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Some Generalize Reimann-Liouville Fractional Estimates Involving Functions Having Exponentially Convexity Property

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Abstract. In this paper, we establish some Trapezoid type inequalities for generalized fractional integral and related inequalities via exponentially convex functions. A novel and new approach is used to obtain these results for general Riemann Liouville fractional integrals. Various special cases are briefly discussed as applications of the main results.

AMS (MOS) Subject Classification Codes: 26D15; 26D10, 90C23
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1. Introduction

Convex functions and their variant forms are being used to study a wide class of unrelated problems which arises in various branches of pure and applied sciences in a natural, unified and general framework. For recent applications, generalizations and other aspects of convex functions and their variant forms, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] and the references therein.

We now recall the well known and basic concepts, which are needed.

Definition 1.1. [13, 22] *The set K in $\mathbb{R}$ is said to be a convex set, if*

$$(1-t)\mu + t\nu \in K, \quad \forall u, v \in K, \quad \zeta \in [0,1].$$

Definition 1.2. [13, 22] *A function $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function, if*

$$f\big(\zeta\mu + (1-\zeta)\nu\big) \leq \zeta f(\mu) + (1-\zeta) f(\nu), \quad \forall \mu, \nu \in K, \quad \zeta \in [0,1]. \tag{1. 1}$$

We say that f is a concave function if $-f$ is a convex function.

It is known that the optimality conditions of the differentiable convex functions on the convex sets can be characterized by a class of variational inequalities, which is itself an interesting field of research. For the applications and other aspects of variational inequalities, see [13, 14, 18] and the references therein.

Let $f : I = [\mu, \nu] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I = [\mu, \nu]$ with $\mu < \nu$. Then the following double inequality [8] holds

$$f\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu} \int_{\mu}^{\nu} f(x)dx \leq \frac{f(\mu)+f(\nu)}{2}, \tag{1. 2}$$

which is known as Hermite-Hadamard inequality and has been investigated extensively. See [3, 4, 5, 6, 7, 8, 9, 10, 13, 19, 22, 24, 26, 32, 33, 34] and the references therein for applications.

We now consider a class of h-convex functions with respect to an arbitrary nonnegative function, which was introduced and investigated by Varosanec[34].

Definition 1.3. *(*[34]*) Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. The function $f : I \to \mathbb{R}$ is said to be a h-convex function with respect to a non-negative function h, if*

$$f(\zeta\mu + (1-\zeta)\nu) \leq h(\zeta) f(\mu) + h(1-\zeta) f(\nu), \quad \forall \mu, \nu \in K, \quad \zeta \in [0,1].$$

We would like to point out that for appropriate and suitable choice of the arbitrary function h, one can obtain a wide class of convex functions and their variant forms as special cases of h-convex functions.

We now consider the definition of exponentially convex functions, which is mainly due to Noor and Noor [15, 16, 17], Antaczak [2] and Dragomir and Gomm[6].

Definition 1.4. *(See* [2, 6]*) A positive function $f : K \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be an exponentially convex function, if*

$$e^{f(\zeta\mu + (1-\zeta)\nu)} \leq \zeta e^{f(\mu)} + (1-\zeta) e^{f(\nu)}, \quad \forall \mu, \nu \in K, \quad \zeta \in [0,1].$$

For the basic properties and applications of the exponentially convex functions, see [14, 15, 16, 24, 25, 26, 27] and the references therein.

We now consider a new class of exponentially convex function with respect to an arbitrary nonnegative functions h.

Definition 1.5. *([26]). A positive function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be an exponentially h-convex function with respect to an arbitrary nonnegative function $h : (0;1) \subseteq J \to \mathbb{R}$, if*

$$e^{f\left(\mu\zeta+(1-\zeta)\nu\right)} \leq h(\zeta)e^{f(\mu)} + h(1-\zeta)e^{f(\nu)}, \quad \forall\zeta \in [0,1], \;\; \mu,\nu \in K.$$

Exponentially convex functions played important role in the study of statistical learning, sequential prediction and stochastic optimization, see [1, 2, 20, 21] and the references therein.

Due to the significance and importance of the exponentially convex functions, Awan et al [3] and Pecaric et al [21] defined some other kind of exponential convex functions. They have shown that these classes of exponential convex functions unify various concepts in different manners.

We now recall some basic concepts and results of the generalized fractional integrals by Sarikaya and Ertugral [32].

$$\int_0^1 \frac{\varphi(\zeta)}{\zeta} d\zeta < \infty. \tag{1. 3}$$

We define the left-sided and right sided generalized fractional integral operators, respectively as:

$$_{\mu^+}I_\varphi f(x) = \int_\mu^x \frac{\varphi(x-\zeta)}{x-\zeta} f(\zeta)d\zeta, \quad x > \mu, \tag{1. 4}$$

$$_{\nu^-}I_\varphi f(x) = \int_x^\nu \frac{\varphi(\zeta-x)}{\zeta-x} f(\zeta)d\zeta, \quad x < \nu. \tag{1. 5}$$

The most important features of generalized fractional integrals is that they include some types of fractional integrals such as Reimann-Liouville fractional integral, k-Reimann Liouville fractional integral, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals as special cases. Some important special cases of the integral operators (1. 4) and (1. 5) are listed below.

