Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(10)(2019) pp. 57-63

Non algebraic limit cycles for family of autonomous polynomial planar differential systems

A.Bendjeddou$^{(1)}$ and A.Kina$^{(2)}$
$^{(1,2)}$University of Setif 1, Department of Mathematics,19 000 Algeria,
$^{(1)}$E-mail: bendjeddou@univ-setif.dz
$^{(2)}$E-mail: abdelkrimkina@gmail.com

Received: 15 April, 2019 / Accepted: 06 June, 2016 / Published online: 01 September, 2019

Abstract. We consider a class of autonomous planar polynomial differential systems on the plane, we provide sufficient conditions for the existence of hyperbolic non algebraic limit cycle. Additionally, limit cycle is explicitly given in polar coordinates.

AMS (MOS) Subject Classification Codes: 34C25, 34A34, 34C05.
Key Words: Autonomous polynomial differential system, first integral, limit cycle.

1. Introduction

An important problem in the qualitative theory of differential equations [5] is to determine the limit cycles and its number of a polynomial differential systems of the form:

$$\begin{cases} x' = \frac{dx}{dt} = F(x,y) \\ y' = \frac{dy}{dt} = G(x,y) \end{cases}, \tag{1. 1}$$

where F and G are real polynomial in the variables x and y. We define the degree of system (1.1) by $p = \max\{\deg(F), \deg(G)\}$.

The idea of limit cycles appeared in the works of the mathematician Henri Poincaré ([14],[15]), the statement of the Sixteen Hilbert's problem [9], and the discovery by Liénard [12]. A limit cycle of a planar vector field given by (1. 1) is an isolated periodic trajectory (isolated compact leaf of the corresponding foliation). In other words, a periodic trajectory of a vector field is a limit cycles, if it has annular neighbourhood free from other periodic trajectories, what's more, it is said to be algebraic [11] if it is contained in the zero level set of a polynomial function, see for example [4], [7], [10]. Generally, the orbits of a system (1. 1) comprised in analytical curve which are non algebraic, see for example [3], [2], [6]. To recognize when a limit cycle is algebraic or not, usually, it is not always easy. Thus, the well-known limit cycle of the van der Pol differential system displayed in1926 (see [16]) was not demonstrated until 1995 by Odani [13] that it was non-algebraic. The Van der Pol differential system can be formulated as a systems (1. 1) as is a degree three, but its limit cycle isn't known explicitly. The initial models were explicit non algebraic limit cycles appeared are those of A.Gasull, Giacomini and Torregrosa [6] and J.Giné and M.Grau [8], Al-Dossary [1] for $p = 5$.

In this work, we are concerned in studying the integrability and the limit cycles of a multi-parameter polynomial differential system

$$\begin{cases} x' = x\left(-h + x^4 + y^4\right)\left(ax^2 + ay^2 - 4bxy\right) - \left(x^2 + y^2\right)\left(-x + 4x^2y^3 + 4y^5\right) \\ y' = y\left(-h + x^4 + y^4\right)\left(ax^2 + ay^2 - 4bxy\right) + \left(x^2 + y^2\right)\left(y + 4x^3y^2 + 4x^5\right) \end{cases}. \tag{1. 2}$$

where a, b, h are real constants. We demonstrate the existence of a non-algebraic limit cycle. Additionally this limit cycle is expressly given in polar coordinates. Concrete example exhibiting the applicability of our result are introduced.

We define the trigonometric functions

$$\varphi(\xi) = \frac{3a + 4\sin 4\xi + a\cos 4\xi - 5b\sin 2\xi - b\sin 6\xi}{\cos 4\xi + 3}. \tag{1. 3}$$

$$\phi(\xi) = \frac{(4 - 4ah + 8bh\sin 2\xi)}{\cos 4\xi + 3}. \tag{1. 4}$$

2. Main result

We prove the following result.

