Punjab University Journal of Mathematics
ISSN: 1016-2526
Vol. 51(11)(2019) pp. 63-75

Oscillating Flows of Fractionalized Second Grade Fluid with Slip Effects

Sanaullah Dehraj*[1], Rajab A. Malookani[2], Muzaffar B. Arain[3], Nasreen Nizamani[4]
[1,2,3]Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology 67480, Sindh, Pakistan
[4]Department of Electronic Engineering, Quaid-e-Awam University of Engineering, Science and Technology 67480, Sindh, Pakistan
Email: sanaullahdehraj@quest.edu.pk *[1], rajibali@quest.edu.pk [2], muzaffararain@quest.edu.pk [3], nasreen_nizamani@hotmail.com [4]

Received: 04 January, 2019 / Accepted: 06 September, 2019 / Published online: 01 October, 2019

Abstract. This paper examines the fractionalized second grade fluid due to oscillating plate under slip condition. The discrete Laplace transform technique is employed to compute the analytical solutions for the equations of motion. The velocity field and shear stress are computed. In order to write them in compact form, the Wright generalized hyper geometric function is used and written as addition of slip and no slip contributions. The closed-form solutions for ordinary second grade and Newtonian fluids carrying out the similar motion are achieved. The computations for fractional and ordinary second grade fluids without slip effect are also achieved as a special case. Furthermore, the impact of various parameters such as the slip, fractional and material parameters on the motion of fractionalized second grade fluid will be explained through graphs. Finally, a comparison among the Newtonian fluids, ordinary second grade fluids and fractionalized second grade fluids is also carried out.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Velocity fields, Shear stress, Ordinary second grade fluid, Slip effects, Fractional derivative, Oscillating flows, Discrete Laplace transform..

1. Introduction

Blood, polymer solutions and certain oils represent many complex fluids, which are expressed via Newtonian constitutive equation that often do not exhibit any relaxation and retardation phenomena. In order to express the dynamics in such fluids, many models are introduced. Particularly, the differential type models have gained much important attention ([1], [2], [3]). The model of second-grade fluid is one of the famous models for non-Newtonian fluids ([4], [5], [6], [7], [8], [9]). The second-grade fluids are associated with the simplest subclass of non-Newtonian fluids whose closed-form solution can easily be found. Recently, the subject of fractional calculus has attained much attention to solve the complicated problems in complex dynamics. Specifically, it is proven to be an important mechanism for tackling the viscoelastic properties. The fractional calculus approach is widely studied (see for instance ([10], [11], [12], [13], [14], [15] [16], [17], [18], [19]). In more recent years, the closed-form solutions describing the flow of various fractionalized non-Newtonian fluids are achieved in ([20], [21], [22], [23], [24], [25], [26]). In most of the applications, it is seen that the fluid renders a loss of adhesion at the wetted wall. The idea of slip of a fluid at a solid wall demonstrates the macroscopic effects of specific molecular phenomenon. The non-continuum effect of fluid such as slip-flow was experimentally obtained by Derek et al. [27]. The boundary conditions which ponder the feasibility of fluid slip at a solid boundary were introduced by Navier [28]. It is expressed in that condition that the fluid speed is directly proportional to the shear stress. An experimental study was first carried out by Beavers and Joseph [29] to examine the fluid flow at the interface between a porous medium and fluid layer. The authors also devised the slip boundary conditions. Some recent studies discussing the closed-form solutions for non-Newtonian fluids under slip effects are shown in Refs. ([30], [31], [32], [33], [34], [35], [36]). Jamil and Najeeb A. Khan [36] examined an unsteady flow of an incompressible Maxwell fluid under slip condition. However, the unsteady flow for second grade fluid under slip condition is not studied for second grade fluid. This paper provides the closed-form solutions for second grade fluids through fractional derivative approach, which is suitable mechanism to express the complex behavior of such fluids with slip effects. Specifically, this work aims to compute the profiles for the shear stress and the velocity field associated to the motion of a fractionalized second grade fluid owing to an oscillating plate. The discrete Laplace transform technique is employed to attain the general solutions for the governed equations, which are expressed in terms of the Wright generalized hyper geometric function ${}_p\psi_q$ and shown as addition of the slip contribution and the associated no-slip contributions. The solutions for Newtonian and ordinary second grade fluids for similar motion are achieved. Furthermore, the solutions for fractional fluids and ordinary second grade fluids are also obtained without slip effect as a special case and they are equivalent with classical Stokes second and first (if frequency of oscillating plane $\omega = 0$) problems. Moreover, the motion of fractionalized second grade fluids under various effects of the slip, fractional and material parameters is also discussed graphically. In the end, the comparison among the fluid models such as Newtonian, fractionalized second grade and ordinary second grade fluid models is carried out with the help of graph.