(1) If $\varphi(\zeta) = \zeta$, then operator (1. 4) and (1. 5) reduce to the Riemann integral:

$$I_{\mu^+} f(x) = \int_\mu^x f(\zeta)d\zeta, \quad x > \mu,$$

$$I_{\nu^-} f(x) = \int_x^\nu f(\zeta)d\zeta, \quad x < \nu.$$

(2) If $\varphi(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then operator (1. 4) and (1. 5) reduce to the Riemann-Liouville fractional integral:

$$I_{\mu^+}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{\mu}^{x}(x-\zeta)^{\alpha-1}f(\zeta)d\zeta, \quad x>\mu,$$

$$I_{\nu^-}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{\nu}(\zeta-x)^{\alpha-1}f(\zeta)d\zeta, \quad x<\nu.$$

(3) If $\varphi(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then operator (1. 4) and (1. 5) reduce to the k-Riemann-Liouville fractional integral [11]:

$$I_{\mu^+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int\limits_{\mu}^{x}(x-\zeta)^{\frac{\alpha}{k}-1}f(\zeta)d\zeta, \quad x>\mu,$$

$$I_{\nu^-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int\limits_{x}^{\nu}(\zeta-x)^{\frac{\alpha}{k}-1}f(\zeta)d\zeta, \quad x<\nu,$$

where

$$\Gamma_k(\alpha) = \int\limits_{0}^{\infty}\zeta^{\alpha-1}e^{\frac{-\zeta^k}{k}}d\zeta, \quad \mathfrak{Re}(\alpha)>0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k}), \quad \mathfrak{Re}(\alpha)>0; k>0.$$

Sarikaya and Ertugral [32] established the Trapezoid inequalities for generalized fractional integrals.

The main motivation of this paper is to establish some Trapezoid type inequalities via generalized fractional integrals for exponentially h-convex functions. These results can be viewed as a significantly different from the previously known results.

2. MAIN RESULTS

For our results, we need the following important fractional integral identity:

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $(e^f)' \in (L[\mu,\nu])$, where $\mu, \nu \in I^\circ$ with $\mu < \nu$. Then*

$$\begin{aligned}
&\frac{\Delta(0)e^{f(\nu)} + \nabla(0)e^{f(\mu)}}{\nu-\mu} - \frac{1}{\nu-\mu}\left[{}_{x^+}I_\varphi e^{f(\nu)} +{}_{x^-} I_\varphi e^{f(\mu)}\right] \\
&= \frac{\nu - x}{\nu-\mu}\int_0^1 \Delta(\zeta) e^{f\left(\zeta x+(1-\zeta)\nu\right)} f'\left(\zeta x + (1-\zeta)\nu\right) d\zeta \\
&\quad - \frac{x-\mu}{\nu-\mu}\int_0^1 \nabla(\zeta) e^{f\left(\zeta x+(1-\zeta)\mu\right)} f'\left(\zeta x + (1-\zeta)\mu\right) d\zeta, \qquad (2.\ 6)
\end{aligned}$$

where

$$\Delta(\zeta) = \int_\zeta^1 \frac{\left(\varphi(x-\mu)u\right)}{u} du < \infty \qquad (2.\ 7)$$

and

$$\nabla(\zeta) = \int_\zeta^1 \frac{\left(\varphi(\nu-x)u\right)}{u} du < \infty. \qquad (2.\ 8)$$

Proof. Consider

$$\begin{aligned}
I \quad &= \frac{\nu - x}{\nu-\mu}\int_0^1 \Delta(\zeta) e^{f\left(\zeta x+(1-\zeta)\nu\right)} f'\left(\zeta x + (1-\zeta)\nu\right) d\zeta \\
&\quad - \frac{x-\mu}{\nu-\mu}\int_0^1 \nabla(\zeta) e^{f\left(\zeta x+(1-\zeta)\mu\right)} f'\left(\zeta x + (1-\zeta)\mu\right) d\zeta.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
I_1 \quad &= \frac{\nu - x}{\nu-\mu}\left[\Delta(\zeta)\frac{e^{f(\zeta x+(1-\zeta)\nu)}}{x-\nu}\Big|_0^1 + \frac{1}{x-\nu}\int_0^1 \frac{\varphi\left((\nu-x)\right)\zeta}{\zeta} e^{f(\zeta x+(1-\zeta)\nu)} d\zeta\right] \\
&= \frac{\nu - x}{\nu-\mu}\left[\frac{\Delta(0)e^{f(\nu)}}{\nu-x} - \frac{1}{\nu-x}\int_x^\nu \frac{\varphi(\nu-s)}{\nu-s} e^{f(s)} ds\right] \\
&= \frac{\Delta(0)e^{f(\nu)} - {}_{x^+}I_\varphi e^{f(\nu)}}{\nu-\mu}, \qquad (2.\ 9)
\end{aligned}$$

and similarly, we have

$$
\begin{aligned}
I_2 &= \frac{x-\mu}{\nu-\mu}\int_0^1 \nabla(\zeta)e^{f(\zeta x+(1-\zeta)\mu)}f'(\zeta x+(1-\zeta)\mu)d\zeta \\
&= \frac{x-\mu}{\nu-\mu}\left[\frac{\nabla(0)e^{f(\mu)}}{x-\mu} - \frac{1}{x-\mu}\int_\mu^x \frac{\varphi(s-\mu)}{s-\mu}e^{f(s)}ds\right] \\
&= \frac{\nabla(0)e^{f(\mu)} - \ _{x^-}I_\varphi e^{f(\mu)}}{\nu-\mu}. \qquad (2.\ 10)
\end{aligned}
$$

By subtracting equation (2. 9) and (2. 10), we have

$$
I = \frac{\Delta(0)e^{f(\nu)}+\nabla(0)e^{f(\mu)}}{\nu-\mu} - \frac{1}{\nu-\mu}\left[_{x^-}I_\varphi e^{f(\mu)} +_{x^+} I_\varphi e^{f(\nu)}\right],
$$

which is the required result. □

Corollary 2.2. *If* $\varphi(\zeta)=\zeta$, *then Lemma 2.1 reduces to a new result*

$$
\begin{aligned}
&\frac{(x-\mu)e^{f(\mu)}+(\nu-x)e^{f(\nu)}}{\nu-\mu} - \frac{1}{\nu-\mu}\int_\mu^\nu e^{f(u)}du \\
&= \frac{(x-\mu)^2}{\nu-\mu}\int_0^1 (\zeta-1)e^{f(\zeta x+(1-\zeta)\mu)}f'(\zeta x+(1-\zeta)\mu)d\zeta \\
&+ \frac{(\nu-x)^2}{\nu-\mu}\int_0^1 (1-\zeta)e^{f(\zeta x+(1-\zeta)\nu)}f'(\zeta x+(1-\zeta)\nu)d\zeta.
\end{aligned}
$$

