L_p[a,b]$ for all $p \geq 1$.

We say that a function $\Psi : [a,b] \to \mathbb{R}$ belong to the class $U(\Phi,k)$ if it admits the representation

$$|\Psi(\xi)| \leq \int_a^\xi k(\xi,\eta)|\Phi(\eta)|d\eta, \tag{1. 1}$$

where Φ is a continuous function and k is an arbitrary non-negative kernel such that $\Phi(\xi) > 0$ implies $\Psi(\eta) > 0$ for every $\xi \in [a,b]$. We also suppose that all integrals under considerations exist and that they are finite.

We start with the definition of Riemann-Liouville in short R-L radial fractional derivative, given in [6].

Definition 1.2. *Assume that B_X stand for the Borel class on space X and define the measure R_N on $((0,\infty), B_{(0,\infty)})$ by*

$$R_N(\Gamma) = \int\limits_{\Gamma} \xi^{N-1} d\xi, \Gamma \in B_{(0,\infty)}. \tag{1. 2}$$

Now suppose $\Phi \in L_1(A) = L_1([R_1, R_2] \times S^{N-1})$. For a fix $\omega \in S^{N-1}$, we define

$$\Psi_\omega(\xi) := \Phi(\xi\omega) = \Phi(x), \tag{1. 3}$$

hence

$$\begin{aligned} &x \in A := B(0, R_2) - \overline{B(0, R_1)}, \\ &0 < R_1 \le \xi \le R_2, \xi = |x|, \omega = \frac{x}{\xi} \in S^{N-1}. \end{aligned} \tag{1. 4}$$

The above bears to the following definition of R-L radial fractional derivative detail is given in [6]. *Assume $\beta > 0, m := [\beta] + 1, \Phi \in L_1(A)$, also A is spherical shell. Then*

$$\frac{\partial^\beta_{R_1} \Phi(x)}{\partial \xi^\beta} = D^\beta_{R_1} \Phi(\xi\omega) = \begin{cases} \frac{1}{\Gamma(m-\beta)} (\frac{\partial}{\partial \xi})^m \int\limits_{R_1}^{\xi} (\xi - \eta)^{m-\beta-1} \Phi(\eta\omega) d\eta, & \omega \in S^{N-1} - K(\Phi), \\ 0, & \omega \in K(\Phi). \end{cases}$$

Here

$$x = \xi\omega \in A, \xi \in [R_1, R_2], \omega \in S^{N-1},$$
$$K(\Phi) =: \{\omega \in S^{N-1} : \Phi(.\omega) \notin L_1([R_1, R_2], B_{[R_1,R_2]}, R_N)\}.$$

If $\beta = 0$, we obtain

$$\frac{\partial^\beta_{R_1} \Phi(x)}{\partial \xi^\beta} := \Phi(x).$$

$\partial^\beta_{R_1} \Phi(x)/\partial \xi^\beta$ *denotes the R-L radial fractional derivative of Φ of order β, see* [6].

Lemma 1.3. *Assume that $\gamma + 1 \le \nu, 0 \le \gamma, n := \lceil \nu \rceil, \Phi := \overline{A} \to \mathbb{R}$ with $\Phi \in L_1(A)$. Suppose that $\Phi(\cdot\omega) \in AC^n([R_1, R_2])$, for all $\omega \in S^{N-1}$, and $\partial^\nu_{R_1} \Phi(\cdot\omega)/\partial \xi^\nu$ is measurable on $[R_1, R_2]$ for all $\omega \in S^{N-1}$. Also suppose that there exist $\partial^\nu_{R_1} \Phi(\xi\omega)/\partial \xi^\nu \in \mathbb{R}$ for every $\xi \in [R_1, R_2]$, and for every $\omega \in S^{N-1}$, and $\partial^\nu_{R_1} \Phi(\omega)/\partial \xi^\nu$ is measurable on $\overline{A}$. If there exist $L > 0$,*

$$\left| \frac{\partial^\nu_{R_1} \Phi(\omega)}{\partial \xi^\nu} \right| \le L,$$

for all

$$(\xi, \omega) \in [R_1, R_2] \times S^{N-1}.$$

Also we assume that $\partial^j \Phi(R_1 \omega)/\partial \xi^j = 0, j = 0, 1, ..., n-1$ for all $\omega \in S^{N-1}$, then

$$D^\gamma_{R_1} \Phi(\xi\omega) = \frac{1}{\Gamma(\nu - \gamma)} \int\limits_{R_1}^{\xi} (\xi - \eta)^{\nu-\gamma-1} (D^\nu_{R_1} \Phi)(\eta\omega) d\eta, \tag{1. 5}$$

is true for all $x \in \overline{A}$, that is correct for all $\xi \in [R_1, R_2]$ and for all $\omega \in S^{N-1}, \gamma > 0$.

Next we define the Caputo radial fractional derivative given in [6].

Definition 1.4. *Suppose* $\Phi : A \to \mathbb{R}, \nu \geq 0, n := \lceil \nu \rceil$, *such that* $\Phi(\cdot\omega) \in AC^n([R_1, R_2])$, *for every* $\omega \in S^{N-1}$, *where* $\overline{A} = [R_1, R_2] \times S^{N-1}$ *for* $N \in \mathbb{N}$ *and* $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. *Then the Caputo radial fractional derivative is defined by*

$$\frac{\partial^{\nu}_{*R_1}\Phi(x)}{\partial \xi^{\nu}} = \frac{1}{\Gamma(n-\nu)} \int\limits_{R_1}^{\xi} (\xi - \eta)^{n-\nu-1} \frac{\partial^n \Phi(\eta\omega)}{\partial \xi^n} d\eta, \tag{1. 6}$$

whence $x \in \overline{A}$, *that is* $x = \xi\omega, \xi \in [R_1, R_2], \omega \in S^{N-1}$. *Obviously,*

$$\frac{\partial^{0}_{*R_1}\Phi(x)}{\partial \xi^{0}} = \Phi(x),$$

$$\frac{\partial^{\nu}_{*R_1}\Phi(x)}{\partial \xi^{\nu}} = \frac{\partial^{\nu}\Phi(x)}{\partial \xi^{\nu}}, \text{ if } \nu \in \mathbb{N}, \text{ the usual radial derivative.}$$

In the upcoming section, we give the Opial-type inequalities involving R-L radial and Caputo radial fractional derivative operators.

2. Opial-type inequalities for Riemann-Liouville radial fractional derivative operator

First we shall give the results for R-L radial fractional derivative operator.