2. MATHEMATICAL MODEL

The mathematical equations governing an incompressible fluid flow including the momentum and continuity equations are given by,

$$\nabla \cdot V = 0\,, \quad \nabla \cdot T = \rho \frac{\partial V}{\partial t} + \rho\,(V.\nabla)\,V \tag{2. 1}$$

where the parameter ρ represents the fluid density, t is the time, V expresses the velocity field, and ∇ is the Nabla operator. The relationship between the Cauchy stress T and the fluid motion is shown in [4-9],

$$T = -pI + S, S = \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \tag{2. 2}$$

where $-pI$ denotes the unspecified section of the stress owing to the incompressibility constraint, S represents the extra stress tensor, α_1 and α_2 expresses the normal stress moduli, μ expresses the dynamic viscosity. The functions A_1, A_2 are the kinematic tensors, which are defined as,

$$A_1 = (\nabla V) + (\nabla V)^T, \;\; A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + (\nabla V)^T A_1 \tag{2. 3}$$

In this study, an extra-stress tensor and a velocity field are presumed in the form

$$V = u\,(y,t)\,i, S = S\,(y,t) \tag{2. 4}$$

where the vector i represents the unit vector in x-direction. The incompressibility constraint for these flows is satisfied automatically. The fluid is considered to be at rest when the time, $t = 0$ then

$$V = V\,(y,0) = 0, \quad S = S\,(y,0) = 0 \tag{2. 5}$$

Eqs. (2. 1)-(2. 5) yield the following coupled system of equations,

$$\begin{cases} \frac{\partial u(y,t)}{\partial t} = \left(v + \alpha \frac{\partial}{\partial t}\right) \frac{\partial^2 u(y,t)}{\partial y^2} \\ \tau\,(y,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \frac{\partial u(y,t)}{\partial y} \end{cases} \tag{2. 6}$$

where $\tau(y,t) = S_{xy}(y,t)$ the non-zero is shear stress and $v = \frac{\mu}{\rho}$ denotes the kinematic viscosity and the viscoelastic characteristic of the second grade fluid. The mathematical equations for fractionalized second grade fluid are given by [18]

$$\begin{cases} \frac{\partial u(y,t)}{\partial t} = \left(v + \alpha D_t^{\beta}\right) \frac{\partial^2 u(y,t)}{\partial y^2} \\ \tau\,(y,t) = \left(\mu + \alpha_1 D_t^{\beta}\right) \frac{\partial u(y,t)}{\partial y}, \end{cases} \tag{2. 7}$$

where the fractional parameter $0 < \beta < 1$ While, the operator so called Caputo fractional operator D_t^{β} (see in [26, 27]) defined by

$$D_t^{\beta} f\,(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} & \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta}} d\tau, \quad 0 \le \beta < 1 \\ \frac{df(t)}{dt} & , \qquad \beta = 1 \end{cases} \tag{2. 8}$$

We observe an incompressible fractionalized second grade fluid over an infinitely extended plate on (x, z)-plane and perpendicular along the y-axis. At the beginning, the fluid is at rest position; the plate starts to oscillate in its own plane. Furthermore, the slip boundary is taken into consideration between the fluid velocity at the plane $u(0, t)$ and the plate speed.

The velocity is taken to be proportional to the shear rate at the plane between $u(0,t)$ and plate. The velocity field is given in Eq.(2.4)$_1$ while the mathematical governed equations of motion are indicated in Eqs. (2.7). Following initial condition and boundary conditions are taken into consideration:

Initial Condition:

$$u\,(y,0)=0 \quad y>0, \tag{2. 9}$$

Boundary Conditions:

$$\begin{cases} u(0,t)=UH(t)sin(\omega t)+\theta H(t)\frac{\partial u(y,t))}{\partial y}\mid_{y=0} \\ u(0,t)=UH(t)cos(\omega t)+\theta H(t)\frac{\partial u(y,t))}{\partial y}\mid_{y=0} \end{cases} \tag{2. 10}$$

where the parameter θ is the slip parameter and the function $H(t)$ is known as the Heaviside function. In addition, the natural or force boundary conditions take the form:

$$u(y,t)\to 0 \text{ as } y\to\infty, \quad t>0\,, \tag{2. 11}$$

are to be satisfied. This is because the fluid is taken to be at rest, when approaches infinity and no shear in the free stream are required. Fig.1 depicts the geometry of the governed equations of motion together with the boundary conditions

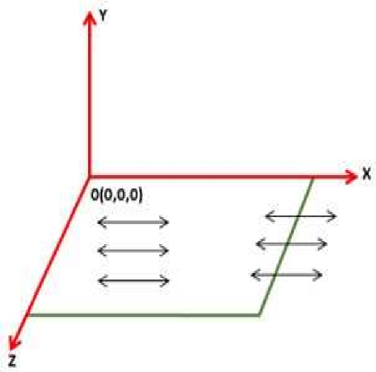

FIGURE 1. geometry of the problem

2.1. **Computation of Velocity Field.**

In the subsequent section, the governing equation of motion Eq. (2.7) is examined through the method of discrete Laplace transform. By taking the Laplace transform to Eq. (2.7)$_1$ together with initial condition (2.9) and boundary condition (2.10)$_1$, we have

$$\left(\frac{\partial^2}{\partial y^2}-\frac{q}{v+\alpha q^{\beta}}\right)\bar{u}\,(y,q)=0, \tag{2. 12}$$

with boundary conditions,

$$\bar{u}(0,q) = \frac{U\omega}{q^2+\omega^2} + \theta \frac{\partial \bar{u}(y,q)}{\partial y}|_{y=0} \; and \; \bar{u}(y,q) \to 0 \; as y \to \infty \tag{2. 13}$$

where $\bar{u}(y,q)$ denotes the Laplace transform of $u(y,t)$ and the parameter q is a kernel of transformation. The solution of Eqs.(2.12)-(2.13) gives,

$$\bar{u}_S\left(y,q\right) = \frac{U\omega}{\left(q^2+\omega^2\right)\left\{1+\theta\left[\frac{q}{v+\alpha q^{\beta}}\right]^{\frac{1}{2}}\right\}} \exp\left\{-\left[\frac{q}{v+\alpha q^{\beta}}\right]^{\frac{1}{2}} y\right\}. \tag{2. 14}$$