Corollary 2.3. *If* $\varphi(\zeta)=\frac{\zeta^\alpha}{\Gamma(\alpha)}$, *then Lemma 2.1 reduces to a new result*

$$
\begin{aligned}
&\frac{(x-\mu)^\alpha e^{f(\mu)}+(\nu-x)^\alpha e^{f(\nu)}}{(\nu-\mu)} - \frac{\Gamma(\alpha+1)}{\nu-\mu}\left[I^\alpha_{x^-}e^{f(\mu)} + I^\alpha_{x^+}e^{f(\nu)}\right] \\
&= \frac{(x-\mu)^{\alpha+1}}{\nu-\mu}\int_0^1 (\zeta^\alpha-1)e^{f(\zeta x+(1-\zeta)\mu)}f'(\zeta x+(1-\zeta)\mu)d\zeta \\
&+ \frac{(\nu-x)^{\alpha+1}}{\nu-\mu}\int_0^1 (1-\zeta^\alpha)e^{f(\zeta x+(1-\zeta)\nu)}f'(\zeta x+(1-\zeta)\nu)d\zeta.
\end{aligned}
$$

Corollary 2.4. *If* $\varphi(t) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then Lemma 2.1 reduces to a new result*

$$
\begin{aligned}
&\frac{(x-\mu)^{\frac{\alpha}{k}} e^{f(\mu)} + (\nu - x)^{\frac{\alpha}{k}} e^{f(\nu)}}{(\nu-\mu)} - \frac{\Gamma_k(\alpha+k)}{\nu-\mu}\left[I^{\alpha}_{x^-,k} e^{f(\mu)} + I^{\alpha}_{x^+,k} e^{f(b)}\right] \\
&= \frac{(x-\mu)^{\frac{\alpha}{k}+1}}{(\nu-\mu)} \int_0^1 (\zeta^{\frac{\alpha}{k}} - 1) e^{f(\zeta x + (1-\zeta)\mu)} f'(\zeta x + (1-\zeta)\mu) d\zeta \\
&+ \frac{(\nu - x)^{\frac{\alpha}{k}+1}}{(\nu-\mu)} \int_0^1 (1 - \zeta^{\frac{\alpha}{k}}) e^{f(\zeta x + (1-\zeta)\nu)} f'(\zeta x + (1-\zeta)\nu) d\zeta.
\end{aligned}
$$

Theorem 2.5. *Let* $f : I = [\mu, \nu] \subseteq \mathbb{R} \to \mathbb{R}$ *be an absolutely continuous mapping on* I° *such that* $(e^f)' \in L([\mu,\nu])$ *with* $\mu < \nu$. *If the mapping* $|(e^f)'|$ *is convex on* $[\mu,\nu]$, *then*

$$
\begin{aligned}
&\frac{\Delta(0) e^{f(\nu)} + \nabla(0) e^{f(\mu)}}{\nu - \mu} - \frac{1}{\nu-\mu}\ {}_{x^+}I_{\varphi} e^{f(\nu)} + {}_{x^-}\ I_{\varphi} e^{f(\mu)} \\
&\leq \frac{\nu - x}{\nu - \mu}\ \ \psi_1\ \ \Delta,\zeta\ \ |e^{f(x)} f'(x)| + \psi_2\ \ \Delta,\zeta\ \ |e^{f(\nu)} f'(\nu)| + \psi_3\ \ \Delta,\zeta\ \ \Theta(x,\nu) \\
&+ \frac{x-\mu}{\nu-\mu}\ \ \psi_4\ \ \nabla,\zeta\ \ |e^{f(x)} f'(x)| + \psi_5\ \ \nabla,\zeta\ \ |e^{f(\mu)} f'(\mu)| + \psi_6\ \ \nabla,\zeta\ \ \Theta(x,\mu)\ \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
&\psi_1\ \ \Delta,\zeta\ \ = \int_0^1 h^2(\zeta)\Delta(\zeta) d\zeta, \qquad \psi_2\ \ \Delta,\zeta\ \ = \int_0^1 h^2(1-\zeta)\Delta(\zeta) d\zeta, \\
&\psi_3\ \ \Delta,\zeta\ \ = \int_0^1 h(\zeta) h(1-\zeta)\Delta(\zeta) d\zeta, \qquad \psi_4\ \ \nabla,\zeta\ \ = \int_0^1 h^2(\zeta)\nabla(\zeta) d\zeta, \\
&\psi_5\ \ \nabla,\zeta\ \ = \int_0^1 h^2(1-\zeta)\nabla(\zeta) d\zeta, \qquad \psi_6\ \ \nabla,\zeta\ \ = \int_0^1 h(\zeta) h(1-\zeta)\nabla(\zeta) d\zeta, \\
&\Theta(x,\mu) = |e^{f(x)} f'(\mu)| + |e^{f(\mu)} f'(x)|, \\
&\Theta(x,\nu) = |e^{f(x)} f'(\nu)| + |e^{f(\nu)} f'(x)|,
\end{aligned}
$$

where Δ *and* ∇ *are given in* (2. 7) *and* (2. 8).