Theorem 2.1. *Suppose* $\gamma \geq 0, D^{\gamma}_{R_1}\Psi(\xi\omega)$ *be the R-L radial fractional derivative operator. Suppose* $\varrho > 0, \lambda \geq 0$ *be measurable functions on* $[R_1, x]$. *If* $s > 1, s > q > 0$ *also* $p \geq 0$ *and* $(D^{\nu}_{R_1}\Phi_1)(\xi\omega), (D^{\nu}_{R_1}\Psi)(\xi\omega) \in L_s[R_1, R_2]$. *Then*

$$\int\limits_{R_1}^{x} \lambda(\xi) \left(|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p |(D^{\nu}_{R_1}\Phi)(\xi\omega)|^q + |D^{\gamma}_{R_1}\Phi(\xi\omega)|^p |(D^{\nu}_{R_1}\Psi)(\xi\omega)|^q \right) d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}} \right)^{\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(\int\limits_{R_1}^{x} [Z(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}}$$

$$\times \left(\int\limits_{R_1}^{x} \varrho(\eta) \left(|(D^{\nu}_{R_1}\Phi)(\eta\omega)|^s + |(D^{\nu}_{R_1}\Psi)(\eta\omega)|^s \right) d\eta \right)^{\frac{p+q}{s}},$$

where

$$Z(\xi) = \frac{1}{[\Gamma(\nu-\gamma)]^p} \lambda(\xi) [\varrho(\xi)]^{\frac{-q}{s}} \left(\int\limits_{R_1}^{\xi} (\xi - \eta)^{\frac{s(\nu-\gamma-1)}{s-1}} [\varrho(\eta)]^{\frac{1}{1-s}} d\eta \right)^{\frac{p(s-1)}{s}}.$$

Proof. Let $\xi \in [R_1, x]$, using the inequality (1. 1), and the identity (1. 5) and applying Hölder's inequality for $\{\frac{s}{s-1}, s\}$ we get

$$
\begin{aligned}
&|D^{\gamma}_{R_1}\Psi(\xi\omega)| \\
&\leq \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{\xi} (\xi-\eta)^{\nu-\gamma-1}[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}|D^{\nu}_{R_1}\Psi(\eta\omega)|d\eta \\
&\leq \frac{1}{\Gamma(\nu-\gamma)} \left(\int_{R_1}^{\xi} (\xi-\eta)^{\frac{s(\nu-\gamma-1)}{s-1}}[\varrho(\eta)]^{\frac{1}{1-s}}d\eta\right)^{\frac{s-1}{s}} \left(\int_{R_1}^{\xi} \varrho(\eta)|D^{\nu}_{R_1}\Psi(\eta\omega)|^s d\eta\right)^{\frac{1}{s}} \\
&= \frac{1}{\Gamma(\nu-\gamma)}[P_1(\xi)]^{\frac{s-1}{s}}[L(\xi)]^{\frac{1}{s}}, \qquad (2.\ 7)
\end{aligned}
$$

where

$$L(\xi) = \int_{R_1}^{\xi} \varrho(\eta)|D^{\nu}_{R_1}\Psi(\eta\omega)|^s d\eta. \qquad (2.\ 8)$$

Let

$$M(\xi) = \int_{R_1}^{\xi} \varrho(\eta)|D^{\nu}_{R_1}\Phi(\eta\omega)|^s d\eta, \qquad (2.\ 9)$$

then

$$M'(\xi) = \varrho(\xi)|D^{\nu}_{R_1}\Phi(\xi\omega)|^s, \qquad (2.\ 10)$$

that is

$$|D^{\nu}_{R_1}\Phi(\xi\omega)|^q = [M'(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \qquad (2.\ 11)$$

Now (2. 7) and (2. 11) imply that

$$\lambda(\xi)|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p|D^{\nu}_{R_1}\Phi(\xi\omega)|^q \leq Z(\xi)[L(\xi)]^{\frac{p}{s}}[M'(\xi)]^{\frac{q}{s}}, \qquad (2.\ 12)$$

where

$$Z(\xi) = \frac{1}{[\Gamma(\nu-\gamma)]^p}\lambda(\xi)[\varrho(\xi)]^{\frac{-q}{s}}[P_1(\xi)]^{\frac{p(s-1)}{s}}. \qquad (2.\ 13)$$

Integrating (2. 12) and applying Hölder's inequality for $\{\frac{s}{s-q}, \frac{s}{q}\}$ we obtain

$$
\begin{aligned}
&\int_{R_1}^{x} \lambda(\xi)|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p|D^{\nu}_{R_1}\Phi(\xi\omega)|^q d\xi \\
&\leq \left(\int_{R_1}^{x} [Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}} \left(\int_{R_1}^{x} [L(\xi)]^{\frac{p}{q}}M'(\xi)d\xi\right)^{\frac{q}{s}}. \qquad (2.\ 14)
\end{aligned}
$$

Similarly we get

$$\int_{R_1}^{x} \lambda(\xi)|D^{\gamma}_{R_1}\Phi(\xi\omega)|^p|D^{\nu}_{R_1}\Psi(\xi\omega)|^q d\xi$$
$$\leq \left(\int_{R_1}^{x}[Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}} \left(\int_{R_1}^{x}[M(\xi)]^{\frac{p}{q}}L'(\xi)d\xi\right)^{\frac{q}{s}}. \quad (2.\ 15)$$

Now we use inequality

$$c_\epsilon(\Delta+\Theta)^\epsilon \leq \Delta^\epsilon+\Theta^\epsilon \leq d_\epsilon(\Delta+\Theta)^\epsilon, (\Delta,\Theta\geq 0), \quad (2.\ 16)$$

where

$$c_\epsilon = \begin{cases} 1, & 0\leq\epsilon\leq 1; \\ 2^{1-\epsilon}, & \epsilon\geq 1. \end{cases}$$

And

$$d_\epsilon = \begin{cases} 2^{1-\epsilon}, & 0\leq\epsilon\leq 1; \\ 1, & \epsilon\geq 1. \end{cases} \quad (2.\ 17)$$

Therefore from (2. 14), (2. 15) and (2. 16), with $s>q$ we have

$$\int_{R_1}^{x} \lambda(\xi)\left(|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p|D^{\nu}_{R_1}\Phi(\xi\omega)|^q d\xi + |D^{\gamma}_{R_1}\Phi(\xi\omega)|^p|D^{\nu}_{R_1}\Psi(\xi\omega)|^q\right)d\xi$$
$$\leq \left(\int_{R_1}^{x}[Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}} \left(\int_{R_1}^{x}\left([L(\xi)]^{\frac{p}{q}}M'(\xi)+[M(\xi)]^{\frac{p}{q}}L'(\xi)\right)d\xi\right)^{\frac{q}{s}}. \quad (2.\ 18)$$

Since $L(R_1)=M(R_1)=0$ then with (2. 16) we have

$$\int_{R_1}^{x}\left[[L(\xi)]^{\frac{p}{q}}M'(\xi)+[M(\xi)]^{\frac{p}{q}}L'(\xi)\right]d\xi$$
$$= \int_{R_1}^{x}\left[[L(\xi)]^{\frac{p}{q}}+[M(\xi)]^{\frac{p}{q}}\right][L'(\xi)+M'(\xi)]d\xi - \int_{R_1}^{x}\left[[L(\xi)]^{\frac{p}{q}}L'(\xi)+[M(\xi)]^{\frac{p}{q}}M'(\xi)\right]d\xi$$
$$\leq d_{\frac{p}{q}}\int_{R_1}^{x}[L(\xi)+M(\xi)]^{\frac{p}{q}}[L(\xi)+M(\xi)]'d\xi - \frac{q}{p+q}\left[L(x)^{\frac{p}{q}+1}+M(x)^{\frac{p}{q}+1}\right]$$
$$= \frac{q}{p+q}d_{\frac{p}{q}}[L(x)+M(x)]^{\frac{p}{q}+1} - \frac{q}{p+q}\left[L(x)^{\frac{p}{q}+1}+M(x)^{\frac{p}{q}+1}\right]$$
$$\leq \frac{q}{p+q}(d_{\frac{p}{q}}-2^{\frac{-p}{q}})[L(x)+M(x)]^{\frac{p}{q}+1}. \quad (2.\ 19)$$