Eq. (2.14) can be rewritten in series form as Eq. (2.16) by using the fact

$$(-1)^p \frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha-p+1\right)} = \frac{\Gamma\left(p-\alpha\right)}{\Gamma\left(-\alpha\right)}, \tag{2. 15}$$

$$\begin{aligned} \bar{u}_S\left(y,q\right) = \frac{U\omega}{q^2+\omega^2} + U\omega \sum_{j=0}^{\infty}\left(-\omega^2\right)^j \sum_{l=1}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{l}{2}\right)}{n!\Gamma\left(\frac{l}{2}\right)}\times \\ \left(\frac{-v}{\alpha}\right)^n \frac{1}{q^{(\beta-1)\frac{l}{2}+n\beta+2j+2}} + U\omega \sum_{j=0}^{\infty}\left(-\omega^2\right)^j \sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l \sum_{r=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r \times \\ \frac{1}{r!}\sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{l+r}{2}\right)}{n!\Gamma\left(\frac{l+r}{2}\right)}\left(\frac{-v}{\alpha}\right)^n \frac{1}{q^{(\beta-1)\left(\frac{l+r}{2}\right)+n\beta+2j+2}} \end{aligned} \tag{2. 16}$$

Taking inverse Laplace transform of Eq. (2.16) on both sides, we obtain

$$\begin{aligned} u_S\left(y,t\right) = UH\left(t\right)\sin\left(\omega t\right) + U\omega H\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j \sum_{l=1}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l \times \\ \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{l}{2}\right)\left(\frac{-v}{\alpha}t^{\beta}\right)^n t^{(\beta-1)\frac{l}{2}+2j+1}}{n!\Gamma\left(\frac{l}{2}\right)\Gamma\left((\beta-1)\frac{l}{2}+n\beta+2j+2\right)} + U\omega H\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j \sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l \sum_{r=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r \times \\ \frac{1}{r!}\sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{l+r}{2}\right)\left(-\frac{v}{\alpha}t^{\beta}\right)^n}{n!\Gamma\left(\frac{l+r}{2}\right)\Gamma\left((\beta-1)\left(\frac{r+l}{2}\right)+n\beta+2j+2\right)} t^{(\beta-1)\left(\frac{l+r}{2}\right)+2j+1} \end{aligned} \tag{2. 17}$$

To obtain the velocity field for sine oscillation, we write Eq. (2.17) in terms of Wright generalized hyper geometric function [37]:

$$\begin{aligned} u_S\left(y,t\right) = UH\left(t\right)\sin\left(\omega t\right) + UH\left(t\right)\omega \sum_{j=0}^{\infty}\left(-\omega^2\right)^j \sum_{l=1}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l \\ t^{(\beta-1)\frac{l}{2}+2j+1} {}_1\Psi_2\left(\frac{-vt^{\beta}}{\alpha}\middle| \begin{matrix}\left(\frac{l}{2},1\right)\\ \left(\frac{l}{2},0\right)\left((\beta-1)\frac{l}{2}+2j+2,\beta\right)\end{matrix}\right) \end{aligned} \tag{2. 18}$$

where the function is defined as

$$ {}_p\Psi_q\left(z\left|\begin{matrix}(a_1,A_1)\ldots(a_p,A_p)\\(b_1,B_1)\ldots(b_q,B_q)\end{matrix}\right.\right)=\sum_{n=0}^{\infty}\frac{z^n\Pi_{j=1}^{p}\Gamma\left(a_j+A_jn\right)}{n!\Pi_{j=1}^{q}\Gamma\left(b_j+B_jn\right)} \tag{2. 19} $$

Furthermore, the velocity field for cosine oscillation can be obtained by using same procedure as earlier and is given in Eq. (2.20). It can easily be seen that obtained solutions in Eqs. (2.18) and (2.20) satisfies both the initial and boundary conditions.

$$ \begin{gathered} u_C\left(y,t\right)=UH\left(t\right)\cos\left(\omega t\right)+UH\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{l=1}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l t^{(\beta-1)\frac{l}{2}+2j}\times \\ {}_1\Psi_2\left(\frac{-vt^\beta}{\alpha}\left|\begin{matrix}\left(\frac{l}{2},1\right)\\\left(\frac{l}{2},0\right)\left((\beta-1)\frac{l}{2}+2j+1,\beta\right)\end{matrix}\right.\right)+UH\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\times \\ \sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l\sum_{r=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\frac{1}{r!}t^{(\beta-1)\frac{l+r}{2}+2j}{}_1\Psi_2\left(\frac{-vt^\beta}{\alpha}\left|\begin{matrix}\left(\frac{l+r}{2},1\right)\\\left(\frac{l+r}{2},0\right)\left((\beta-1)\frac{l+r}{2}+2j+1,\beta\right)\end{matrix}\right.\right) \end{gathered} \tag{2. 20} $$

2.2. **Computation of Shear Stress.**

To calculate the shear stress, we employ the Laplace transform on both sides of Eq.$(2.7)_2$, we get

$$ \bar{\tau}\left(y,q\right)=\left(\mu+\alpha_1q^\beta\right)\frac{\partial\bar{u}\left(y,q\right)}{\partial y} \tag{2. 21} $$

Using Eq. (2.14) in (2.21), we find that

$$ \bar{\tau}_S\left(y,q\right)=\frac{-\rho U\omega\sqrt{q}\left(v+\alpha q^\beta\right)^{\frac{1}{2}}}{\left(q^2+\omega^2\right)\left[1+\theta\sqrt{\frac{q}{v+\alpha q^\beta}}\right]}\exp\left(-\left[\frac{q}{v+\alpha q^\beta}\right]^{\frac{1}{2}}y\right) \tag{2. 22} $$