Proof. From Lemma 2.1 and convexity of $|(e^f)'|$ on $[\mu,\nu]$, we get

$$\frac{\Delta(0)e^{f(\nu)}+\nabla(0)e^{f(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\ {}_{x^+}I_{\varphi}e^{f(\nu)}+{}_{x^-}I_{\varphi}e^{f(\mu)}$$

$$\leq \frac{\nu-x}{\nu-\mu}\int_0^1 \Delta(\zeta)e^{f(\zeta x+(1-\zeta)\nu)}f'(\zeta x+(1-\zeta)\nu)d\zeta$$

$$-\frac{x-\mu}{\nu-\mu}\int_0^1 \nabla(\zeta)e^{f(\zeta x+(1-\zeta)\mu)}f'(\zeta x+(1-\zeta)\mu)d\zeta$$

$$=\frac{\nu-x}{\nu-\mu}\int_0^1 |\Delta(\zeta)|\ e^{f(\zeta x+(1-\zeta)\nu)}f'(\zeta x+(1-\zeta)\nu)\ d\zeta+$$

$$\frac{x-\mu}{\nu-\mu}\int_0^1 |\nabla(\zeta)|\ e^{f(\zeta x+(1-\zeta)\mu)}f'(\zeta x+(1-\zeta)\mu)\ d\zeta$$

$$=\frac{\nu-x}{\nu-\mu}\int_0^1 |\Delta(\zeta)|\ h(\zeta)|e^{f(x)}|+h(1-\zeta)|e^{f(\nu)}|\quad h(\zeta)|f'(x)|+h(1-\zeta)|f'(\nu)|\ d\zeta$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1 |\nabla(\zeta)|\ h(\zeta)|e^{f(x)}|+h(1-\zeta)|e^{f(\mu)}|\quad h(\zeta)|f'(x)|+h(1-\zeta)|f'(\mu)|\ d\zeta$$

$$=\frac{\nu-x}{\nu-\mu}\int_0^1 |\Delta(\zeta)|\quad h(\zeta)\ {}^2|e^{f(x)}f'(x)|+\ h(1-\zeta)\ {}^2|e^{f(\nu)}f'(\nu)|$$

$$+h(\zeta)h(1-\zeta)\ \ |e^{f(x)}f'(\nu)|+|e^{f(\nu)}f'(x)|\quad d\zeta$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1 |\nabla(\zeta)|\quad h(\zeta)\ {}^2|e^{f(x)}f'(x)|+h\ \ (1-\zeta)\ {}^2|e^{f(\mu)}f'(\mu)|$$

$$+h(\zeta)h(1-\zeta)\ \ |e^{f(x)}f'(\mu)|+|e^{f(\mu)}f'(x)|\quad d\zeta$$

$$=\frac{\nu-x}{\nu-\mu}\int_0^1 |\Delta(\zeta)|\quad h^2(\zeta)|e^{f(x)}f'(x)|+h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|+h(\zeta)h(1-\zeta)\Theta(x,\nu)\quad d\zeta$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1 |\nabla(\zeta)|\quad h^2(\zeta)|e^{f(x)}f'(x)|+h^2(1-\zeta)|e^{f(\mu)}f'(\mu)|+h(\zeta)h(1-\zeta)\Theta(x,\mu)\quad d\zeta$$

$$=\frac{\nu-x}{\nu-\mu}\ \ \psi_1\ \ \Delta,\zeta\ \ |e^{f(x)}f'(x)|+\psi_2\ \ \Delta,\zeta\ \ |e^{f(\nu)}f'(\nu)|+\psi_3\ \ \Delta,\zeta\ \ \Theta(x,\nu)$$

$$+\frac{x-\mu}{\nu-\mu}\ \ \psi_4\ \ \nabla,\zeta\ \ |e^{f(x)}f'(x)|+\psi_5\ \ \nabla,\zeta\ \ |e^{f(\mu)}f'(\mu)|+\psi_6\ \ \nabla,\zeta\ \ \Theta(x,\mu)\ \ .$$

This completes the proof. □

Remark 2.6. *Under assumptions of Theorem 2.5,*

(1) *if* $h(\zeta)=\zeta$, *then Theorem 2.5 reduces to Theorem 2.2 in* [23].

(2) *If* $\varphi(\zeta)=h(\zeta)=\zeta$, *then under the assumption of Theorem 2.5 reduces to Corollary 2.4 in* [23].

(3) *if* $\varphi(\zeta)=\frac{\zeta^{\alpha}}{\Gamma(\alpha)}$ *and* $h(\zeta)=\zeta$ *then under the assumption of Theorem 2.5 reduces to Corollary 2.5 in* [23].

(4) *if* $\varphi(\zeta)=\frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $h(\zeta)=\zeta$ *then under the assumption of Theorem 2.5 reduces to Corollary 2.6 in* [23].

Corollary 2.7. *If* $\varphi(\zeta)=\zeta$ *and* $h(\zeta)=\zeta^{s}$, *then under the assumption of Theorem 2.5 reduces to a new result*

$$\frac{(\nu-x)e^{f(\nu)}+(x-\mu)e^{f(\mu)}}{\nu-\mu}-\frac{1}{\nu-a}\int_{\mu}^{\nu}e^{f(x)}dx \le \frac{(x-\mu)^2+(x-\nu)^2}{(2s+1)(2s+2)(\nu-\mu)}\quad|e^{f(x)}f'(x)|$$
$$+\frac{(x-\mu)^2|e^{f(\mu)}f'(\mu)|+(\nu-x)^2|e^{f(\nu)}f'(\nu)|}{(2s+2)(\nu-\mu)}+\frac{\Gamma(s+1)\ ^2\ \Theta(x,\mu)+\Theta(x,\nu)}{(2s+1)\Gamma(2s+1)}.$$

