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Caputo-Fabrizio Fractionalized Second Grade Fluid in a Circular Cylinder with Uniform Magnetic Field

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Abstract. In this communication, magnetohydrodynamics (MHD) fractional second grade fluid in a circular pipe is observed. Initially the whole system which consists on a circular cylinder filled with fluid having infinite length is at rest. Suddenly, the pipe set in translational motion along its axis due to stress i.e time dependent, on the boundary of pipe. The fluid inside the circular pipe which is electrically conducting, gradually starts moving by the effect of cylinder's motion. In the governing equations of second grade fluid, an innovative formulation of fractional derivative (FD) with specific kernel which is without singularity in a prescribed domain, given by Caputo and Fabrizio jointly is used, which is more suitable for viscoelastic and electromagnetic systems as compare to usual fractional derivative. To analyze above flow problem we use Laplace transform and modified Bessel functions. For the calculation of inverse Laplace transform, we apply Stehfest's algorithm with the help of MATHCAD software instead of lengthy and complicated calculations. For the validity of the results, a very good comparison between existing analytical solution and our approximate results is obtained, which shows that both the solutions are equivalent. Finally, the effect of different parameters and their comparison are explained graphically.

AMS (MOS) Subject Classification Code: 76A05
Key Words: Magnetohydrodynamics; Caputo-Fabrizio fractional derivative; Second grade fluid; Circular cylinder; Semi analytical solution; Velocity field; Shear stress.

1. INTRODUCTION

In the last few decades, researchers show notable interest in the flow problems of non-Newtonian fluids (blood, cosmetic fluids, paint, polymer, fresh concrete, sludge, multi grade oils, mud flows and greases, etc) instead of Newtonian fluids. Most important thing is that non-Newtonian fluids are used on large scale in industry like in food industry, petroleum industry, chemical engineering, geophysics, biological analysis etc. Fluids have vast applications in our practical life. Many fluids that are used in industry are much different from Newtonian fluids in their rheology. For those fluids stress shows nonlinear behaviour for rate of strain. The response of various viscoelastic fluids can not be studied by using Newtonian fluid model. The critical thinking about the flow behaviour of non-Newtonian fluids is essential because it involves in many aspects of life. Since non-Newtonian fluids have complex structure and typical properties therefore a large number of fluid models proposed in literature for them such as integral, differential [10] and rate type [26]. Among them the differential type fluids model are specially concentrated by researchers due to their several technological applications.

The study of the movement of a fluid in a translating, oscillating or rotating cylinders is very important and interesting. The study of the mechanism of viscoelastic fluids is very critical. It has lot of applications in many walks of industry, such as chemical industry, bio engineering and oil exploitation.

Viscoelasticity is the property of fluids which has major effect on the motion of the fluids or movement of the bodies through fluids even at micro level. It is practically verified that differential equations with non-integer order is very reasonable to describe the viscoelasticity and some other properties of the fluids. Many researchers are interested in generalizing the flow problems by using FDs. They have made various formulations of FD like Riemann-Liouville, Capto and Capto-Fabrizio fractional derivative (CFFD) [11, 17, 12, 9]. We know commonly, the derivative of constant terms must be zero but Riemann-Liouville FD does not follow this rule. We come to know that when we apply Laplace transform upon Riemann-Liouville FD it gives some insignificant terms in result. The new definition of FD which wipe out these obstacles and having very smooth and non-singular kernel, is given by Capto and Fabrizio [9]. To solve our flow problem we also used this new definition because it is most reasonable for Laplace transform. Some investigations with FDs can be seen [13, 31, 5, 7, 8, 15, 19, 23, 22, 28, 25, 16, 33, 34, 35, 36, 37, 3, 18, **?**].

The flow of fluid which is electrically conducted and transversal magnetic field present, through a circular pipe or channel is very important due to its usage in MHD generators, accelerators, pumps and flow meters it becomes essential for life. Some important investigations about MHD flowing fluids are [20, 2, 4, 21, 14, 1, 27, 29].

The aspiration of this communication is to find out velocity and stress fields of an incompressible MHD second grade fluid (MHD-SGF) with CFFD, in a circular cylinder by using semi analytical technique. Initially the cylinder and fluid both are at rest. At the moment $(t = 0^+)$, the cylinder pulled by longitudinal shear which is time dependent. The well known integral transform namely Laplace transform is used to simplify the above flow model analytically which becomes very complicated. To solve this problem we used Stehfest's algorithm with the help of MATHCAD software.