From (2. 18) and (2. 19) we conclude

$$\int_{R_1}^{x} \lambda(\xi) \left(|D^{\gamma}_{R_1} \Psi(\xi\omega)|^p |D^{\nu}_{R_1} \Phi(\xi\omega)|^q + |D^{\gamma}_{R_1} \Phi(\xi\omega)|^p |D^{\nu}_{R_1} \Psi(\xi\omega)|^q \right) d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{\frac{-p}{q}} \right)^{\frac{q}{s}} \left(\int_{R_1}^{x} [Z(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}}$$

$$\times \left(\int_{R_1}^{x} \varrho(\eta) \left(|D^{\nu}_{R_1} \Phi(\eta\omega)|^s + |D^{\nu}_{R_1} \Psi(\eta\omega)|^s \right) d\eta \right)^{\frac{p+q}{s}} .$$

This complete the proof. □

If we take $\nu = 1$ and $\gamma = 0$ in the above theorem we get the next corollary.

Corollary 2.2. *Assume that $\gamma \geq 0$, and let $\varrho > 0, \lambda \geq 0$ be measurable functions on $[R_1, x]$. If $s > 1, s > q > 0$ and $p \geq 0$ and $\Phi'(\xi\omega), \Psi'(\xi\omega) \in L_s[R_1, R_2]$. Then*

$$\int_{R_1}^{x} \lambda(\xi) \left(|\Psi(\xi\omega)|^p |\Phi'(\xi\omega)|^q + |\Phi(\xi\omega)|^p |\Psi'(\xi\omega)|^q \right) d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}} \right)^{\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(\int_{R_1}^{x} [Z(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}}$$

$$\times \left(\int_{R_1}^{x} \varrho(\eta) \left(|\Phi'(\eta\omega)|^s + |\Psi'(\eta\omega)|^s \right) d\eta \right)^{\frac{p+q}{s}} , \qquad (2.\ 20)$$

where Z and $d_{\frac{p}{q}}$ are define in (2. 13), (2. 17) respectively.

Example 2.3. *If we take $\lambda(\xi) = 1, \varrho(\xi) = 1, s = 2, R_1 = 0, \gamma = 0,$ in (2. 20) we obtain*

$$Z(\xi) = \frac{1}{[\Gamma(1)]^p} \left(\int_{0}^{\xi} d\eta \right)^{\frac{p}{2}} = (\xi)^{\frac{p}{2}},$$

and

$$\left(\int_{0}^{x} [Z(\xi)]^{\frac{2}{2-q}} d\xi \right)^{\frac{2-q}{2}} = \left(\int_{0}^{x} \xi^{\frac{p}{2-q}} d\xi \right)^{\frac{2-q}{2}} = \frac{x^{\frac{p-q+2}{2}}}{\left(\frac{p}{2-q} + 1 \right)^{\frac{2-q}{2}}},$$

where $0 < q < 2$ *and* $p \geq 0$. *If* $\Phi', \Psi' \in L_2[a, b]$, *then for all* $x \in [a, b]$, *we get*

$$\int_0^x \left(|\Psi(\xi\omega)|^p |\Phi'(\xi\omega)|^q + |\Phi(\xi\omega)|^p |\Psi'(\xi\omega)|^q\right) d\xi$$
$$\leq 2^{1-\frac{q}{2}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{2}} \left(\frac{q}{p+q}\right)^{\frac{q}{2}} \frac{x^{\frac{p-q+2}{2}}}{\left(\frac{p}{2-q} + 1\right)^{\frac{2-q}{2}}}$$
$$\times \left(\int_0^x \left(|\Phi'(\eta\omega)|^2 + |\Psi'(\eta\omega)|^2\right) d\eta\right)^{\frac{p+q}{2}}. \tag{2. 21}$$

The related extreme case of Theorem 2.1 is given in next theorem.

Theorem 2.4. *Assume that* $\nu > \gamma_1, \gamma_2 \geq 0$ *and* $D^{\gamma}_{R_1} \Psi(\xi\omega)$ *be the R-L fractional derivative operator. suppose* $\lambda \geq 0$ *be a measurable function on* $[R_1, x]$. *If* $p, l_1, l_2 \geq 0$ *and suppose* $D^{\nu}_{R_1}\Phi, D^{\nu}_{R_1}\Psi \in L_\infty[R_1, R_2]$. *Then*

$$\int_{R_1}^x \lambda(\xi)[|D^{\gamma_1}_{R_1}\Phi(\xi\omega)|^{l_1} |D^{\gamma_2}_{R_1}\Psi(\xi\omega)|^{l_2} |D^{\nu}_{R_1}\Phi(\xi\omega)|^p$$
$$+ |D^{\gamma_2}_{R_1}\Phi(\xi\omega)|^{l_2} |D^{\gamma_1}_{R_1}\Psi(\xi\omega)|^{l_1} |D^{\nu}_{R_1}\Psi(\xi\omega)|^p] d\xi$$
$$\leq N_1 \left[\|D^{\nu}_{R_1}\Phi\|_\infty^{2(l_1+p)} + \|D^{\nu}_{R_1}\Phi\|_\infty^{2l_2} + \|D^{\nu}_{R_1}\Psi\|_\infty^{2l_2} + \|D^{\nu}_{R_1}\Psi\|_\infty^{2(l_1+p)}\right],$$

where

$$N_1 = \frac{(x-R_1)^{l_1(\nu-\gamma_1)+l_2(\nu-\gamma_2)+1} \|\lambda\|_\infty}{2\left[\Gamma(\nu-\gamma_1+1)\right]^{l_1}\left[\Gamma(\nu-\gamma_2+1)\right]^{l_2}\left[l_1(\nu-\gamma_1)+l_2(\nu-\gamma_2)+1\right]}.$$

Proof. Let $\xi \in [R_1, x]$, using identity (1. 5) the triangle inequality and Hölder's inequality, for $i = 1, 2$ we have

$$|D^{\gamma_i}_{R_1}\Phi(\xi\omega)|^{l_i} \leq \frac{1}{[\Gamma(\nu-\gamma_i)]^{l_i}} \left(\int_{R_1}^{\xi} (\xi-\eta)^{\nu-\gamma_i-1} |D^{\nu}_{R_1}\Phi(\eta)| d\eta\right)^{l_i}$$
$$\leq \frac{1}{[\Gamma(\nu-\gamma_i)]^{l_i}} \left(\int_{R_1}^{\xi} (\xi-\eta)^{\nu-\gamma_i-1} d\eta\right)^{l_i} \|D^{\nu}_{R_1}\Phi\|_\infty^{l_i}$$
$$= \frac{(\xi-R_1)^{l_i(\nu-\gamma_i)}}{[\Gamma(\nu-\gamma_i+1)]^{l_i}} \|D^{\nu}_{R_1}\Phi\|_\infty^{l_i}.$$