Corollary 2.8. *If* $\varphi(\zeta)=\frac{\zeta^{\alpha}}{\Gamma(\alpha)}$ *and* $h(\zeta)=\zeta^{s}$, *then under the assumption of Theorem 2.5 reduces to a new result*

$$\left|\frac{(\nu-x)e^{f(\nu)}+(x-\mu)e^{f(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}e^{f(x)}dx\right|$$
$$\le\frac{\alpha|e^{f(x)}f'(x)|}{(2s+1)(2s+\alpha+1)}\left\{\frac{(x-\mu)^{\alpha+1}+(\nu-x)^{\alpha+1}}{\nu-\mu}\right\}$$
$$+\left[\frac{1}{2s+1}-\frac{\Gamma(\alpha+1)\Gamma(2s+1)}{\Gamma(2s+\alpha+2)}\right]\left\{\frac{(x-\mu)^{\alpha+1}|e^{f(\mu)}f'(\mu)|+(\nu-x)^{\alpha+1}|e^{f(\nu)}f'(\nu)|}{(\nu-\mu)}\right\}$$
$$+\left[\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+2)}-\frac{\Gamma(s+1)\Gamma(s+\alpha+1)}{\Gamma(2s+\alpha+2)}\right]\big\{\Theta(x,\mu)+\Theta(x,\nu)\big\}.$$

Corollary 2.9. *If* $\varphi(\zeta)=\frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $h(\zeta)=\zeta^{s}$, *then under the assumption of Theorem 2.5 reduces to a new result*

$$\left|\frac{(\nu-x)e^{f(\nu)}+(x-\mu)e^{f(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}e^{f(x)}dx\right|$$
$$\le\frac{\frac{\alpha}{k}|e^{f(x)}f'(x)|}{(2s+1)(\frac{\alpha}{k}+2s+1)}\left\{\frac{(x-\mu)^{\frac{\alpha}{k}+1}+(\nu-x)^{\frac{\alpha}{k}+1}}{\nu-\mu}\right\}$$
$$+\left[\frac{1}{2s+1}-\frac{\Gamma(\frac{\alpha}{k}+1)\Gamma(2s+1)}{\Gamma(\frac{\alpha}{k}+2s+2)}\right]\left\{\frac{(x-\mu)^{\frac{\alpha}{k}+1}|e^{f(\mu)}f'(\mu)|+(\nu-x)^{\frac{\alpha}{k}+1}|e^{f(\nu)}f'(\nu)|}{(\nu-\mu)}\right\}$$
$$+\left[\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+2)}-\frac{\Gamma(s+1)\Gamma(\frac{\alpha}{k}+s+1)}{\Gamma(\frac{\alpha}{k}+2s+2)}\right]\big\{\Theta(x,\mu)+\Theta(x,\nu)\big\}.$$

Theorem 2.10. *Let $f : I = [\mu, \nu] \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $(e^f)' \in L([\mu, \nu])$, where $\mu, \nu \in I^\circ$ with $\mu < \nu$. If the mapping $|(e^f)'|^q,\ q > 1$ with $p^{-1} + q^{-1} = 1$ is convex on $[\mu, \nu]$, then*

$$\frac{\Delta(0)e^{f(\nu)} + \nabla(0)e^{f(\mu)}}{\nu - \mu} - \frac{1}{\nu - \mu}\ {}_{x^+}I_\varphi e^{f(\nu)} + {}_{x^-}I_\varphi e^{f(\mu)}$$
$$\leq \frac{\nu - x}{\nu - \mu}\left(\int_0^1 |\Delta(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 h^2(\zeta)|e^{f(x)}f'(x)|^q + h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q + h(\zeta)h(1-\zeta)\Phi(x,\nu)\ d\zeta\right)^{\frac{1}{q}}$$
$$+\frac{x - \mu}{\nu - \mu}\left(\int_0^1 |\nabla(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 h^2(\zeta)|e^{f(x)}f'(x)|^q + h^2(1-\zeta)|e^{f(\mu)}f'(\mu)|^q + h(\zeta)h(1-\zeta)\Phi(x,\mu)\ d\zeta\right)^{\frac{1}{q}},$$

where

$$\Phi(x, \mu) = |e^{f(x)}f'(\mu)|^q + |e^{f(\mu)}f'(x)|^q, \tag{2. 11}$$

$$\Phi(x, \nu) = |e^{f(x)}f'(\nu)|^q + |e^{f(\nu)}f'(x)|^q, \tag{2. 12}$$

and Δ and ∇ are given in (2. 7) *and* (2. 8).