Theorem 1. *Consider a multi-parameter polynomial differential system (1. 2)*

Then the following statements hold

1) The system (1. 2) is Darboux integrable with the Liouvillan first integral

$$L(x,y) = \left(x^2 + y^2\right)^2 \exp\left(-\int_0^{\arctan\frac{y}{x}} \varphi(\xi)d\xi\right) - \left(\int_0^{\arctan\frac{y}{x}} \phi(\xi)\exp\left(-\int_0^{\xi}\varphi(w)dw\right)d\xi\right).$$

2) If $3a + 4 + |a| + 6|b| < 0$ and $4 - 4ah - 8|bh| > 0$, then the system (1. 2) has limit cycle which non algebraic explicitly presented in polar coordinates (τ, θ) by:

$$\tau(\theta, \tau_*) = \left(\exp\left(\int_0^{\theta}\varphi(\xi)d\xi\right)\left(\tau_*^4 + \int_0^{\theta}\phi(\xi)\exp\left(-\int_0^{\xi}\varphi(w)dw\right)d\xi\right)\right)^{\frac{1}{4}}, \tag{2. 5}$$

where

$$\tau_* = \left(\frac{\int_0^{2\pi}\phi(\xi)\exp\left(-\int_0^{\xi}\varphi(w)dw\right)d\xi}{\exp\left(-\int_0^{2\pi}\varphi(\xi)d\xi\right) - 1}\right)^{\frac{1}{4}}.$$

Additionally, this limit cycle is unstable hyperbolic limit cycle.

Proof. Firstly we have

$$y'x - x'y = \left((2x)^2 + (2y)^2\right)\left(x^2y^2(x^2 + y^2) + x^6 + y^6\right),$$

hence the point $O(0,0)$ is the unique critical point of the differential system (1. 2) at finite distance.

1) To demonstrate our results we write the differential system (1. 2) in polar coordinates (τ,θ), defined by $x=\tau\cos\theta$, $y=\tau\sin\theta$, then the differential system (1. 2) becomes.

$$\begin{cases} \tau' = \frac{1}{4}\tau^3\left(4-4ha+8bh\sin 2\theta+\left(4\sin 4\theta+3a+a\cos 4\theta-5b\sin 2\theta-b\sin 6\theta\right)\tau^4\right), \\ \theta' = \tau^6\left(\cos 4\theta+3\right). \end{cases} \tag{2. 6}$$

We take an independent variable the coordinate θ, then the differential system (2. 6) becomes

$$\frac{d\tau}{d\theta} = \frac{\left(4\sin 4\theta+3a+a\cos 4\theta-5b\sin 2\theta-b\sin 6\theta\right)}{4\left(\cos 4\theta+3\right)}\tau+\frac{4-4ha+8bh\sin 2\theta}{4\left(\cos 4\theta+3\right)}\frac{1}{\tau^3}, \tag{2. 7}$$

that is a Bernoulli equation. By changing variables $\rho=\tau^4$ we get the linear equation

$$\frac{d\rho}{d\theta} = \frac{4\sin 4\theta+3a+a\cos 4\theta-5b\sin 2\theta-b\sin 6\theta}{\cos 4\theta+3}\rho+\frac{4-4ha+8bh\sin 2\theta}{\cos 4\theta+3}, \tag{2. 8}$$

the general solution of linear differential equation (2. 8) is

$$\rho(\theta)=\exp\left(\int_0^{\theta}\varphi(\xi)ds\right)\left(k+\int_0^{\theta}\phi(\xi)\exp\left(-\int_0^{\xi}\varphi(w)dw\right)d\xi\right), \tag{2. 9}$$

where φ, ϕ two trigonometric functions defined in (1. 3) and (1. 4) respectively. Then

$$\tau(\theta)=\left(\exp\left(\int_0^{\theta}\varphi(\xi)d\xi\right)\left(k+\int_0^{\theta}\frac{\phi(\xi)d\xi}{\exp\left(\int_0^{\xi}\varphi(w)dw\right)}\right)\right)^{\frac{1}{4}}, \tag{2. 10}$$

where $k\in\mathbb{R}$, from these solution we can obtain a first integral in the variables (x,y) of the form

$$L(x,y)=\left(x^2+y^2\right)^2\exp\left(-\int_0^{\arctan\frac{y}{x}}\varphi(\xi)d\xi\right)-\left(\int_0^{\arctan\frac{y}{x}}\frac{\phi(\xi)d\xi}{\exp\left(\int_0^{s}\varphi(w)dw\right)}\right).$$

For the reason that this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and accordingly system 1. 2) is Darbouxt integrable.

The curve $L=l$ with $l\in\mathbb{R}$, which are shaped by trajectories of the differential system (1. 2), in Cartesian coordinates are formulated as

$$\left(x^2+y^2\right)^2=\exp\left(\int_0^{\arctan\frac{y}{x}}\varphi(\xi)d\xi\right)\left(\int_0^{\arctan\frac{y}{x}}\frac{\varphi(\xi)d\xi}{\exp\left(\int_0^{s}\varphi(w)dw\right)}\right)+l\exp\left(\int_0^{\arctan\frac{y}{x}}\varphi(\xi)d\xi\right).$$

Hence, statement (1) is proved.