By analogy for $i = 1, 2$ we get

$$|D^{\gamma_i}_{R_1}\Psi(\xi\omega)|^{l_i} \leq \frac{(\xi-R_1)^{l_i(\nu-\gamma_i)}}{[\Gamma(\nu-\gamma_i+1)]^{l_i}} \|D^{\nu}_{R_1}\Psi\|_\infty^{l_i}. \tag{2. 22}$$

Also

$$|D^{\nu}_{R_1}\Phi(\xi\omega)|^p \leq \|D^{\nu}_{R_1}\Phi\|^p_{\infty},$$

and

$$|D^{\nu}_{R_1}\Psi(\xi\omega)|^p \leq \|D^{\nu}_{R_1}\Psi\|^p_{\infty}.$$

Hence

$$\begin{aligned} &|D^{\gamma_1}_{R_1}\Phi(\xi\omega)|^{l_1}|D^{\gamma_2}_{R_1}\Psi(\xi\omega)|^{l_2}|D^{\nu}_{R_1}\Phi(\xi\omega)|^p \\ &\leq \frac{(\xi-R_1)^{l_1(\nu-\gamma_1)+l_2(\nu-\gamma_2)}}{[\Gamma(\nu-\gamma_1+1)]^{l_1}[\Gamma(\nu-\gamma_2+1)]^{l_2}}\|D^{\nu}_{R_1}\Phi\|^{l_1+p}_{\infty}\|D^{\nu}_{R_1}\Psi\|^{l_2}_{\infty}. \end{aligned} \tag{2. 23}$$

Similarly

$$\begin{aligned} &|D^{\gamma_2}_{R_1}\Phi(\xi\omega)|^{l_2}|D^{\gamma_1}_{R_1}\Psi(\xi\omega)|^{l_1}|D^{\nu}_{R_1}\Psi(\xi\omega)|^p \\ &\leq \frac{(\xi-R_1)^{l_2(\nu-\gamma_2)+l_1(\nu-\gamma_1)}}{[\Gamma(\nu-\gamma_1+1)]^{l_1}[\Gamma(\nu-\gamma_2+1)]^{l_2}}\|D^{\nu}_{R_1}\Phi\|^{l_2}_{\infty}\|D^{\nu}_{R_1}\Psi\|^{l_1+p}_{\infty}. \end{aligned} \tag{2. 24}$$

From (2. 23) and (2. 24) follows

$$\begin{aligned} &\int_{R_1}^{x}\lambda(\xi)[|D^{\gamma_1}_{R_1}\Phi(\xi\omega)|^{l_1}|D^{\gamma_2}_{R_1}\Psi(\xi\omega)|^{l_2}|D^{\nu}_{R_1}\Phi(\xi\omega)|^p \\ &+|D^{\gamma_2}_{R_1}\Phi(\xi\omega)|^{l_2}|D^{\gamma_1}_{R_1}\Psi(\xi\omega)|^{l_1}|D^{\nu}_{R_1}\Psi(\xi\omega)|^p]d\xi \\ &\leq \frac{(x-R_1)^{l_1(\nu-\gamma_1)+l_2(\nu-\gamma_2)+1}\|\lambda\|_{\infty}}{[\Gamma(\nu-\gamma_1+1)]^{l_1}[\Gamma(\nu-\gamma_2+1)]^{l_2}[l_1(\nu-\gamma_1)+l_2(\nu-\gamma_2)+1]} \\ &\times\frac{1}{2}\left[\|D^{\nu}_{R_1}\Phi\|^{2(l_1+p)}_{\infty}+\|D^{\nu}_{R_1}\Phi\|^{2l_2}_{\infty}+\|D^{\nu}_{R_1}\Psi\|^{2l_2}_{\infty}+\|D^{\nu}_{R_1}\Psi\|^{2(l_1+p)}_{\infty}\right] \\ &\leq N_1\left[\|D^{\nu}_{R_1}\Phi\|^{2(l_1+p)}_{\infty}+\|D^{\nu}_{R_1}\Phi\|^{2l_2}_{\infty}+\|D^{\nu}_{R_1}\Psi\|^{2l_2}_{\infty}+\|D^{\nu}_{R_1}\Psi\|^{2(l_1+p)}_{\infty}\right]. \end{aligned}$$

The proof is complete. □

In upcoming the counterpart of Theorem 2.1 is given.

Theorem 2.5. *Assume* $\gamma \geq 0$, $D^{\gamma}_{R_1}\Psi(\xi\omega)$ *be the R-L radial fractional derivative operator. Suppose* $\lambda \geq 0, \varrho > 0$ *be measurable functions on* $[R_1, x]$. *If* $s < 0, q > 0$ *also* $p \geq 0$. *If* $(D^{\nu}_{R_1}\Phi), (D^{\nu}_{R_1}\Psi) \in L_s[R_1, R_2]$, *each of which is of fixed sign a.e. on* $[R_1, R_2]$ *with*

$\frac{1}{(D^{\nu}_{R_1}\Phi)(\xi\omega)}, \frac{1}{(D^{\nu}_{R_1}\Phi)(\xi\omega)} \in L_s[R_1, R_2]$. *Then*

$$\int_{R_1}^{x} \lambda(\xi)\left(|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p |(D^{\nu}_{R_1}\Phi)(\xi\omega)|^q + |D^{\gamma}_{R_1}\Phi(\xi\omega)|^p |(D^{\nu}_{R_1}\Psi)(\xi\omega)|^q\right) d\xi$$

$$\geq 2^{1-\frac{q}{s}}\left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(\int_{R_1}^{x}[Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}$$

$$\times\left(\int_{R_1}^{x}\varrho(\eta)\left(|(D^{\nu}_{R_1}\Phi)(\eta\omega)|^s + |(D^{\nu}_{R_1}\Psi)(\eta\omega)|^s\right)d\eta\right)^{\frac{p+q}{s}}.$$

Proof. Let $\xi \in [R_1, x]$ using the identity (1. 5) and reverse Hölder's inequality for $\{\frac{s}{s-1}, s\}$ we get

$$\begin{aligned}
&|D^{\gamma}_{R_1}\Psi(\xi\omega)| \\
&= \frac{1}{\Gamma(\nu-\gamma)}\int_{R_1}^{\xi}(\xi-\eta)^{\nu-\gamma-1}[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}|D^{\nu}_{R_1}\Psi(\eta\omega)|d\eta \\
&\geq \frac{1}{\Gamma(\nu-\gamma)}\left(\int_{R_1}^{\xi}(\xi-\eta)^{\frac{s(\nu-\gamma-1)}{s-1}}[\varrho(\eta)]^{\frac{1}{1-s}}d\eta\right)^{\frac{s-1}{s}}\left(\int_{R_1}^{\xi}\varrho(\eta)|D^{\nu}_{R_1}\Psi(\eta\omega)|^s d\eta\right)^{\frac{1}{s}} \\
&= \frac{1}{\Gamma(\nu-\gamma)}[P_1(\xi)]^{\frac{s-1}{s}}[L(\xi)]^{\frac{1}{s}}, \qquad (2.\ 25)
\end{aligned}$$

where $L(\xi)$ is defined by (2.8). Since $M(\xi)$ and $M^{'}(\xi)$ are defined by (2. 9) and (2. 10) respectively therefore we can have

$$|D^{\nu}_{R_1}\Phi(\xi\omega)|^q = [M^{'}(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \qquad (2.\ 26)$$