Proof. From Lemma 2.1 and by Hölder's inequality, we get

$$\frac{\Delta(0)e^{f(\nu)} + \nabla(0)e^{f(\mu)}}{\nu - \mu} - \frac{1}{\nu - \mu}\ {}_{x^+}I_\varphi e^{f(\nu)} +{}_{x^-} I_\varphi e^{f(\mu)}$$
$$\leq \frac{\nu - x}{\nu - \mu}\left(\int_0^1 |\Delta(\zeta)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |e^{f(\zeta x + (1-\zeta)\nu)} f'(\zeta x + (1-\zeta)\nu)|^q d\zeta\right)^{\frac{1}{q}}$$
$$+\frac{x - \mu}{\nu - \mu}\left(\int_0^1 |\nabla(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 |e^{f(\zeta x + (1-\zeta)\mu)} f'(\zeta x + (1-\zeta)\nu)|^q d\zeta\right)^{\frac{1}{q}}$$
$$\leq \frac{\nu - x}{\nu - \mu}\left(\int_0^1 |\Delta(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 \left(h(\zeta)|e^{f(x)}|^q + h(1-\zeta)|e^{f(\nu)}|^q\right)\left(\zeta|f'(x)|^q + (1-\zeta)|f'(\nu)|^q\right) d\zeta\right)^{\frac{1}{q}}$$
$$+\frac{x - \mu}{\nu - \mu}\left(\int_0^1 |\nabla(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 \left(h(\zeta)|e^{f(x)}|^q + h(1-\zeta)|e^{f(\mu)}|^q\right)\left(h(\zeta)|f'(x)|^q + h(1-\zeta)|f'(\mu)|^q\right) d\zeta\right)^{\frac{1}{q}}$$
$$= \frac{\nu - x}{\nu - \mu}\left(\int_0^1 |\Delta(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 \Big(h^2(\zeta)|e^{f(x)}f'(x)|^q + h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q\right.$$
$$\left. +h(\zeta)h(1-\zeta)\left(|e^{f(x)}f'(\nu)|^q + |e^{f(\nu)}f'(x)|^q\right)\Big) d\zeta\right)^{\frac{1}{q}}$$
$$+\frac{x - \mu}{\nu - \mu}\left(\int_0^1 |\nabla(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 \Big(h^2(\zeta)|e^{f(x)}f'(x)|^q + h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q\right.$$
$$\left. +h(\zeta)h(1-\zeta)\left(|e^{f(x)}f'(\nu)|^q + |e^{f(\nu)}f'(x)|^q\right)\Big) d\zeta\right)^{\frac{1}{q}}$$
$$= \frac{\nu - x}{\nu - \mu}\left(\int_0^1 |\Delta(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 h^2(\zeta)|e^{f(x)}f'(x)|^q + h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q + h(\zeta)h(1-\zeta)\Phi(x,\nu)\ d\zeta\right)^{\frac{1}{q}}$$
$$+\frac{x - \mu}{\nu - \mu}\left(\int_0^1 |\nabla(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 h^2(\zeta)|e^{f(x)}f'(x)|^q + h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q + h(\zeta)h(1-\zeta)\Phi(x,\mu)\ d\zeta\right)^{\frac{1}{q}},$$

which is the required result. □

Remark 2.11. *Under the assumption of Theorem 2.10,*

(1) *if* $h(\zeta)=\zeta$*, then Theorem 2.10 reduces to Theorem 2.3 in* [23].

(2) *if* $\varphi(\zeta)=h(\zeta)=\zeta$*, then Theorem 2.10 reduces to Corollary 2.7 in* [23].

(3) *if* $\varphi(\zeta)=\frac{\zeta^{\alpha}}{\Gamma(\alpha)}$ *and* $h(\zeta)=\zeta$*, then Theorem 2.10 reduces to Corollary 2.8 in* [23].

(4) *if* $\varphi(\zeta)=\frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $h(\zeta)=\zeta$*, then Theorem 2.10 reduces to Corollary 2.9 in* [23].

Corollary 2.12. *If* $\varphi(\zeta)=\zeta$ *and* $h(\zeta)=\zeta^s$*, then under the assumption of Theorem, 2.10 reduces to a new result*

$$\begin{aligned}&\left|\frac{(\nu-x)e^{f(\nu)}+(x-\mu)e^{f(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}e^{f(x)}dx\right|\\&\leq\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left[\frac{(x-\mu)^2}{(\nu-\mu)}\left(\frac{|e^{f(x)}f'(x)|^q+|e^{f(\mu)}f'(\mu)|^q}{2s+1}+\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+1)}\Phi(x,\mu)\right)^{\frac{1}{q}}\right.\\&\left.+\frac{(\nu-x)^2}{(\nu-\mu)}\left(\frac{|e^{f(x)}f'(x)|^q+|e^{f(\nu)}f'(\nu)|^q}{2s+1}+\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+1)}\Phi(x,\nu)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 2.13. *If* $\varphi(\zeta)=\frac{\zeta^{\alpha}}{\Gamma(\alpha)}$ *and* $h(\zeta)=\zeta^s$*, then under the assumption of Theorem 2.10 reduces to a new result*

$$\begin{aligned}&\left|\frac{(\nu-x)^{\alpha}e^{f(\nu)}+(x-\mu)^{\alpha}e^{f(\mu)}}{\nu-\mu}-\frac{\Gamma(\alpha+1)}{\nu-\mu}\left[I_{x^-}e^{f(\mu)}+I_{x^+}e^{f(\nu)}\right]\right|\\&\leq\left(\frac{1}{1+\alpha p}\right)^{\frac{1}{p}}\left[\frac{(x-\mu)^{\alpha+1}}{(\nu-\mu)}\left(\frac{|e^{f(x)}f'(x)|^q+|e^{f(\mu)}f'(\mu)|^q}{2s+1}+\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+1)}\Phi(x,\mu)\right)^{\frac{1}{q}}\right.\\&\left.+\frac{(\nu-x)^{\alpha+1}}{(\nu-\mu)}\left(\frac{|e^{f(x)}f'(x)|^q+|e^{f(\nu)}f'(\nu)|^q}{2s+1}+\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+1)}\Phi(x,\nu)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 2.14. *If* $\varphi(\zeta)=\frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $h(\zeta)=\zeta^s$*, then under the assumption of Theorem 2.10 reduces to a new result*

$$\begin{aligned}&\left|\frac{(\nu-x)^{\frac{\alpha}{k}}e^{f(\nu)}+(x-\mu)^{\frac{\alpha}{k}}e^{f(\mu)}}{(\nu-\mu)}-\frac{\Gamma_k(\alpha+k)}{\nu-\mu}\left[I_{x^-,k}e^{f(\mu)}+I_{x^+,k}e^{f(\nu)}\right]\right|\\&\leq\left(\frac{k}{k+\alpha p}\right)^{\frac{1}{p}}\left[\frac{(x-\mu)^{\frac{\alpha}{k}+1}}{(\nu-\mu)}\left(\frac{|e^{f(x)}f'(x)|^q+|e^{f(\mu)}f'(\mu)|^q}{2s+1}+\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+1)}\Phi(x,\mu)\right)^{\frac{1}{q}}\right.\\&\left.+\frac{(\nu-x)^{\frac{\alpha}{k}+1}}{(\nu-\mu)}\left(\frac{|e^{f(x)}f'(x)|^q+|e^{f(\nu)}f'(\nu)|^q}{2s+1}+\frac{\big(\Gamma(s+1)\big)^2}{\Gamma(2s+1)}\Phi(x,\nu)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 2.15. *Let* $f:I=[\mu,\nu]\subseteq\mathbb{R}\to\mathbb{R}$ *be an absolutely continuous mapping on* I° *such that* $(e^f)'\in L([\mu,\nu])$*, where* $\mu,\nu\in I^{\circ}$ *with* $\mu<\nu$*. If the mapping* $|(e^f)'|^q,\ q>1$