2- We note that the system (2. 6) has a periodic orbit if and only if equation (2. 7) has a strictly positive 2π periodic solution. This, moreover, is equivalent to the existence of a solution of (2. 7) that satisfies $\tau(0,\tau_*)=\tau(2\pi,\tau_*)$ and $\tau(\theta,\tau_*)>0$ for any $\theta\in[0,2\pi]$.

It is anything but difficult to watch that the solution $\tau(0,\tau_0)=\tau_0$ is

$$\tau(\theta,\tau_0)=\left(\exp\left(\int_0^\theta \varphi(\xi)d\xi\right)\left(\tau_0^4+\int_0^\theta \phi(\xi)\exp\left(-\int_0^\xi \varphi(w)dw\right)d\xi\right)\right)^{\frac{1}{4}}, \tag{2. 11}$$

where $\tau_0(0)=\tau_0>0$. A periodic solution of system (2. 6) must satisfy the condition $\tau(2\pi,\tau_0)=\tau(0,\tau_0)$, which leads to a single value $\tau_0=\tau_*$ given by

$$\tau_*=\left(\frac{\int_0^{2\pi}\phi(\xi)\exp\left(-\int_0^\xi \varphi(w)dw\right)d\xi}{\exp\left(-\int_0^{2\pi}\varphi(\xi)d\xi\right)-1}\right)^{\frac{1}{4}}>0, \tag{2. 12}$$

because $3a+4+|a|+6|b|<0$, $4-4ah-8|bh|>0$ we have

$$\varphi(\xi)=\frac{3a+4\sin 4\xi+a\cos 4\xi-5b\sin 2\xi-b\sin 6\xi}{\cos 4\xi+3}<0,$$

and

$$\phi(\xi)=\frac{(4-4ah+8bh\sin 2\xi)}{\cos 4\xi+3}>0,$$

for all $\xi\in\mathbb{R}$, hence $1-\exp\left(-\int_0^{2\pi}\varphi(\xi)d\xi\right)<0$ and $\phi(\xi)\exp\left(-\int_0^\xi \varphi(w)dw\right)>0$.
After the substitution of the value τ_* in (2. 11), we obtain

$$\begin{aligned}\tau(\theta,\tau_*) &= \left(\left(\frac{\int_0^{2\pi}\phi(\xi)\exp\left(-\int_0^\xi \varphi(w)dw\right)d\xi}{\exp\left(-\int_0^{2\pi}\varphi(\xi)d\xi\right)-1}\right)+\int_0^\theta \frac{\phi(\xi)}{\exp\left(\int_0^\xi \varphi(w)dw\right)}d\xi\right)^{\frac{1}{4}}\\ &\quad\times\left(\exp\left(\int_0^\theta \varphi(\xi)d\xi\right)\right)^{\frac{1}{4}}.\end{aligned}$$

We have $\tau(\theta,\tau_*)>0$, for all $\theta\in[0,2\pi]$, because $3a+4+|a|+6|b|<0$, $4-4ah-8|bh|>0$ hence

$$\frac{\int_0^{2\pi}-\phi(\xi)\exp\left(-\int_0^\xi \varphi(w)dw\right)d\xi}{1-\exp\left(-\int_0^{2\pi}\varphi(\xi)d\xi\right)}>0 \text{ and } \phi(\xi)>0 \text{ for all } \xi\in\mathbb{R}.$$

More precisely, in Cartesian coordinates $\tau^4(\theta,\tau_*)=(x^2+y^2)^2$ and $\theta=\arctan\left(\frac{x}{y}\right)$, the curve equation determined by this limit cycle it is as follows

$$u(x,y)=(x^2+y^2)^2-\exp\left(\int_0^{\arctan\frac{x}{y}}\varphi(\xi)d\xi\right)\left(r_*^4+\int_0^{\arctan\frac{x}{y}}\phi(\xi)\exp\left(-\int_0^\xi \varphi(w)dw\right)d\xi\right)=0.$$

But there is no integer n for which both $\frac{\partial^{(n)} f}{\partial x^n}(x,y)$ and $\frac{\partial^{(n)} f}{\partial y^n}(x,y)$ vanish identically. To be convinced by this fact, one has to compute for example $\frac{\partial u}{\partial x}(x,y)$, that is