2. GOVERNING EQUATIONS

The flow considered, have the specific velocity field say **V** and extra stress tensor **P** of the form [6]

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\mathbf{e_z}, \qquad \mathbf{P} = \mathbf{P}(r,t), \tag{2. 1}$$

where $\mathbf{e_z}$ is the unit vector in the z-direction of the cylindrical coordinates system r, θ and z. The constraints of incompressibility for above flows are automatically satisfied. Since initially the whole system is at rest so

$$\mathbf{v}(r,0) = \mathbf{0}, \qquad \mathbf{P}(r,0) = \mathbf{0}. \tag{2. 2}$$

A set of equations which define MHD-SGF is [14, 1]

$$\frac{\partial v(r,t)}{\partial t} = \left(\nu + \alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) v(r,t) - M\,v(r,t); \tag{2. 3}$$

$$\tau(r,t) = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial r}, \tag{2. 4}$$

where

$\tau(r,t) = p_{rz}$	only non zero shear stress
α_1	normal stress module
$\alpha = \frac{\alpha_1}{\rho}$	material constant
μ	dynamic viscosity
$\nu = \frac{\mu}{\rho}$	kinematic viscosity
ρ	constant density
$M = \frac{\sigma B_0^2}{\rho}$	magnetic parameter
σ	coefficient of electric conductivity
B_0	transverse magnetic field

Equations representing incompressible MHD fractional second grade fluid (MHD-FSGF) are obtained by using CFFD [9]

$$D_t^{\varepsilon} g(t) = \begin{cases} \frac{L(\varepsilon)}{(1-\varepsilon)}\int_{t_0}^{t} \dot{g(\tau)} \exp\left[\frac{-\varepsilon(t-\tau)}{1-\varepsilon}\right] d\tau, & t > t_0, \quad \varphi \in (0,1) \\ \frac{d}{dt}\, g(t), & \varphi = 1, \end{cases} \tag{2. 5}$$

as a replacement for inner time derivative in Eqs. (2.3) and (2.4), where $L(\varepsilon)$ is a normalization function such that $L(0) = 1 = L(1)$. Consequently, the governing equations for an incompressible MHD-FSGF model are

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha D_t^{\eta})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) v(r,t) - M\,v(r,t); \tag{2. 6}$$

$$\tau(r,t) = (\mu + \alpha_1 D_t^{\eta})\frac{\partial v(r,t)}{\partial r}. \tag{2. 7}$$

3. FLOW THROW AN ANNULUS

Suppose a circular cylinder filled with incompressible MHD-FSGF is at rest initially. Further we consider R be the radius of an infinite cylinder. At $(t = 0^+)$, the cylinder is suddenly moved with shear stress that is time depended. The fluid steadily moved because of effect of the shear. The system of governing equations are given by Eqs. (2.6) and (2.7) and the initial and boundary conditions are

$$v(r,0) = 0 = \tau(r,0), \quad r \in [0, R], \tag{3. 8}$$

$$\tau(R,t) = (\mu + \alpha_1 D_t^{\eta}) \frac{\partial v(r,t)}{\partial r}|_{r=R} = At, \tag{3. 9}$$

where A is any constant. For the solution of above problem there exits a number of techniques in literature but we use most efficient, systematic and powerful integral transform technique named as Laplace transform. We evaluate inverse Laplace transform by using Stehfest's algorithm with the help of MATHCAD software instead of huge calculation and writing material.

3.1. **Calculation of the velocity field.** Keeping in mind initial condition Eq. (3.8), we apply the Laplace transform to Eqs. (2.6) and (3.9)

$$s\breve{v}(r,s) = \left[\nu + \frac{\alpha s}{\eta + (1-\eta)s}\right]\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\breve{v}(r,s) - M\breve{v}(r,s), \tag{3. 10}$$

$$\frac{\partial \breve{v}(r,s)}{\partial r}|_{r=R} = \frac{A}{s^2\left[\mu + \frac{\alpha_1 s}{\eta + (1-\eta)s}\right]}, \tag{3. 11}$$

where, $\breve{v}(r,s)$ represents the Laplace transforms of the function $v(r,t)$ and s is Laplace transform parameter. Eqs. (3.10) and (3.11) can be written as

$$\left[\frac{(s+M)[\eta + (1-\eta)s]}{\nu\eta + \eta_2 s}\right]\breve{v}(r,s) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\breve{v}(r,s), \tag{3. 12}$$

$$\frac{\partial \breve{v}(r,s)}{\partial r}|_{r=R} = \frac{A[\eta + (1-\eta)s]}{s^2[\eta\mu + \eta_1 s]} \tag{3. 13}$$

where $\eta_2 = \nu(1-\eta) + \alpha$ and $\eta_1 = \mu(1-\eta) + \alpha_1$.