We write the Eq. (2.22) in series form as under

$$ \begin{gathered} \bar{\tau}_S\left(y,q\right)=-\rho U\omega\sqrt{\alpha}\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l\sum_{r=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\times \\ \frac{1}{r!}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{l+r-1}{2}\right)\left(\frac{-v}{\alpha}\right)^n}{n!\Gamma\left(\frac{l+r-1}{2}\right)}\frac{1}{q^{(\beta-1)\left(\frac{l+r-1}{2}\right)+n\beta+2j+1}} \end{gathered} \tag{2. 23} $$

Eq. (2.23) is solved by taking the inverse Laplace transform on both sides and then by using the Eq. (2.19), we shall find the shear-stress $\tau_s(y,t)$ for sine oscillation:

$$ \begin{gathered} \tau_S\left(y,t\right)=-\rho U\omega\sqrt{\alpha}H\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l\sum_{r=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\frac{1}{r!} \\ t^{(\beta-1)\left(\frac{l+r-1}{2}\right)+2j}{}_1\Psi_2\left(-\frac{vt^\beta}{\alpha}\left|\begin{matrix}\left(\frac{l+r-1}{2},1\right)\\\left(\frac{l+r-1}{2},0\right)\left(\left(\frac{l+r-1}{2}\right)(\beta-1)+2j+2,\beta\right)\end{matrix}\right.\right) \end{gathered} \tag{2. 24} $$

Shear stress for cosine oscillation can be obtained in same manner and is given below,

$$\tau_c\left(y,t\right)=-\rho U\sqrt{\alpha}H\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^l\sum_{r=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\frac{1}{r!}\times \\ t^{(\beta-1)\left(\frac{l+r-1}{2}\right)+2j}{}_1\Psi_2\left(-\frac{vt^\beta}{\alpha}\left|\begin{smallmatrix}\left(\frac{l+r-1}{2},1\right)\\ \left(\frac{l+r-1}{2},0\right)\left(\left(\frac{l+r-1}{2}\right)(\beta-1)+2j+2,\beta\right)\end{smallmatrix}\right.\right) \tag{2. 25}$$

3. SPECIAL CASES

3.1. **Ordinary Second Grade Fluid with Slip Effect when $\beta \to 1$.**

By putting $\beta \to 1$ into Eqs. (2.18), (2.20), (2.24) and (2.25), we shall acquire the velocity field and associated shear stress for ordinary second grade fluid under slip effect.

3.2. **Fractionalized Second Grade Fluid without Slip Effects.**

By letting $\theta \to 0$ in Eqs. (2.18), (2.20), (2.24) and (2.25), the shear stress and the velocity field without slip effect are given as under,

$$u_S\left(y,t\right)=UH\left(t\right)\sin\left(\omega t\right)+U\omega H\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{r=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\times \\ \frac{1}{r!}t^{(\beta-1)\left(\frac{r}{2}\right)+2j+1}{}_1\Psi_2\left(-\frac{vt^\beta}{\alpha}\left|\begin{smallmatrix}\left(\frac{r}{2},1\right)\\ \left(\frac{r}{2},0\right)\left(\frac{r}{2}(\beta-1)+2j+2,\beta\right)\end{smallmatrix}\right.\right) \tag{3. 26}$$

$$u_c\left(y,t\right)=UH\left(t\right)\cos\left(\omega t\right)+UH\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{r=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\times \\ \frac{1}{r!}t^{(\beta-1)\left(\frac{r}{2}\right)+2j+1}{}_1\Psi_2\left(\frac{-vt^\beta}{\alpha}\left|\begin{smallmatrix}\left(\frac{r}{2},1\right)\\ \left(\frac{r}{2},0\right)\left((\beta-1)\frac{r}{2}+2j+1,\beta\right)\end{smallmatrix}\right.\right) \tag{3. 27}$$

$$\tau_s\left(y,t\right)=-\rho U\sqrt{\alpha}H\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{r=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\frac{1}{r!}t^{(\beta-1)\left(\frac{r-1}{2}\right)+2j} \\ \times{}_1\Psi_2\left(\frac{-vt^\beta}{\alpha}\left|\begin{smallmatrix}\left(\frac{r-1}{2},1\right)\\ \left(\frac{r-1}{2},0\right)\left(\left(\frac{r-1}{2}\right)(\beta-1)+2j+1,\beta\right)\end{smallmatrix}\right.\right) \tag{3. 28}$$

$$\tau_c\left(y,t\right)=-\rho U\sqrt{\alpha}H\left(t\right)\sum_{j=0}^{\infty}\left(-\omega^2\right)^j\sum_{r=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^r\frac{1}{r!}t^{(\beta-1)\left(\frac{r-1}{2}\right)+2j} \\ \times{}_1\Psi_2\left(\frac{-vt^\beta}{\alpha}\left|\begin{smallmatrix}\left(\frac{r-1}{2},1\right)\\ \left(\frac{r-1}{2},0\right)\left(\left(\frac{r-1}{2}\right)(\beta-1)+2j+1,\beta\right)\end{smallmatrix}\right.\right) \tag{3. 29}$$

It is also worth to note that the substitution of frequency of oscillating plate $\omega = 0$, in Eqs. (3.28) and (3.29), yields the similar solutions as were obtained in [18] for fractionalized second grade fluid.