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y) \quad &= \quad \frac{y\varphi(\arctan\frac{x}{y})\exp\left(\int_0^{\arctan\frac{x}{y}}\varphi(\xi)d\xi\right)}{x^2+y^2}\left(\tau_*^4+\int_0^{\arctan\frac{x}{y}}\phi(\xi)\exp\left(-\int_0^{\xi}\varphi(w)dw\right)d\xi\right)\\ &+\exp\left(\int_0^{\arctan\frac{x}{y}}\varphi(\xi)d\xi\right)\left(\frac{y\phi(\arctan\frac{x}{y})}{x^2+y^2}\exp\left(-\int_0^{\arctan\frac{x}{y}}\varphi(w)dw\right)\right)+4x(x^2+y^2).\end{aligned}$$

The expression

$$\exp\left(\int_0^{\arctan\frac{x}{y}}\varphi(\xi)d\xi\right)\left(\tau_*^4+\int_0^{\arctan\frac{x}{y}}\phi(\xi)\exp\left(-\int_0^{\xi}\varphi(w)dw\right)d\xi\right),$$

already exists in $u(x,y)$ and still reappears when partial derivatives of arbitrary order are performed, which means that $u(x,y)$ is not a polynomial and that this limit cycle are non algebraic.

So as to demonstrate the hyperbolicity of the limit cycle notice that the Poincaré return map $\tau_0 \to \prod(2\pi,\tau_0) = r(2\pi,\tau_0)$, for more details see ([5]). We compute
$\frac{d\prod}{d\tau_0}(2\pi,\tau_0)\Big|_{\tau_0=\tau_*} = \exp\left(\int_0^{2\pi}\varphi(\xi)d\xi\right) < 1,$
because $\int_0^{2\pi}\varphi(\xi)d\xi < 0$ for all ξ in $[0,2\pi]$.

Accordingly the limit cycle of the Bernoulli equation (2. 7) is stable and hyperbolic, for more details see ([5]). This finishes the confirmation of articulation (2) of theorem (1). □

3. Examples

Example 1. *If we take $a=-6$ and $h=b=1$, then system (1. 2) reads*

$$\left\{\begin{array}{l} x' = x\left(-1+x^4+y^4\right)\left(-6x^2-6y^2-4xy\right)-\left(x^2+y^2\right)\left(-x+4x^2y^3+4y^5\right), \\ y' = y\left(-1+x^4+y^4\right)\left(-6x^2-6y^2-4xy\right)+\left(x^2+y^2\right)\left(y+4x^3y^2+4x^5\right). \end{array}\right. \tag{3. 13}$$

The system (3. 13) possess a limit cycle which non algebraic whose articulation in polar coordinates is (τ,θ)

$$\begin{aligned}\tau(\theta,\tau_*) \quad &= \quad \left(\tau_*^4+\int_0^{\theta}\frac{28+8\sin 2\xi}{\cos 4\xi+3}\exp\left(-\int_0^{s}\frac{-18+4\sin 4w-6\cos 4w-5\sin 2w-\sin 6w}{\cos 4w+3}dw\right)d\xi\right)^{\frac{1}{4}}\\ &\times\exp\left(\frac{1}{4}\int_0^{\theta}\frac{-18+4\sin 4\xi-6\cos 4\xi-5\sin 2\xi-\sin 6\xi}{\cos 4\xi+3}d\xi\right),\end{aligned}$$

where $\theta\in\mathbb{R}$, what's more the intersection of the limit cycle with the horizontal axis is

$$\tau_* = \left(\frac{\int_0^{2\pi}-\frac{28+8\sin 2\xi}{\cos 4\xi+3}\exp\left(-\int_0^{\xi}\frac{-18+4\sin 4w-6\cos 4w-5\sin 2w-\sin 6w}{\cos 4w+3}dw\right)d\xi}{1-\exp\left(-\int_0^{2\pi}\frac{-18+4\sin 4\xi-6\cos 4\xi-5\sin 2\xi-\sin 6\xi}{\cos 4\xi+3}d\xi\right)}\right)^{\frac{1}{4}} \simeq 2.14484.$$

FIGURE 1. The phase portrait in the Poincar disc of the system (3. 13)

Example 2. *If we take* $a = -4, b = 0$ *and* $h = 1$*, then system (1. 2) reads*

$$\begin{cases} x' = x\left(-1 + x^4 + y^4\right)\left(-4x^2 - 4y^2\right) - \left(x^2 + y^2\right)\left(-x + 4x^2y^3 + 4y^5\right), \\ y' = y\left(-1 + x^4 + y^4\right)\left(-4x^2 - 4y^2\right) + \left(x^2 + y^2\right)\left(y + 4x^3y^2 + 4x^5\right). \end{cases} \tag{3. 14}$$