Now (2. 25) and (2. 26) imply that

$$\lambda(\xi)|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p|D^{\nu}_{R_1}\Phi(\xi\omega)|^q \geq Z(\xi)[L(\xi)]^{\frac{p}{s}}[M^{'}(\xi)]^{\frac{q}{s}}, \qquad (2.\ 27)$$

where $Z(\xi)$ defined by (2. 13). Integrating (2. 27) and applying reverse Hölder's inequality for $\{\frac{s}{s-q}, \frac{s}{q}\}$ we obtain

$$\begin{aligned}
&\int_{R_1}^{x}\lambda(\xi)|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p|D^{\nu}_{R_1}\Phi(\xi\omega)|^q d\xi \\
&\geq \left(\int_{R_1}^{x}[Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_{R_1}^{x}[L(\xi)]^{\frac{p}{q}}M^{'}(\xi)d\xi\right)^{\frac{q}{s}}. \qquad (2.\ 28)
\end{aligned}$$

Similarly we get

$$\int_{R_1}^{x} \lambda(\xi)|D^{\gamma}_{R_1}\Phi(\xi\omega)|^p|D^{\nu}_{R_1}\Psi(\xi\omega)|^q d\xi$$

$$\geq \left(\int_{R_1}^{x}[Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}} \left(\int_{R_1}^{x}[M(\xi)]^{\frac{p}{q}}L^{'}(\xi)d\xi\right)^{\frac{q}{s}}. \tag{2. 29}$$

The inequality for negative power is given

$$\Delta^{\chi}+\Theta^{\chi}\geq 2^{1-\chi}(\Delta+\Theta)^{\chi}, (\chi<0;\Delta,\Theta\geq 0), \tag{2. 30}$$

thus from (2. 28), (2. 29) and (2. 30) for $\chi<0$ with $\frac{q}{s}<0$ we have

$$\int_{R_1}^{x} \lambda(\xi)\left[|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p|D^{\nu}_{R_1}\Phi(\xi\omega)|^q d\xi+|D^{\gamma}_{R_1}\Phi(\xi\omega)|^p|D^{\nu}_{R_1}\Psi(\xi\omega)|^q\right]d\xi$$

$$\geq \left(\int_{R_1}^{x}[Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}} \left(\int_{R_1}^{x}\left[[L(\xi)]^{\frac{p}{q}}M^{'}(\xi)+[M(\xi)]^{\frac{p}{q}}L^{'}(\xi)\right]d\xi\right)^{\frac{q}{s}}. \tag{2. 31}$$

For $\frac{p}{q}>0$ and since $L(R_1)=M(R_1)=0$ then with (2. 16) we have

$$\int_{R_1}^{x}\left[[L(\xi)]^{\frac{p}{q}}M^{'}(\xi)+[M(\xi)]^{\frac{p}{q}}L^{'}(\xi)\right]d\xi\geq\frac{q}{p+q}(c_{\frac{p}{q}}-2^{\frac{-p}{q}})[L(x)+M(x)]^{\frac{p}{q}+1}. \tag{2. 32}$$

From (2. 31) and (2. 32) we conclude that

$$\int_{R_1}^{x} \lambda(\xi)\left[|D^{\gamma}_{R_1}\Psi(\xi\omega)|^p|D^{\nu}_{R_1}\Phi(\xi\omega)|^q+|D^{\gamma}_{R_1}\Phi(\xi\omega)|^p|D^{\nu}_{R_1}\Psi(\xi\omega)|^q\right]d\xi$$

$$\geq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(c_{\frac{p}{q}}-2^{\frac{-p}{q}}\right)^{\frac{q}{s}}\left(\int_{R_1}^{x}[Z(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}$$

$$\times\left(\int_{R_1}^{x}\varrho(\eta)\left[|D^{\nu}_{R_1}\Phi(\eta\omega)|^s+|D^{\nu}_{R_1}\Psi(\eta\omega)|^s\right]d\eta\right)^{\frac{p+q}{s}}.$$

This complete the proof. □

3. OPIAL-TYPE INEQUALITIES FOR CAPUTO RADIAL FRACTIONAL DERIVATIVE OPERATOR

In this section we shall give inequalities involving Caputo radial fractional derivative.

Theorem 3.1. *Let* $\frac{\partial^{\nu}_{*R_1}}{\partial\xi^{\nu}}$ *denote the Caputo radial fractional derivative operator. Suppose* $\varrho > 0, \lambda \geq 0$ *be measurable functions on* $[R_1, x]$*. If* $s > 1, s > q > 0$ *also* $p \geq 0$*. Suppose* $\frac{\partial^n\Psi(\xi\omega)}{\partial\xi^n}, \frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n} \in L_s[R_1, R_2]$*. Then*

$$\int_{R_1}^{x} \lambda(\xi)\left(\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu}}\right|^p\left|\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\right|^q + \left|\frac{\partial^{\nu}_{*R_1}\Phi(\xi)}{\partial\xi^{\nu}}\right|^p\left|\frac{\partial^n\Psi(\xi\omega)}{\partial\xi^n}\right|^q\right)d\xi$$

$$\leq 2^{1-\frac{q}{s}}\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(\int_{R_1}^{x}[C(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}$$

$$\times\left(\int_{R_1}^{x}\varrho(\eta)\left(\left|\frac{\partial^n\Phi(\eta\omega)}{\partial\xi^n}\right|^s + \left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|^s\right)d\eta\right)^{\frac{p+q}{s}}, \tag{3. 33}$$

where

$$C(\xi) = \frac{1}{[\Gamma(n-\nu)]^p}\lambda(\xi)[\varrho(\xi)]^{\frac{-q}{s}}\left(\int_{R_1}^{\xi}(\xi-\eta)^{\frac{s(n-\nu-1)}{s-1}}[\varrho(\eta)]^{\frac{1}{1-s}}d\eta\right)^{\frac{p(s-1)}{s}}.$$

Proof. Let $\xi \in [R_1, x]$, using the inequality (1. 1), the identity (1. 6) and Hölder's inequality for $\{\frac{s}{s-1}, s\}$ we get.

$$\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu}}\right|$$

$$\leq \frac{1}{\Gamma(n-\nu)}\int_{R_1}^{\xi}(\xi-\eta)^{n-\nu-1}[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}\left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|d\eta$$

$$\leq \frac{1}{\Gamma(n-\nu)}\left(\int_{R_1}^{\xi}(\xi-\eta)^{\frac{s(n-\nu-1)}{s-1}}[\varrho(\eta)]^{\frac{1}{1-s}}d\eta\right)^{\frac{s-1}{s}}\left(\int_{R_1}^{\xi}\varrho(\eta)\left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|^s d\eta\right)^{\frac{1}{s}}$$

$$= \frac{1}{\Gamma(n-\nu)}[P_2(\xi)]^{\frac{s-1}{s}}[A(\xi)]^{\frac{1}{s}}, \tag{3. 34}$$

where

$$A(\xi) = \int_{R_1}^{\xi}\varrho(\eta)\left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|^s d\eta,$$

and

$$B(\xi) = \int_{R_1}^{\xi}\varrho(\eta)\left|\frac{\partial^n\Phi(\eta\omega)}{\partial\xi^n}\right|^s d\eta.$$