Above Eqs. (3.12) and (3.13) can be written as

$$\frac{\partial^2 \breve{v}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \breve{v}(r,s)}{\partial r} - \phi(s)\breve{v}(r,s) = 0, \tag{3. 14}$$

$$\frac{\partial \breve{v}(r,s)}{\partial r}|_{r=R} = \psi(s), \tag{3. 15}$$

where $\phi(s) = \left[\frac{(s+M)[\eta+(1-\eta)s]}{\nu\eta + \eta_2 s}\right]$ and $\psi(s) = \frac{A[\eta+(1-\eta)s]}{s^2[\eta\mu + \eta_1 s]}$.

Take $q = r\sqrt{\phi(s)}$, in Eq. (3.14), we obtain Bessel equation of zero order

$$q^2\frac{d^2\breve{v}}{dq^2} + q\frac{d\breve{v}}{dq} - (q^2 - 0^2)\breve{v} = 0, \tag{3. 16}$$

which is the specific case of generalized Bessel equation of order 'n' and it's general solution is of the form

$$\check{v}(q,s) = C_1 I_0(q) + C_2 K_0(q). \tag{3. 17}$$

In above relation C_1, C_2 are functions of parameter s and $I_0(q), K_0(q)$ represent modified bessel functions of first and second kind respectively. For the finite solution at $q = 0(r = 0)$, C_2 must be zero. Then by solving Eq. (3.17) with the help of Eq. (3.15) we get

$$C_1 = \frac{\psi(s)}{\sqrt{\phi(s)} I_1(R\sqrt{\phi(s)})}. \tag{3. 18}$$

By using the values of C_1 and C_2 in Eq. (3.17) we find

$$\check{v}(r,s) = \left(\frac{\psi(s)}{\sqrt{\phi(s)}}\right) \frac{I_0(r\sqrt{\phi(s)})}{I_1(R\sqrt{\phi(s)})}. \tag{3. 19}$$

Eq. (3.19) is in term of modified Bessel functions $I_n(.)$ of first kind with order zero and one depending upon the value of $n = 0, 1$ respectively. Since above expression is very complicated and is in complex form, whose direct Laplace inverse is approximately impossible. That is why we use alternate numerical technique which is given by Stehfest, called Stehfest's algorithm [24], with the help of MATHCAD software.

3.2. **Calculation of the shear stress.**

After the application of Laplace transformation on Eq. (2.7), we get

$$\check{\tau}(r,s) = \left[\mu + \frac{\alpha_1 s}{\eta + (1-\eta)s}\right] \frac{\partial \check{v}(r,s)}{\partial r}, \tag{3. 20}$$

above equation can be written as

$$\check{\tau}(r,s) = \left[\frac{\mu\eta + [\mu(1-\eta) + \alpha_1]s}{\eta + (1-\eta)s}\right] \frac{\partial \check{v}(r,s)}{\partial r}, \tag{3. 21}$$

Eq. (3.21) becomes

$$\check{\tau}(r,s) = \left[\frac{\mu\eta + \eta_1 s}{\eta + (1-\eta)s}\right] \frac{\partial \check{v}(r,s)}{\partial r}, \tag{3. 22}$$

by using Eq. (3.19) in Eq. (3.22), we get

$$\check{\tau}(r,s) = \left[\frac{\mu\eta + \eta_1 s}{\eta + (1-\eta)s}\right] \psi(s) \frac{I_1(r\sqrt{\phi(s)})}{I_1(R\sqrt{\phi(s)})}. \tag{3. 23}$$

Eq. (3.23) is also in very complicated form like Eq. (3.19). To find the solution for shear stress we use Stehfest's algorithm instead of any analytical technique.