with $p^{-1}+q^{-1}=1$ *is convex on* $[\mu,\nu]$, *then*

$$\frac{\Delta(0)e^{f(\nu)}+\nabla(0)e^{f(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\ {}_{x^+}I_\varphi e^{f(\nu)}+{}_{x^-}\ I_\varphi e^{f(\mu)}$$

$$\leq\frac{\nu-x}{\nu-\mu}\int_0^1\Delta\quad h^2(\zeta)|e^{f(x)}f'(x)|^q+h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q\quad+h(\zeta)h(1-\zeta)\Phi(x,\nu)\ \ dt\Big)^{\frac{1}{q}}$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1\nabla\quad h(\zeta)^2|e^{f(x)}f'(x)|^q+h^2(1-\zeta)|e^{f(\mu)}f'(\mu)|^q\quad+h(\zeta)h(1-\zeta)\Phi(x,\mu)\ \ dt\Big)^{\frac{1}{q}},$$

where Δ, ∇, $\Phi(x,\mu)$ *and* $\Phi(x,\nu)$ *are given in* (2. 7), (2. 8), (2. 11) *and* (2. 12) *respectively.*

Proof. From Lemma 2.1 and by Hölder's inequality, we get

$$\frac{\Delta(0)e^{f(\nu)}+\nabla(0)e^{f(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\ {}_{x^+}I_\varphi e^{f(\nu)}+{}_{x^-}\ I_\varphi e^{f(\mu)}$$

$$\leq\frac{\nu-x}{\nu-\mu}\int_0^1 1d\zeta^{\ \frac{1}{p}}\int_0^1\Delta(\zeta)|e^{f\ \ \zeta x+(1-\zeta)\nu}\ f'\ \ \zeta x+(1-\zeta)\nu\ \ |^q d\zeta^{\ \frac{1}{q}}$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1 1d\zeta^{\ \frac{1}{p}}\int_0^1\nabla(\zeta)|e^{f\ \ \zeta x+(1-\zeta)\mu}\ f'\ \ \zeta x+(1-\zeta)\nu\ \ |^q d\zeta^{\ \frac{1}{q}}$$

$$\leq\frac{\nu-x}{\nu-\mu}\int_0^1\Delta\ \ h(\zeta)|e^{f(x)}|^q+h(1-\zeta)|e^{f(\nu)}|^q\quad h(\zeta)|f'(x)|^q+h(1-\zeta)|f'(\nu)|^q\ \ d\zeta^{\ \frac{1}{q}}$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1\nabla\ \ h(\zeta)|e^{f(x)}|^q+h(1-\zeta)|e^{f(\mu)}|^q\quad h(\zeta)|f'(x)|^q+h(1-\zeta)|f'(\mu)|^q\ \ d\zeta^{\ \frac{1}{q}}$$

$$=\frac{\nu-x}{\nu-\mu}\int_0^1\Delta\quad h^2(\zeta)|e^{f(x)}f'(x)|^q+h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q$$

$$+h(\zeta)h(1-\zeta)\ \ |e^{f(x)}f'(\nu)|^q+|e^{f(\nu)}f'(x)|^q\quad dt\Big)^{\frac{1}{q}}$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1\nabla\quad h^2(\zeta)|e^{f(x)}f'(x)|^q+h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q$$

$$+h(\zeta)h(1-\zeta)\ \ |e^{f(x)}f'(\nu)|^q+|e^{f(\nu)}f'(x)|^q\quad dt\Big)^{\frac{1}{q}}$$

$$=\frac{\nu-x}{\nu-\mu}\int_0^1\Delta\quad h^2(\zeta)|e^{f(x)}f'(x)|^q+h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q\quad+h(\zeta)h(1-\zeta)\Phi(x,\nu)\ \ d\zeta\Big)^{\frac{1}{q}}$$

$$+\frac{x-\mu}{\nu-\mu}\int_0^1\nabla\quad h(\zeta)^2|e^{f(x)}f'(x)|^q+h^2(1-\zeta)|e^{f(\nu)}f'(\nu)|^q\quad+h(\zeta)h(1-\zeta)\Phi(x,\mu)\ \ d\zeta\Big)^{\frac{1}{q}},$$

which is the required result. □

Remark 2.16. (1) *If $h(\zeta) = \zeta$, then Theorem 2.15 reduces to Theorem 2.4 in* [23].

(2) *If $\varphi(\zeta) = h(\zeta) = \zeta$, then Theorem 2.15 reduces to Corollary 2.10 in* [23].

(3) *If $\varphi(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$ and $h(\zeta) = \zeta$, then Theorem 2.15 reduces to Corollary 2.11 in* [23].

(4) *If $\varphi(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $h(\zeta) = \zeta$, then Theorem 2.15 reduces to Corollary 2.12 in* [23].