Then

$$B^{'}(\xi) = \varrho(\xi)\left|\frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n}\right|^s,$$

that is

$$\left|\frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n}\right|^q = [B^{'}(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \tag{3. 35}$$

Now (3. 34) and (3. 35) imply that

$$\lambda(\xi)\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial \xi^{\nu}}\right|^p \left|\frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n}\right|^q \leq C(\xi)[A(\xi)]^{\frac{p}{s}}[B^{'}(\xi)]^{\frac{q}{s}}, \tag{3. 36}$$

where

$$C(\xi) = \frac{1}{[\Gamma(n-\nu)]^p}\lambda(\xi)[\varrho(\xi)]^{\frac{-q}{s}}[P_2(\xi)]^{\frac{p(s-1)}{s}}.$$

Integrating (3. 36) and applying Hölder's inequality for $\{\frac{s}{s-q}, \frac{s}{q}\}$ we obtain

$$\int_{R_1}^{x}\lambda(\xi)\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial \xi^{\nu}}\right|^p \left|\frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n}\right|^q d\xi$$
$$\leq \left(\int_{R_1}^{x}[C(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_{R_1}^{x}[A(\xi)]^{\frac{p}{q}}B^{'}(\xi)d\xi\right)^{\frac{q}{s}}. \tag{3. 37}$$

Similarly we get

$$\int_{R_1}^{x}\lambda(\xi)\left|\frac{\partial^{\nu}_{*R_1}\Phi(\xi)}{\partial \xi^{\nu}}\right|^p \left|\frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n}\right|^q d\xi$$
$$\leq \left(\int_{R_1}^{x}[C(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_{R_1}^{x}[B(\xi)]^{\frac{p}{q}}A^{'}(\xi)d\xi\right)^{\frac{q}{s}}. \tag{3. 38}$$

Therefore from (3. 37), (3. 38) and (2. 16) with $s > q$ we have

$$\int_{R_1}^{x}\lambda(\xi)\left[\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial \xi^{\nu}}\right|^p \left|\frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n}\right|^q + \left|\frac{\partial^{\nu}_{*R_1}\Phi(\xi)}{\partial \xi^{\nu}}\right|^p \left|\frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n}\right|^q\right] d\xi$$
$$\leq \left(\int_{R_1}^{x}[C(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}}\left(\int_{R_1}^{x}\left[[A(\xi)]^{\frac{p}{q}}B^{'}(\xi) + [B(\xi)]^{\frac{p}{q}}A^{'}(\xi)\right]d\xi\right)^{\frac{q}{s}}. \tag{3. 39}$$

Since $A(R_1) = B(R_1) = 0$ then with (2. 16) we have

$$\int_{R_1}^{x}\left[[A(\xi)]^{\frac{p}{q}}B^{'}(\xi) + [B(\xi)]^{\frac{p}{q}}A^{'}(\xi)\right]d\xi \leq \frac{q}{p+q}(d_{\frac{p}{q}} - 2^{\frac{-p}{q}})[A(x)+B(x)]^{\frac{p}{q}+1}. \tag{3. 40}$$

From(3. 39) and (3. 40) we conclude

$$\int_{R_1}^{x} \lambda(\xi) \left[\left| \frac{\partial^{\nu}_{*R_1} \Psi(\xi)}{\partial \xi^{\nu}} \right|^{p} \left| \frac{\partial^{n} \Phi(\xi\omega)}{\partial \xi^{n}} \right|^{q} + \left| \frac{\partial^{\nu}_{*R_1} \Phi(\xi)}{\partial \xi^{\nu}} \right|^{p} \left| \frac{\partial^{n} \Psi(\xi\omega)}{\partial \xi^{n}} \right|^{q} \right] d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{\frac{-p}{q}} \right)^{\frac{q}{s}} \left(\int_{R_1}^{x} [C(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}}$$

$$\times \left(\int_{R_1}^{x} \varrho(\eta) \left[\left| \frac{\partial^{n} \Phi(\eta\omega)}{\partial \xi^{n}} \right|^{s} + \left| \frac{\partial^{n} \Psi(\eta\omega)}{\partial \xi^{n}} \right|^{s} \right] d\eta \right)^{\frac{p+q}{s}} .$$

The proof is complete. □

Remark 3.2. If $\nu \in \mathbb{N}$, then the inequality (3. 33) takes the form:

$$\int_{R_1}^{x} \lambda(\xi) \left(\left| \frac{\partial^{\nu} \Psi(\xi)}{\partial \xi^{\nu}} \right|^{p} \left| \frac{\partial^{n} \Phi(\xi\omega)}{\partial \xi^{n}} \right|^{q} + \left| \frac{\partial^{\nu} \Phi(\xi)}{\partial \xi^{\nu}} \right|^{p} \left| \frac{\partial^{n} \Psi(\xi\omega)}{\partial \xi^{n}} \right|^{q} \right) d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}} \right)^{\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(\int_{R_1}^{x} [C(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}}$$

$$\times \left(\int_{R_1}^{x} \varrho(\eta) \left(\left| \frac{\partial^{n} \Phi(\eta\omega)}{\partial \xi^{n}} \right|^{s} + \left| \frac{\partial^{n} \Psi(\eta\omega)}{\partial \xi^{n}} \right|^{s} \right) d\eta \right)^{\frac{p+q}{s}} .$$

and particularly for $\nu = 0$ we get

$$\int_{R_1}^{x} \lambda(\xi) \left(|\Psi(\xi)|^{p} |\Phi(\xi\omega)|^{q} + |\Phi(\xi)|^{p} |\Psi(\xi\omega)|^{q} \right) d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}} \right)^{\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(\int_{R_1}^{x} [C(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}}$$

$$\times \left(\int_{R_1}^{x} \varrho(\eta) \left(|\Phi(\eta\omega)|^{s} + |\Psi(\eta\omega)|^{s} \right) d\eta \right)^{\frac{p+q}{s}} .$$

The next theorem is the related extreme case of the Theorem 3.1 for $s = \infty$

Theorem 3.3. *Assume $n > \gamma_1, \gamma_2 \geq 0$ and $\frac{\partial^{\nu}_{*R_1}}{\partial \xi^{\nu}}$ be the Caputo radial fractional differential operator. Let $\lambda \geq 0$ be a measurable function on $[R_1, x]$. Suppose $p, l_1, l_2 \geq 0$ and let*