3.3. **Ordinary Second Grade Fluid without Slip Effects $\beta \to 1$.**

When letting $\beta \to 1$ into Eqs. (3.26)- (3.29), the solutions for second problem of Stokes for ordinary second grade fluid without slip effect can be achieved.

3.4. Newtonian Fluid under Slip Effects $\alpha \to 0$.

By taking $\alpha \to 0$ into Eqs. (2.14) and (2.22), under slip effect, the solutions for a Newtonian fluid corresponding to sine oscillation is obtained as under,

$$\begin{aligned} u_S(y,t) &= UH(t)\sin(\omega t) + UH(t)\,\omega t \sum_{l=1}^{\infty}\left(\frac{-\theta}{\sqrt{tv}}\right)^{l} {}_1\Psi_1\left(-(\omega t)^2\Big|_{\left(\frac{-l}{2}+2,2\right)}^{(1,1)}\right) \\ &+UH(t)\,\omega t\sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{tv}}\right)^{l}\sum_{r=1}^{\infty}\left(\frac{-y}{\sqrt{tv}}\right)^{r}\frac{1}{r!}{}_1\Psi_1\left(-(\omega t)^2\Big|_{\left(-\frac{l+r}{2}+2,2\right)}^{(1,1)}\right) \end{aligned} \tag{3. 30}$$

$$\tau_S(y,t) = -\sqrt{\rho}U\sqrt{\frac{\mu}{t}}\omega H(t)\sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{tv}}\right)^{l}\sum_{r=0}^{\infty}\left(\frac{-y}{\sqrt{tv}}\right)^{r}\frac{1}{r!}{}_1\Psi_1\left(-(\omega t)^2\Big|_{\left(-\frac{l+r-1}{2}+1,2\right)}^{(1,1)}\right) \tag{3. 31}$$

Similarly, the shear stress profiles and velocity field profiles for cosine oscillations are given by,

$$\begin{aligned} u_c(y,t) &= UH(t)\cos(\omega t) + UH(t)\sum_{l=1}^{\infty}\left(\frac{-\theta}{\sqrt{tv}}\right)^{l} {}_1\Psi_1\left(-(\omega t)^2\Big|_{\left(-\frac{-l}{2}+1,2\right)}^{(1,1)}\right) \\ &+UH(t)\sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{tv}}\right)^{l}\sum_{r=1}^{\infty}\left(\frac{-y}{\sqrt{tv}}\right)^{r}\frac{1}{r!}{}_1\Psi_1\left(-(\omega t)^2\Big|_{\left(-\frac{l+r}{2}+1,2\right)}^{(1,1)}\right) \end{aligned} \tag{3. 32}$$

and

$$\tau_c(y,t) = -\sqrt{\rho}H(t)\,U\sqrt{\frac{\mu}{t}}\sum_{l=0}^{\infty}\left(\frac{-\theta}{\sqrt{tv}}\right)^{l}\sum_{r=0}^{\infty}\left(\frac{-y}{\sqrt{tv}}\right)^{r}\frac{1}{r!}{}_1\Psi_1\left(-(\omega t)^2\Big|_{\left(-\frac{l+r-1}{2},2\right)}^{(1,1)}\right) \tag{3. 33}$$

3.5. Newtonian Fluid without Slip Effects $\theta \to 0$.

Finally, taking $\theta \to 0$ into Eqs. (3.30)-(3.33), the solutions for Newtonian fluid without slip effect can be achieved.

4. Results and Discussions

This section discusses the implementation and correlated physical aspect of our obtained solutions. Much attention has been paid on analyzing the velocity field $u_s(y,t)$ for sine oscillations in the presence (i.e. $\theta = 5.0$) and absence (i.e. $\theta = 0.0$) of slip effects. Using the Mathcad software the graphs for the velocity field profiles $u_s(y,t)$ have been drawn against y under different material constants α, v, θ the frequency ω and fractional parameter β with fixed $\theta = 5.0$ and $\theta = 0.0$ Fig. 2 depicts the effects of time on velocity field $u_s(y,t)$ against y treating the slip parameter $\theta = 5.0$ (Fig.2a) and $\theta = 0.0$ (Fig. 2b) fixed with different material constants, α, v, θ the frequency ω and fractional parameter β. It is clear in Fig.2a that the fluid motion increases in the beginning, then decreases and remains constant as y increases with the presence of slip condition. While, in Fig. 2b, the similar behavior is observed except $t = 6$ and larger values of t. The effects of material parameter α and kinematic viscosity v on the velocity field are, respectively, sketched in Fig. 3 and Fig. 4. It is shown in Fig. 3 that the velocity increases and remains constant for larger values of α with and without slip effects. The similar behavior for velocity field against for various values of kinematic velocity $u_s(y)$ is observed. Fig. 5 exhibits the effects of fractional parameter β on the velocity field with respect to y. It is shown that the velocity grows initially and then remains constant as y increases for larger values of fractional parameter β. Fig. 6 exhibits the effects of different frequency-values ω on the fluid motion in terms of velocity field. It can be seen that there is no effect of slip parameter on the velocity field under various frequency-values ω. The effects of a parameter on the velocity field with respect to time t is shown in Fig. 7. The decreasing oscillatory behavior of velocity field is seen as the time t progresses. Finally, the effects of a fractional parameter β for fractionalized, ordinary and Newtonian fluid with respect to y is depicted in Fig. 8. It is shown that the velocity field behaves alike for both the fractionalized and ordinary fluids except slightly different from Newtonian fluid.