FIGURE 2. The phase portrait in the Poincar disc of the system (3. 14)

The system (3. 14) possess a limit cycle which non algebraic whose articulation in polar coordinates is (τ, θ)

$$\begin{aligned} \tau(\theta, \tau_*) \quad = \quad & \left(\tau_*^4 + \int_0^\theta \frac{20}{\cos 4\xi + 3} \exp\left(-\int_0^\xi \frac{-12 + 4\sin 4w - 4\cos 4w}{\cos 4w + 3} dw\right) d\xi\right)^{\frac{1}{4}} \\ & \times \exp\left(\frac{1}{4}\int_0^\theta \frac{-12 + 4\sin 4\xi - 4\cos 4\xi}{\cos 4\xi + 3} d\xi\right), \end{aligned}$$

where $\theta \in \mathbb{R}$, what's more the intersection of the limit cycle with the horizontal axis is

$$\tau_* = \left(\frac{\int_0^{2\pi} -\frac{20}{\cos 4\xi + 3} \exp\left(-\int_0^\xi \frac{-12+4\sin 4w - 4\cos 4w}{\cos 4w + 3} dw\right) d\xi}{1 - \exp\left(-\int_0^{2\pi} \frac{-12+4\sin 4\xi - 4\cos 4s}{\cos 4\xi + 3} d\xi\right)}\right)^{\frac{1}{4}} \simeq 1.99214.$$

REFERENCES

[1] K.T. Al-Dosary, *Non-algebraic limit cycles for parametrized planar polynomial systems*, Int. J. Math. **18**, No.2 (2007) 179-189.

[2] R. Benterki, J. Llibre, *Polynomial differential systems with explicit non-algebraic limit cycles*, Elect. J. of Diff. Equ. **2012**, No.78 (2012) 1-6.

[3] A.Bendjeddou, R.Boukoucha, *Explicit limit cycles of a cubic polynomial differential systems*, Studia Universitatis Babes-Bolyai, Mathematica. **61**, No.1 (2016) 77-85.

[4] A. Bendjeddou and R. Cheurfa, *Coexistence of algebraic and non-algebraic limit cycle for quintic polynomial differential systems,* Elect.J.of.differ. equ. appl. **2017**, No.71 (2017) 1-7.

[5] F. Dumortier, J. Llibre, J. Artés; *Qualitative Theory of Planar Differential Systems*, (Universitex) Berlin, Springer 2006.

[6] A. Gasull, H. Giacomini and J. Torregrosa, *Explicit non-algebraic limit cycles for polynomial systems,* J. Comput. Appl. Math. **200**, No.1 (2007) 448-457.

[7] J. Gine and J Llibre, I*ntegrability and algebraic limit cycles for polynomial differential systems with homogeneous non-linearities*, J. Differential Equations. **197**, No.1 (2004) 147 – 161.

[8] J. Giné and M. Grau, *Coexistence of algebraic and non-algebraic limit cycles explicitly given*, using Riccati equations. Nonlinearity. **19**, No.8 (2006) 1939-1950.

[9] D. Hilbert, *Mathematical Problems*, lecture delivered before the International Congress of Mathematicians at Paris (MW Newson, trans.), Bulletin of the American Mathematical Society. **8**, No.10 (1902) 437-479.

[10] J. Llibre and Y. Zhao, *Algebraic Limit Cycles in Polynomial Systems of Differential Equations*, J. Phys. A: Math. Theor. **40**, No.47 (2007) 14207-14222.

[11] J.Llibre and G. Rodrıguez, *Configurations of limit cycles and planar polynomial vector fields*, Journal of Differential Equations. **198**, No.2 (2004) 374-380.

[12] A. Liénard, *Etude des oscillations entrenues*, Revue Génerale de l'Electricité. **23**, No.21 (1928) 946–954.

[13] K. Odani, *The limit cycle of the van der Pol equation is not algebraic*, J. of Diff. Equ. **115**, No.1 (1995) 146-152.

[14] H. Poincare, *Mémoire sur les courbes définies par une équation différentielle*, J.Maths.Pures Appl. **7**, (1881) 375–422.

[15] H. Poincare, Mém*oire sur les courbes définies par une équation différentielle*, J. Maths.Pures Appl. **3**, No.8 (1882) 251–296.

[16] B. van der Pol; O*n rel axation -oscil lation s*, The London, Edinburgh and Dublin Phil. Mag. & J. of Sci. **7** No.2, (1926) 978–992.