$\frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n}, \frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n} \in L_\infty[R_1, R_2]$. *Then*

$$\int_{R_1}^{x} \lambda(\xi) \left[\left| \frac{\partial^{\nu_1}_{*R_1} \Phi(\xi)}{\partial \xi^{\nu_1}} \right|^{l_1} \left| \frac{\partial^{\nu_2}_{*R_1} \Psi(\xi)}{\partial \xi^{\nu_2}} \right|^{l_2} \left| \frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n} \right|^{p} \right.$$
$$\left. + \left| \frac{\partial^{\nu_2}_{*R_1} \Phi(\xi)}{\partial \xi^{\nu_2}} \right|^{l_2} \left| \frac{\partial^{\nu_1}_{*R_1} \Psi(\xi)}{\partial \xi^{\nu_1}} \right|^{l_1} \left| \frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n} \right|^{p} \right] d\xi$$
$$\leq N_2 \left[\| \frac{\partial^n \Phi}{\partial \xi^n} \|_\infty^{2(l_1+p)} + \| \frac{\partial^n \Phi}{\partial \xi^n} \|_\infty^{2l_2} + \| \frac{\partial^n \Psi}{\partial \xi^n} \|_\infty^{2l_2} + \| \frac{\partial^n \Psi}{\partial \xi^n} \|_\infty^{2(l_1+p)} \right],$$

where

$$N_2 = \frac{(x-R_1)^{l_1(n-\nu_1)+l_2(n-\nu_2)+1} \|\lambda\|_\infty}{2 \left[\Gamma(n-\nu_1+1)\right]^{l_1} \left[\Gamma(n-\nu_2+1)\right]^{l_2} \left[l_1(n-\nu_1)+l_2(n-\nu_2)+1\right]}.$$

Proof. Let $\xi \in [R_1, x]$, using inequality (1. 1) and the identity (1. 6) and Hölder's inequality, for $i = 1, 2$ we have

$$\left| \frac{\partial^{\nu_i}_{*R_1} \Phi(\xi)}{\partial \xi^{\nu_i}} \right|^{l_i} \leq \frac{1}{\left[\Gamma(n-\nu_i)\right]^{l_i}} \left(\int_{R_1}^{\xi} (\xi-\eta)^{n-\nu_i-1} \left| \frac{\partial^n \Phi(\eta)}{\partial \xi^n} \right| d\eta \right)^{l_i}$$
$$\leq \frac{1}{\left[\Gamma(n-\nu_i)\right]^{l_i}} \left(\int_{R_1}^{\xi} (\xi-\eta)^{n-\nu_i-1} d\eta \right)^{l_i} \| \frac{\partial^n \Phi}{\partial \xi^n} \|_\infty^{l_i}$$
$$= \frac{(\xi-R_1)^{l_i(n-\nu_i)}}{\left[\Gamma(n-\nu_i+1)\right]^{l_i}} \| \frac{\partial^n \Phi}{\partial \xi^n} \|_\infty^{l_i}.$$

By analogy for $i = 1, 2$ we get

$$\left| \frac{\partial^{\nu_i}_{*R_1} \Psi(\xi)}{\partial \xi^{\nu_i}} \right|^{l_i} \leq \frac{(\xi-R_1)^{l_i(n-\nu_i)}}{\left[\Gamma(n-\nu_i+1)\right]^{l_i}} \| \frac{\partial^n \Psi}{\partial \xi^n} \|_\infty^{l_i}. \tag{3. 41}$$

Also

$$\left| \frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n} \right|^{p} \leq \| \frac{\partial^n \Phi}{\partial \xi^n} \|_\infty^{p},$$

and

$$\left| \frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n} \right|^{p} \leq \| \frac{\partial^n \Psi}{\partial \xi^n} \|_\infty^{p}.$$

Hence

$$\left| \frac{\partial^{\nu_1}_{*R_1} \Phi(\xi)}{\partial \xi^{\nu_1}} \right|^{l_1} \left| \frac{\partial^{\nu_2}_{*R_1} \Psi(\xi)}{\partial \xi^{\nu_2}} \right|^{l_2} \left| \frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n} \right|^{p}$$
$$\leq \frac{(\xi-R_1)^{l_1(n-\nu_1)+l_2(n-\nu_2)}}{\left[\Gamma(n-\nu_1+1)\right]^{l_1} \left[\Gamma(n-\nu_2+1)\right]^{l_2}} \| \frac{\partial^n \Phi}{\partial \xi^n} \|_\infty^{l_1+p} \| \frac{\partial^n \Psi}{\partial \xi^n} \|_\infty^{l_2}. \tag{3. 42}$$

Likewise we can write

$$\left|\frac{\partial^{\nu_2}_{*R_1}\Phi(\xi)}{\partial\xi^{\nu_2}}\right|^{l_2}\left|\frac{\partial^{\nu_1}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu_1}}\right|^{l_1}\left|\frac{\partial^n\Psi(\xi\omega)}{\partial\xi^n}\right|^p$$
$$\leq \frac{(\xi-R_1)^{l_2(n-\nu_2)+l_1(n-\nu_1)}}{[\Gamma(n-\nu_1+1)]^{l_1}\,[\Gamma(n-\nu_2+1)]^{l_2}}\|\frac{\partial^n\Phi}{\partial\xi^n}\|^{l_2}_\infty\|\frac{\partial^n\Psi}{\partial\xi^n}\|^{l_1+p}_\infty. \qquad (3.\ 43)$$

From (3. 42) and (3. 43) follows

$$\int_{R_1}^{x}\lambda(\xi)\left[\left|\frac{\partial^{\nu_1}_{*R_1}\Phi(\xi)}{\partial\xi^{\nu_1}}\right|^{l_1}\left|\frac{\partial^{\nu_2}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu_2}}\right|^{l_2}\left|\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\right|^p\right.$$
$$\left.+\left|\frac{\partial^{\nu_2}_{*R_1}\Phi(\xi)}{\partial\xi^{\nu_2}}\right|^{l_2}\left|\frac{\partial^{\nu_1}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu_1}}\right|^{l_1}\left|\frac{\partial^n\Psi(\xi\omega)}{\partial\xi^n}\right|^p\right]d\xi$$
$$\leq \frac{(x-R_1)^{l_1(v-\gamma_1)+l_2(n-\nu_2)+1}\|\lambda\|_\infty}{[\Gamma(n-\nu_1+1)]^{l_1}\,[\Gamma(n-\nu_2+1)]^{l_2}\,[l_1(n-\nu_1)+l_2(n-\nu_2)+1]}$$
$$\times\frac{1}{2}\left[\|\frac{\partial^n\Phi}{\partial\xi^n}\|^{2(l_1+p)}_\infty+\|\frac{\partial^n\Phi}{\partial\xi^n}\|^{2l_2}_\infty+\|\frac{\partial^n\Psi}{\partial\xi^n}\|^{2l_2}_\infty+\|\frac{\partial^n\Psi}{\partial\xi^n}\|^{2(l_1+p)}_\infty\right]$$
$$\leq N_2\left[\|\frac{\partial^n\Phi}{\partial\xi^n}\|^{2(l_1+p)}_\infty+\|\frac{\partial^n\Phi}{\partial\xi^n}\|^{2l_2}_\infty+\|\frac{\partial^n\Psi}{\partial\xi^n}\|^{2l_2}_\infty+\|\frac{\partial^n\Psi}{\partial\xi^n}\|^{2(l_1+p)}_\infty\right].$$

This complete the proof. □

Now for $s<0$ we give the counterpart of the Theorem 3.1.