5. CONCLUSION

The closed-form solutions for the governing equations motion describing the fractionalized second grade fluid under slip effect are achieved. By employing the method of discrete Laplace transform, the solutions for velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ corresponding to sine and cosine oscillations are gained. These solutions are shown in series form in term of function ${}_p\psi_q$ and expressed as sum between no-slip and slip contribution, satisfying all given constraints. The associated solutions for ordinary second grade fluid with and without slip effects are also found from the general solutions for $\beta \to 1$ and $\theta \to 0$. For $\alpha \to 0$, the Newtonian solutions are achieved as special cases from the general solutions. Stokes second problem of fractionalized and ordinary second grade fluid is obtained by setting $\theta \to 0$ as a special case. Furthermore, by making $\theta = 0$ and $\omega = 0$ in Eqs. (2.18) and (2.24), we recovered [18] the classical solutions for Stokes first problem for fractionalized second grade fluid. The distinction among the various fluid models such as Newtonian, fractionalized second and ordinary second grade fluid models has also been pointed out. The important conclusions from present study are given below.

- The general solutions (2.18), (2.20), (2.24) and (2.25) are shown as a sum of the slip and related no-slip contributions. The attained solutions can easily be specified to display the same solutions for ordinary second grade fluid.
- It is shown that the fluid motion slows down when the parameter grows.
- The amplitude of the velocity field is seen to be increasing with respect to time in beginning then decreases slowly over the time.
- The amplitude of the velocity field is shown to be increasing with respect to the material parameter and kinematic viscosity , whether slip effect is present or not.
- The fractional parameter and frequency of the oscillating plate have similar effects on the motion of fluid.
- The highest amplitude of the Newtonian and fractionalized second grade fluids is witnessed respectively with and without presence of the slip effect.

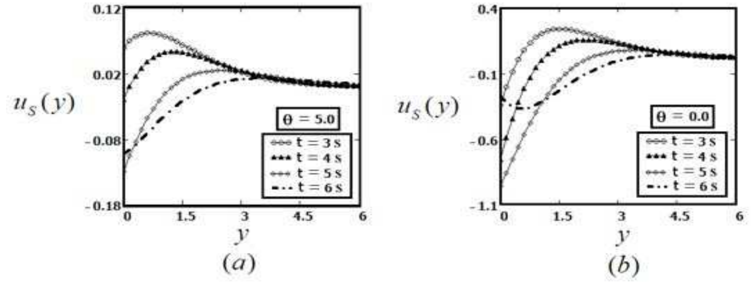

FIGURE 2. Velocity field profiles $u_s(y,t)$ verses y for various values of t with $U = 1, v = 0.295, \mu = 26, \alpha = 0.5, \beta = 0.2, \omega = 1$

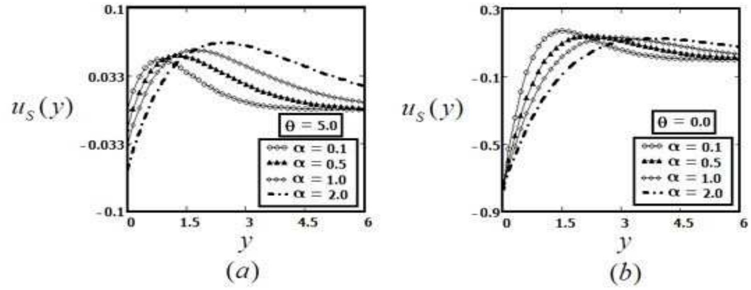

FIGURE 3. Velocity field profiles $u_s(y,t)$ verses y for different parameter α, with $U = 1$, $\beta = 0.2$, $\rho = 88$,$\beta = 0.2$, $\omega = 1$, $t = 4s$

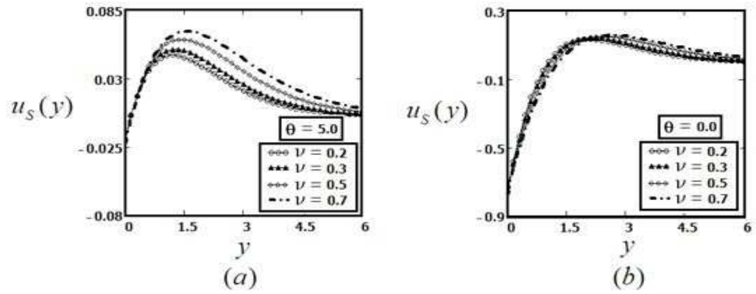

FIGURE 4. Velocity field profiles $u_s(y,t)$ verses y for different parameter α, with $U = 1$, $\rho = 88$, $\beta = 0.2$,$\omega = 1$, $t = 4s$

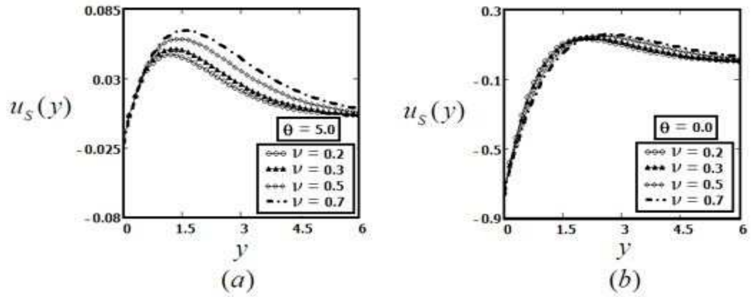

FIGURE 5. Velocity field profiles $u_s(y,t)$ verses y for different parameter α, with $U = 1$, $\rho = 88$, $\beta = 0.2$,$\omega = 1$, $t = 4s$