Corollary 2.17. *If $\varphi(\zeta) = \zeta$ and $h(\zeta) = \zeta^s$, then under the assumption of Theorem 2.15 reduces to a new result*

$$\left|\frac{(\nu - x)e^{f(\nu)} + (x-\mu)e^{f(\mu)}}{\nu-\mu} - \frac{1}{\nu-\mu}\int_\mu^\nu e^{f(x)}dx\right|$$

$$\leq \left[\frac{(x-\mu)^2}{(\nu-\mu)}\left(\frac{\left\{|e^{f(x)}f'(x)|^q + (2s+2)|e^{f(\mu)}f'(\mu)|^q\right\}}{(2s+1)(2s+2)} + \frac{\Gamma(s+1)\Gamma(s+2)}{\Gamma(2s+3)}\Phi(x,\mu)\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\nu-x)^2}{(\nu-\mu)}\left(\frac{\left\{|e^{f(x)}f'(x)|^q + (2s+2)|e^{f(\nu)}f'(\nu)|^q\right\}}{(2s+1)(2s+2)} + \frac{\Gamma(s+1)\Gamma(s+2)}{\Gamma(2s+3)}\Phi(x,\nu)\right)^{\frac{1}{q}}\right].$$

Corollary 2.18. *If $\varphi(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$ and $h(\zeta) = \zeta^s$, then under the assumption of Theorem 2.15 reduces to a new result*

$$\frac{(\nu-x)^\alpha e^{f(\nu)} + (x-\mu)^\alpha e^{f(\mu)}}{\nu-\mu} - \frac{\Gamma(\alpha+1)}{\nu-\mu} \; I_{x-} e^{f(\mu)} + I_{x+} e^{f(\nu)}$$
$$\leq \; \frac{(x-\mu)^{\alpha+1}}{(\nu-\mu)} \; \frac{\alpha|e^{f(x)}f'(x)|^q}{(2s+1)(2s+\alpha+1)} + \; \frac{1}{2s+1} - \frac{\Gamma(\alpha+1)\Gamma(2s+1)}{\Gamma(\alpha+2s+2)} \; |e^{f(\mu)}f'(\mu)|^q$$
$$+ \; \frac{\Gamma(s+1)}{\Gamma(2s+2)}^{\,2} - \frac{\Gamma(\alpha+s+2)\Gamma(s+1)}{\Gamma(\alpha+2s+2)} \; \Phi(x,\mu) \;^{\frac{1}{q}} + \frac{(\nu-x)^{\alpha+1}}{(\nu-\mu)} \; \frac{\alpha|e^{f(x)}f'(x)|^q}{(2s+1)(2s+\alpha+1)}$$
$$+ \; \frac{1}{2s+1} - \frac{\Gamma(\alpha+1)\Gamma(2s+1)}{\Gamma(\alpha+2s+2)} \; |e^{f(\nu)}f'(\nu)|^q + \; \frac{\Gamma(s+1)}{\Gamma(2s+2)}^{\,2} - \frac{\Gamma(\alpha+s+2)\Gamma(s+1)}{\Gamma(\alpha+2s+2)} \; \Phi(x,\nu) \;^{\frac{1}{q}} \; .$$

Corollary 2.19. *If $\varphi(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $h(\zeta) = \zeta^s$, then under the assumption of Theorem 2.15 reduces to a new result*

$$\frac{(\nu-x)^{\frac{\alpha}{k}} e^{f(\nu)} + (x-\mu)^{\frac{\alpha}{k}} e^{f(\mu)}}{(\nu-\mu)} - \frac{\Gamma_k(\alpha+k)}{\nu-\mu} \; I_{x-,k} e^{f(\mu)} + I_{x+,k} e^{f(\nu)}$$
$$\leq \; \frac{(x-\mu)^{\frac{\alpha}{k}+1}}{(\nu-\mu)} \; \frac{\frac{\alpha}{k}|e^{f(x)}f'(x)|^q}{(2s+1)(2s+\frac{\alpha}{k}+1)} + \; \frac{1}{2s+1} - \frac{\Gamma(\frac{\alpha}{k}+1)\Gamma(2s+1)}{\Gamma(\frac{\alpha}{k}+2s+2)} \; |e^{f(a)}f'(a)|^q$$
$$+ \; \frac{\Gamma(s+1)}{\Gamma(2s+2)}^{\,2} - \frac{\Gamma(\frac{\alpha}{k}+s+2)\Gamma(s+1)}{\Gamma(\frac{\alpha}{k}+2s+2)} \; \Phi(x,a) \;^{\frac{1}{q}} + \frac{(\nu-x)^{\frac{\alpha}{k}+1}}{(\nu-a)} \; \frac{\frac{\alpha}{k}|e^{f(x)}f'(x)|^q}{(2s+1)(2s+\frac{\alpha}{k}+1)}$$
$$+ \; \frac{1}{2s+1} - \frac{\Gamma(\frac{\alpha}{k}+1)\Gamma(2s+1)}{\Gamma(\frac{\alpha}{k}+2s+2)} \; |e^{f(\nu)}f'(\nu)|^q + \; \frac{\Gamma(s+1)}{\Gamma(2s+2)}^{\,2} - \frac{\Gamma(\frac{\alpha}{k}+s+2)\Gamma(s+1)}{\Gamma(\frac{\alpha}{k}+2s+2)} \; \Phi(x,\nu) \;^{\frac{1}{q}} \; .$$

Conclusion: In this paper, we have derived the Trapezoid type inequalities via generalized fractional operators for exponentially convex functions that incorporate a certain type of generalized fractional integrals involving generalizes the classical exponentially h-convex function and well known Riemann-Liouville fractional integrals. We obtained results are the refinements and new generalizations by adopting the technique of Mohammed [12]. We studied the conditions under which these results are stable in the framework of classical, Reimann-Liouville fractional integral and k-Reimann-Liouville fractional integral. Our results can be viewed as the generalizations of the previous studies on exponentially convex functions associated with generalized fractional integrals.

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