4. NUMERICAL RESULTS AND DISCUSSION

Comparison of shear stress with existing result (Table.1)			
radius of cylinder r	Nazar *et al.* [37] result	current numerical	Error
0.00	0.0000	0.0000	0.0000
0.05	0.0320	0.033	-0.0010
0.10	0.0641	0.0661	-0.0020
0.15	0.0965	0.0993	-0.0028
0.20	0.1294	0.1326	-0.0032
0.25	0.1629	0.1663	-0.0034
0.30	0.1971	0.2001	-0.0030
0.35	0.2321	0.2346	-0.0025
0.40	0.2679	0.2693	-0.0014
0.45	0.3046	0.3046	0.0000
0.50	0.3422	0.3408	0.0014
0.55	0.3807	0.3777	0.0030
0.60	0.4200	0.4151	0.0049
0.65	0.4601	0.4540	0.0061
0.70	0.5009	0.4928	0.0081
0.75	0.5422	0.5337	0.0085
0.80	0.5839	0.5746	0.0093
0.85	0.6259	0.6181	0.0078
0.90	0.6680	0.6596	0.0084
0.95	0.7102	0.7058	0.0044
1	0.7521	0.7515	0.0006

In this article, the incompressible, unsteady flow of MHD second grade fluid with FD given by Caputo and Fabrizio is studied by semi-analytical technique. Circular pipe exerts a stress which develop translational motion in fluid filled in it. By applying Laplace transform, partial differential equation changed into Ordinary differential equation. The final expressions in terms of modified Bessel functions become very complex and impossible to solve it analytically. This problem resolved by applying Stehfest's algorithm through MATHCAD software. Semi-numerical solutions for velocity and stress fields under uniform magnetic effect have been obtained. For the validity and authenticity purpose, a good comparison between current approximate solution and existing analytical result has been achieved. A comprehensive analysis of different physical parameters given in the end in graphical form.

Figs. 1-6 indicate influence of various physical parameters specifically magnetic, fractional, material and time. Fig. 1 represent the impact of time on velocity and stress function. It is clear from Fig. 1(a) and 1(b) that both have zero value at the center of pipe and they smoothly increase up to the maximum value at the surface of cylinder.By increasing the effect of magnetic field both stress and velocity field decreases as expected which is clear from Fig. 3(b) and Fig. 3(a) respectively. Figs. 4, 5 and 6 indicates the influence of material, fractional and viscous parameters respectively. It is very much interesting that by increasing the values of all three parameters (material, fractional, viscous) shear stress and velocity are increases. It is clear that the influence of α, η and ν is opposite to magnetic parameter M. Fig. 2 represents the effect of constant A due to which both velocity and stress increases. A comparison between existing analytical solution [24] and current

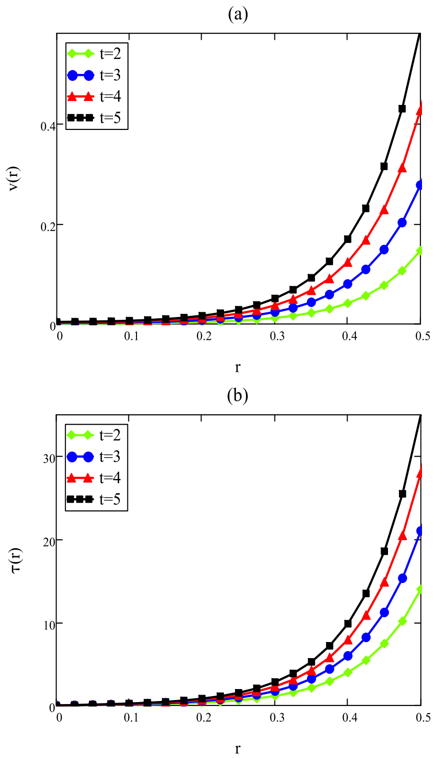

FIGURE 1. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (2.19) and (3.23), for different values of t and $[R = 0.5, A = 7, M = 0.6, \mu = 3, \nu = 0.003, \rho = 1000, \alpha = 0.005, \eta = 0.8]$