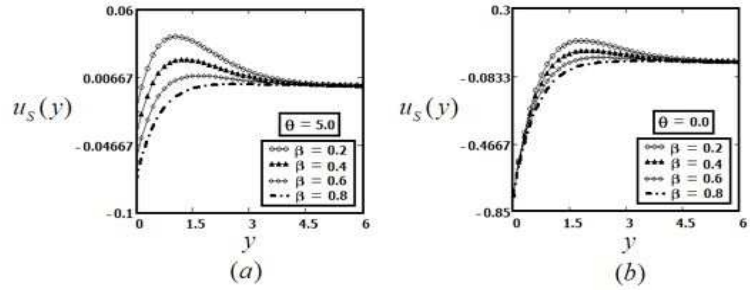

FIGURE 6. Velocity field profiles $u_s(y,t)$ verses y for various values of fractional parameters β, with $U = 1$, $v = 0.0295$, $\mu = 26$, $\alpha = 0.5$, $\omega = 1$, $t = 4s$

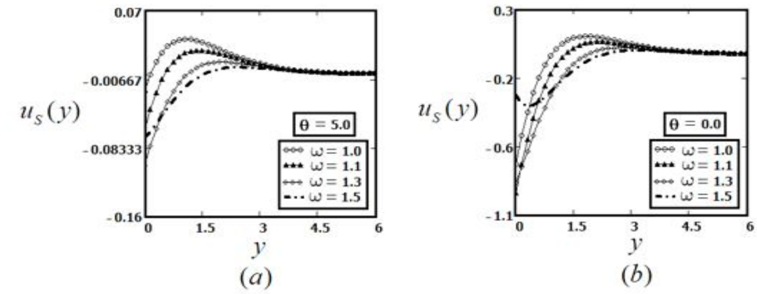

FIGURE 7. Velocity field $u_s(y,t)$ verses y for different values of frequency ω, with $U = 1$, $v = 0.0295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2$, $t = 4s$

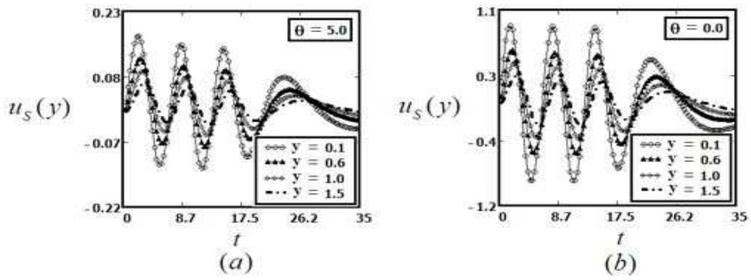

FIGURE 8. Velocity field $u_s(y,t)$ verses t for different values of frequency y, with $U = 1$, $v = 0.295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2$, $\omega = 1$

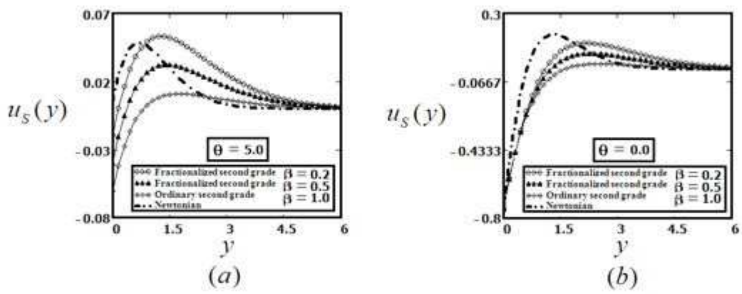

FIGURE 9. Comparison among different models with $U = 1$, $v = 0.295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2, 0.5, 1$, $\omega = 1$ and $t = 4s$

6. Acknowledgments

The authors acknowledge the Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah, Pakistan for providing financial support and the unknown referees for reviewing and giving important suggestions to improve our paper.