Theorem 3.4. *Let* $\frac{\partial^{\nu}_{*R_1}}{\partial\xi^\nu}$ *be the Caputo radial fractional derivative operator. Suppose* $\varrho>0,\lambda\geq 0$ *be measurable functions on* $[R_1,x]$. *Also suppose that* $s<0, q>0, p\geq 0$ *and* $\frac{\partial^n\Psi(\xi\omega)}{\partial\xi^n},\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\in L_s[R_1,R_2]$, *each of which is of fixed sign a.e. on* $[R_1,R_2]$. *Assume* $\frac{1}{\frac{\partial^n\Psi(\xi\omega)}{\partial\xi^n}},\frac{1}{\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}}\in L_s[R_1,R_2]$. *Then*

$$\int_{R_1}^{x}\lambda(\xi)\left(\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu}}\right|^p\left|\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\right|^q+\left|\frac{\partial^{\nu}_{*R_1}\Phi(\xi)}{\partial\xi^{\nu}}\right|^p\left|\frac{\partial^n\Psi(\xi\omega)}{\partial\xi^n}\right|^q\right)d\xi$$
$$\geq 2^{1-\frac{q}{s}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(\int_{R_1}^{x}[C(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}$$
$$\times\left(\int_{R_1}^{x}\varrho(\eta)\left(\left|\frac{\partial^n\Phi(\eta\omega)}{\partial\xi^n}\right|^s+\left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|^s\right)d\eta\right)^{\frac{p+q}{s}}.$$

Proof. Let $\xi \in [R_1, x]$, using the identity (1. 6) and the reverse Hölder's inequality for $\{\frac{s}{s-1}, s\}$ we get.

$$\begin{aligned}
&\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu}}\right| \\
&= \frac{1}{\Gamma(n-\nu)}\int_{R_1}^{\xi}(\xi-\eta)^{n-\nu-1}[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}\left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|d\eta \\
&\geq \frac{1}{\Gamma(n-\nu)}\left(\int_{R_1}^{\xi}(\xi-\eta)^{\frac{s(n-\nu-1)}{s-1}}[\varrho(\eta)]^{\frac{1}{1-s}}d\eta\right)^{\frac{s-1}{s}}\left(\int_{R_1}^{\xi}\varrho(\eta)\left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|^s d\eta\right)^{\frac{1}{s}} \\
&= \frac{1}{\Gamma(n-\nu)}[P_2(\xi)]^{\frac{s-1}{s}}[A(\xi)]^{\frac{1}{s}}, \qquad (3.\ 44)
\end{aligned}$$

where

$$A(\xi) = \int_{R_1}^{\xi}\varrho(\eta)\left|\frac{\partial^n\Psi(\eta\omega)}{\partial\xi^n}\right|^s d\eta,$$

and

$$B(\xi) = \int_{R_1}^{\xi}\varrho(\eta)\left|\frac{\partial^n\Phi(\eta\omega)}{\partial\xi^n}\right|^s d\eta.$$

Then

$$B^{'}(\xi) = \varrho(\xi)\left|\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\right|^s,$$

that is

$$\left|\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\right|^q = [B^{'}(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \qquad (3.\ 45)$$

Now (3. 44) and (3. 45) imply that

$$\lambda(\xi)\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu}}\right|^p\left|\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\right|^q \geq C(\xi)[A(\xi)]^{\frac{p}{s}}[B^{'}(\xi)]^{\frac{q}{s}}, \qquad (3.\ 46)$$

where

$$C(\xi) = \frac{1}{[\Gamma(n-\nu)]^p}\lambda(\xi)[\varrho(\xi)]^{\frac{-q}{s}}[P_2(\xi)]^{\frac{p(s-1)}{s}}.$$

Integrating (3. 46) and applying reverse Hölder's inequality for $\{\frac{s}{s-q}, \frac{s}{q}\}$ we obtain

$$\begin{aligned}
&\int_{R_1}^{x}\lambda(\xi)\left|\frac{\partial^{\nu}_{*R_1}\Psi(\xi)}{\partial\xi^{\nu}}\right|^p\left|\frac{\partial^n\Phi(\xi\omega)}{\partial\xi^n}\right|^q d\xi \\
&\geq \left(\int_{R_1}^{x}[C(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_{R_1}^{x}[A(\xi)]^{\frac{p}{q}}B^{'}(\xi)d\xi\right)^{\frac{q}{s}}. \qquad (3.\ 47)
\end{aligned}$$

Similarly we get

$$\int_{R_1}^{x} \lambda(\xi) \left| \frac{\partial^{\nu}_{*R_1} \Phi(\xi)}{\partial r^{\nu}} \right|^p \left| \frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n} \right|^q d\xi$$
$$\geq \left(\int_{R_1}^{x} [C(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}} \left(\int_{R_1}^{x} [B(\xi)]^{\frac{p}{q}} A^{'}(\xi) d\xi \right)^{\frac{q}{s}}. \qquad (3.\ 48)$$

Therefore from (3. 47), (3. 48) and (2. 30), for $\chi < 0$ with $\frac{q}{s} < 0$ we have

$$\int_{R_1}^{x} \lambda(\xi) \left[\left| \frac{\partial^{\nu}_{*R_1} \Psi(\xi)}{\partial \xi^{\nu}} \right|^p \left| \frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n} \right|^q + \left| \frac{\partial^{\nu}_{*R_1} \Phi(\xi)}{\partial \xi^{\nu}} \right|^p \left| \frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n} \right|^q \right] d\xi$$
$$\geq \left(\int_{R_1}^{x} [C(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}} \left(\int_{R_1}^{x} \left[[A(\xi)]^{\frac{p}{q}} B^{'}(\xi) + [B(\xi)]^{\frac{p}{q}} A^{'}(\xi) \right] d\xi \right)^{\frac{q}{s}}. \qquad (3.\ 49)$$

For $\frac{p}{q} > 0$ and $A(R_1) = B(R_1) = 0$ then with (2. 16) we have

$$\int_{R_1}^{x} \left[[A(\xi)]^{\frac{p}{q}} B^{'}(\xi) + [B(\xi)]^{\frac{p}{q}} A^{'}(\xi) \right] d\xi$$
$$\geq \left(\frac{q}{p+q} \right) \left(c_{\frac{p}{q}} - 2^{\frac{-p}{q}} \right) [A(x) + B(x)]^{\frac{p}{q}+1}. \qquad (3.\ 50)$$

From (3. 49) and (3. 50) we conclude

$$\int_{R_1}^{x} \lambda(\xi) \left[\left| \frac{\partial^{\nu}_{*R_1} \Psi(\xi)}{\partial \xi^{\nu}} \right|^p \left| \frac{\partial^n \Phi(\xi\omega)}{\partial \xi^n} \right|^q + \left| \frac{\partial^{\nu}_{*R_1} \Phi(\xi)}{\partial \xi^{\nu}} \right|^p \left| \frac{\partial^n \Psi(\xi\omega)}{\partial \xi^n} \right|^q \right] d\xi$$
$$\geq 2^{1-\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(c_{\frac{p}{q}} - 2^{\frac{-p}{q}} \right)^{\frac{q}{s}} \left(\int_{R_1}^{x} [C(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}}$$
$$\times \left(\int_{R_1}^{x} \varrho(\eta) \left[\left| \frac{\partial^n \Phi(\eta\omega)}{\partial \xi^n} \right|^s + \left| \frac{\partial^n \Psi(\eta\omega)}{\partial \xi^n} \right|^s \right] d\eta \right)^{\frac{p+q}{s}}.$$

The proof is complete. □

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