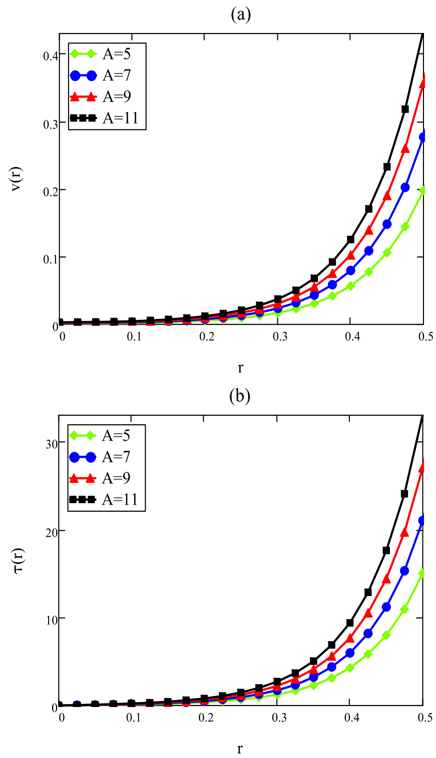

FIGURE 2. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (2.19) and (3.23), for different values of A and $[R = 0.5, t = 3, M = 0.6, \mu = 3, \nu = 0.003, \rho = 1000, \alpha = 0.005, \eta = 0.8]$

approximate result without magnetic effect indicated by Fig. 7 and Table 1. In current analysis the notable things are

- The velocity of fractional second grad fluid goes to slow down by increasing the strength of magnetic field M as expected.
- By increasing fractional parameter, both stress and velocity fields have been increased.
- With the passage of time, the stress and thus velocity of fluid increased.
- Its velocity increases by increasing kinematic viscosity.
- By decreasing radius of the cylinder, the value of shear stress decreases and thus velocity also

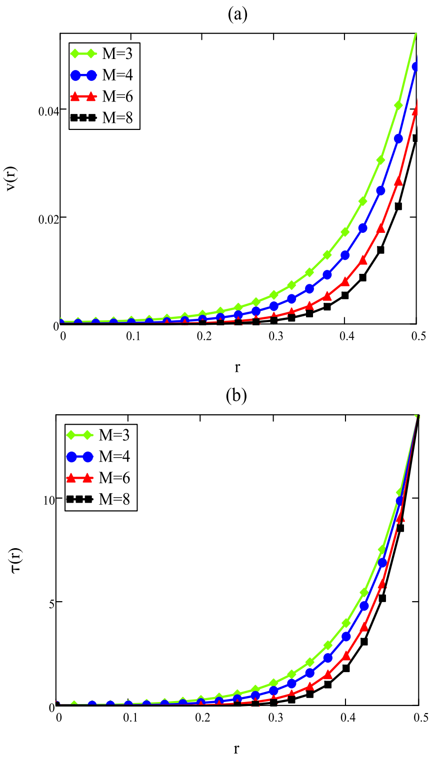

FIGURE 3. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (2.19) and (3.23), for different values of M and $[R = 0.5, A = 7, \mu = 3, \nu = 0.003, t = 2, \rho = 1000, \alpha = 0.005, \eta = 0.8]$

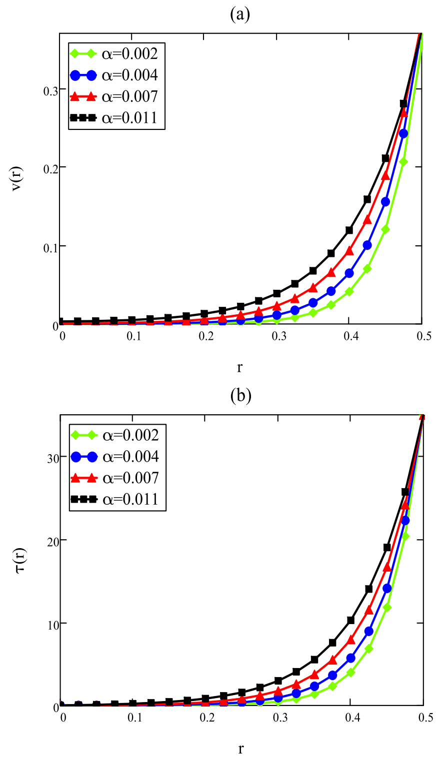

FIGURE 4. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (2.19) and (3.23), for different values of α and $[R = 0.5, A = 7, \mu = 3, \nu = 0.0003, t = 5, \rho = 1000, \eta = 0.003, \alpha = 0.005, M = 0.6]$

- Coefficient A also have effect similar to fractional parameter.
- Our obtained solution from new definition i.e. CFFD given in [9], derived from Stehfest's algorithm are equivalent to analytical solution obtained by Nazar *et al.* [24].
- In all Figs., the value of stress and velocity is maximum at the boundary of the pipe and smoothly reduce up to zero at the center of the cylinder. S.I units are used in all Figs.