References

[1] R. L. Bagley, *A theoretical basis for the application of fractional calculus to viscoelasticity*, Journal of Rheology, **27** (1983) 201-210.
[2] G. S. Beavers and D. D. Joseph, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech, **30** (1967) 197-207.
[3] C. Derek, D.C. Tretheway and C.D. Meinhart, *A generating mechanism for apparent fluid slip in hydrophobic micro channels*, Phys. Fluids **14** (2002) L9.
[4] A. Ebaid, *Effects of magnetic field and wall slip conditions on the peristaltic transport of a Newtonian fluid in an asymmetric channel*, Physics Letters A, **372** (2008) 4493-4499.
[5] R. Ellahi, T. Hayat and F.M. Mahomed, *Generalized Couette flow of a third-grade fluid with slip: The exact solutions*, Z. Naturforsch. **65a** (2010) 1071-1076.
[6] M. E. Erdogan, *Plane surface suddenly set in motion in a non-Newtonian fluid*, Acta Mech., **108** (1995) 179-187.
[7] C. Fetecau and Corina Fetecau, *Starting solutions for some unsteady unidirectional flows of a second grade fluid*, Int. J. Eng. Sci, **43** (2005) 781-789.
[8] C. Fetecau and Corina Fetecau, *Starting solutions for the motion of second grade fluid due to longitudinal and torsional oscillations of a circular cylinder*, Int. J. Eng. Sci, **44** (2006) 788-796.
[9] C. Fetecau, Corina Fetecau and J. Zierep, *Decay of potential vortex and propagation of a heat wave in a second grade fluid*, Int. J. Non-linear Mech, **37** (2002) 1051-1056.
[10] C. Fetecau, A. Mahmood and M. Jamil, *Exact solutions for the flow of a viscoelastic fluid induced by a circular cylinder subject to a time dependent shear stress*, Commun Nonlinear Sci Numer Simulat, **15** (2010) 3931-3938.
[11] C. H. R. Friedrich, *Relaxation and retardation function of the Maxwell model with fractional derivatives*, Rheological Acta, **30** (1991) 151-158.
[12] N. C. Ghosh, B. C. Ghosh and L. Debnath, *The hydromagnetic flow of a dusty viscoelastic fluid between two infinite parallel plates*, Comput. Math. Appl, **39** (2000) 103-116.
[13] H. Guangyu, H. Junki and L Ciqun, *General second order fluid flow in a pipe, Appl. Math*. Mech. (English Ed.) **14** (1995) 825-831.
[14] T. Hayat, R. Ellahi, S. Asghar and A.M.Siddiqui, *Flow induced by a non-coaxial rotation of a porous disk executing non-torsional oscillations and a second grade fluid at infinity*, Appl. Math. Model, **28** (2004) 591-605.
[15] T. Hayat, M. Khan and M. Ayub, *Some analytical solutions for second grade fluid flows for cylindrical geometries*, Math. and Comput. Modelling, **43** (2006) 16-29.
[16] T. Hayat, S. Najam, M. Sajid, M. Ayub and S. Mesloub, *On exact solutions for oscillatory flows in a generalized burgers fluid with slip condition*, Z. Naturforsch. **65a** (2010) 381-391.
[17] M. Jamil, A. Rauf, A.A. Zafar and N.A Khan, *New exact analytical solutions for Stokes first problem of Maxwell fluid with fractional derivative approach*, Comput. Math. App, **62** (2011) 1013-1023.
[18] M. Jamil, N.A. Khan and A.A. Zafar, *Translational flows of an Oldroyd-B fluid with fractional derivatives*, Comput. Math. Appl, **62** (2011) 1013-1023.
[19] M. Jamil and N. A. Khan, *Slip effects on fractional viscoelastic fluids*, International Journal of Differential Equations, (2011) Article ID 193813.
[20] H. Junki, H. Guangyu and L. Ciqun, *Analysis of general second order fluid flow in double cylinder rheometer*, Science in China Series A, **40** (1997) 183-190.
[21] M. Khan, *Partial slip effects on the oscillatory flows of a fractional Jeffrey fluid in a porus medium*, J. Porus Media, **10** (2007) 473-488.

[22] Y. Liu, L. Zheng and X. Zhang, *Unsteady MHD Couette flow of a generalized Oldroyd-B fluid with fractional derivative*, Comput. Math. Appl, **61** (2011) 443-450.

[23] A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-Functions: Theory and Applications*, Springer, New York, (2010).

[24] A. Mahmood, C. Fetecau, N.A Khan and M. Jamil, *Some exact solutions of the oscillatory motion of a generalized second grade fluid in an annular region of two cylinders*, Acta Mech Sin. **26** (2010) 541-550.

[25] C. L. M. H Navier and Mem. *Memoire surles du movement des Acad*. Sci. Inst. France **1** (1823) 414.

[26] K. R. Rajagopal,*Longitudinal and torsional oscillations of a rod in a non-Newtonian fluid, Acta Mech*. **49** (1983) 281-285.

[27] K. R. Rajagopoal, *On the creeping flow of the second order fluid*, J. of Non-Newtonian Fluid Mech, **15** (1984) 239-246.

[28] W. C. Tan, W. X. Pan and M. Y Xu, *A note on unsteady flows of a viscoelastic fluid with fractional Maxwell model between two parallel plates*, Int. J. Non-linear Mech, **38** (2003) 645-650.

[29] W. C Tan, F. Xian and L. Wei, *An exact solution of unsteady Couette flow of generalized second grade fluid*, Chinese Science Bulletin, **47** (2002) 1783-1785.

[30] W. C Tan and Xu M.Y, *The impulsive motion of flat plate in a generalized second grade fluid*, Mech. Res. Comm, **29** (2002) 3-9.

[31] W. C Tan and Xu M. Y, *Unsteady flows of a generalized second grade fluid with the fractional derivative model between two parallel plates*, Acta Mech Sin, **20** (2004) 471-476.

[32] D. Tripathi, S.K. Pandey and S. Das, *Peristaltic transport of a generalized Burgers fluid, Application to the movement of chime in small intestine*, Acta Astronautica, **69** (2011) 30-38.

[33] D. Tripathi, P. K. Gupta and S. Das, *Influence of slip condition on peristaltic transport of a viscoelastic fluid with fractional burgers model*, Thermal Science (2010) DOI: 10.2298/TSCI090801037G.

[34] D. Tripathi, *Peristaltic transport of a viscoelastic fluid in a channel*, Acta Astronautica, **68** (2011) 1379-1385.

[35] M. Y. Xu and W Tan, *Theoretical analysis of the velocity field, stress field and vortex sheet of generalized second order fluid with fractional anomalous diffusion*, Sci. China Ser. A, **44** (2001) 1387-1399.

[36] M. Y. Xu and W. C. Tan, *Representation of the constitutive equation of viscoelastic materials by the generalized fractional element networks and its generalized solution*, Sci. China Ser. G. textbf46 (2003) 145-157.

[37] L. Zheng, Y. Liu and X. Zhang, *Slip effects on MHD flow of a generalized Oldroyd-B fluid with fractional derivative*, Nonlinear Anal.: Real World Appl, **13** (2012) 513-523.