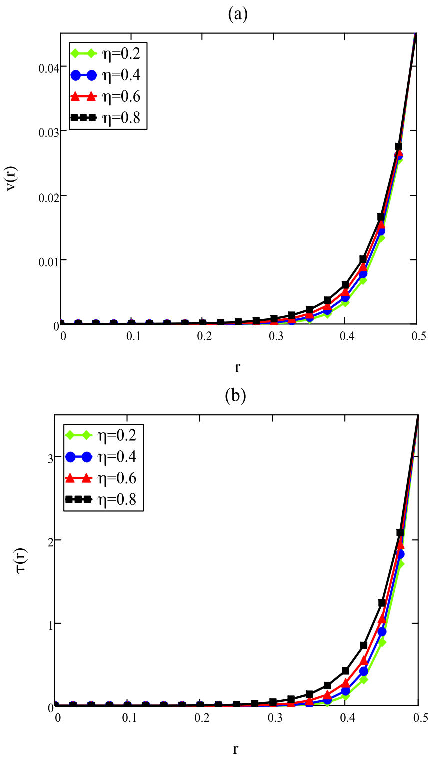

FIGURE 5. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (2.19) and (3.23), for different values of η and $[R = 0.5, A = 7, \mu = 3, t = 0.5, \rho = 1000, \alpha = 0.003, \nu = 0.0003, M = 0.6]$

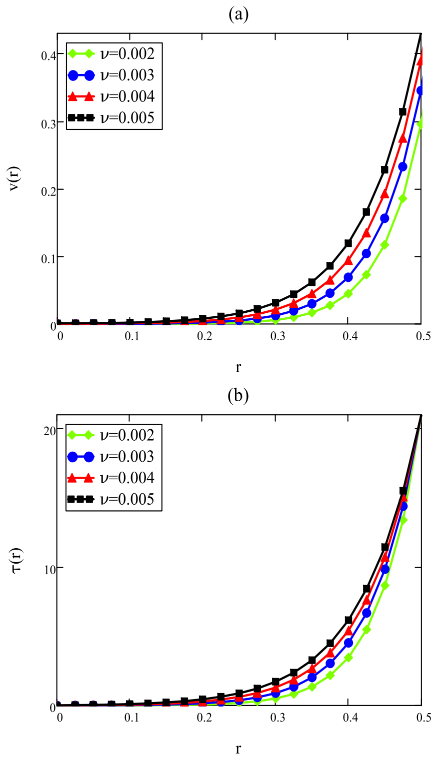

FIGURE 6. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (2.19) and (3.23), for different values of ν and $[R = 0.5, A = 7, \mu = 3, t = 3, \rho = 1000, \alpha = 0.001, \eta = 0.2, M = 0.6]$

5. DECLARATIONS

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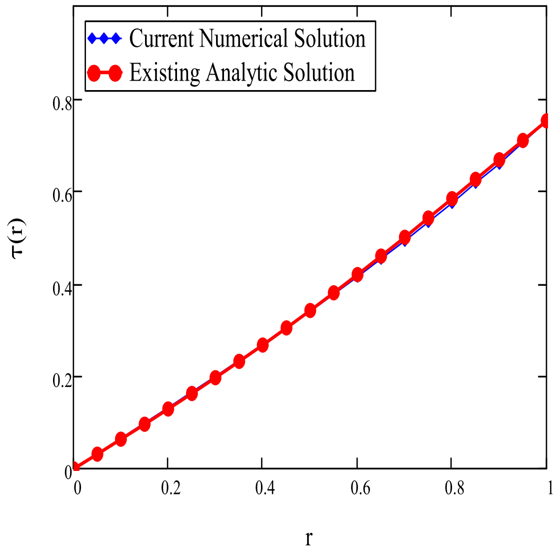

FIGURE 7. Variation in shear stress $\tau(r)$ given by Eq. (3.23) and solution obtained by Nazar *et al.* [24], for $[R = 0.5, A = 0.5, \mu = 3, t = 1.5, \rho = 1000, \alpha = 0.003, \eta = 0.8, M = 0]$

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