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Some Hermite-Hadamard type integral inequalities whose n-times differentiable functions are s-logarithmically convex functions

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Abstract. In this paper, the authors have tried to prove some new results of Hermite-Hadamard type integral inequality for n-times differentiable s-logarithmically convex functions and as a consequences the authors have concluded some well-known inequalities for such type of the functions.

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1. Introduction

Inequalities and theory of convex functions have a great dependency on each other. This relationship is the main sanity behind the vast literature published using convex functions.

The following double inequality holds:

$$\omega\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\omega(t)dt \leq \frac{\omega(\alpha)+\omega(\beta)}{2}, \tag{1. 1}$$

for convex functions $\omega : J \subseteq \mathbb{R} \to \mathbb{R}$, know as the Hermite-Hadamard inequality. The inequality (1. 1) holds in reverse direction if ω is a concave function. A number of the papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications and the references cited therein [5]-[8], [10]-[13], [15]-[22], [25]-[31] and [36].

Recently, several authors have worked on the generalization of classical inequalities through different mathematical approaches. One of the most popular and useful way is the use of s-convex functions. Dragomir et al. [9] derived Hermite-Hadamard type inequalities by s-convex function in second sense. Xi et al. [33] considered a new extension and produced Hermite-Hadamard inequalities with the help of (s, m)-convex functions. Jiang et al. [35] proved generalization of Hermite-Hadamard integral inequalities for a class of n-times differentiable functions via (s, m)-convex functions in second sense. Latif et al. studied Hermite-Hadamard type integral inequality for n-times differentiable functions s-logarithmically and (α,m)-logarithmically convex functions for more detail see, [23, 24]. Zafar et al. [38] and Zhang et al. [34] made significant contributions and have produced some Hermite-Hadamard types inequalities for (ρ,m)-geometrically and s-geometrically convex functions. The role of fractional integral can be found as one of the best ways to generalize the classical inequalities. Al-Mdallal et al. [3] proposed algorithm is a spectral Galerkin method based on fractional-order Legendre functions. For more information about fractional integral and fractional differentiable equation see [1, 2, 4, 32].

In this papers, we have established some new Hermite - Hadamard type inequalities for n-times differentiable s-logarithmically convex functions. We have divided the paper in three main sections. This section is for the literature review. In the second section, we dicuss some relevant definitions from the available literature. In the third section, we have given the proofs of our main results and as a consequence we have concluded some well-known inequalities for such type of the functions.

2. Preliminaries

Many mathematicians are trying to generalize the classical convexity in a number of ways defined by: A function $\omega : J \to \mathbb{R}$ is called convex on J, if

$$\omega(t\alpha + (1-t)\beta) \leq t\omega(\alpha) + (1-t)\omega(\beta), \tag{2. 2}$$

holds for $\alpha, \beta \in J$ and $0 \leq t \leq 1$. The inequality (2.2) holds in reverse direction if ω is a concave function. Hudzik et al. [14] defined the class of functions known as $s-$convex functions in the second sense as:

Definition 2.1. *A function $\omega : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$\omega(t\alpha + (1-t)\beta) \leq t^s\omega(\alpha) + (1-t)^s\omega(\beta), \tag{2. 3}$$

holds for $\alpha, \beta \in [0, \infty)$, $0 \leq t \leq 1$ and $0 < s \leq 1$.

Definition 2.2. [37] *A function $\omega : J \subseteq \mathbb{R} \to (0, \infty)$ is said to be logarithmically convex on J, if*

$$\omega(t\alpha + (1-t)\beta) \leq [\omega(\alpha)]^t[\omega(\beta)]^{1-t}, \tag{2. 4}$$

holds for $\alpha, \beta \in J$ and $0 \leq t \leq 1$. If the inequality (2.4) holds in reverse order, then ω is called logarithmically concave on J.

In [37], Xi et al. defined the cocept of $s-$logarithmically convex functions and derived some Hermite-Hadamard type integral inequalities for such type of functions.

Definition 2.3. [37] *A positive function $\omega : J \subseteq \mathbb{R} \to (0, \infty)$ defined as:*

$$\omega(t\alpha + (1-t)\beta) \leq [\omega(\alpha)]^{t^s}[\omega(\beta)]^{(1-t)^s}$$

is called $s-$logarithmically convex on J for $\alpha, \beta \in J$, $0 \leq t \leq 1$ and $0 < s \leq 1$.

It may be noted that for $s = 1$ Definition 2.3, reduces to Definition 2.2.

3. Main Results

The following Lemma is useful to establish our main results.

Lemma 3.1. [38] *Let ω be a real valued n-times differentiable function on (α, β) such that $w^{(n)}(z)$ is absolutely continuous on $[\alpha, \beta]$; let $\psi(z) : [\alpha, \beta] \to [\alpha, \beta]$ and $\phi(z) : [\alpha, \beta] \to [\alpha, \beta]$ be such that $\psi(z) \le z \le \phi(z)$, , then*

$$\begin{aligned}&(\beta-\alpha)^{n+1}\int_0^1 K_n(z,\lambda)\omega^{(n)}(\lambda\alpha+(1-\lambda)\beta)d\lambda\\&=\int_\alpha^\beta \omega(t)dt-\sum_{\bar{m}=1}^{n}\frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z)+S_{\bar{m}}(z)\right]\qquad \forall z\in[\alpha,\beta],\end{aligned} \tag{3. 5}$$

provided that kernel $K_n : [\alpha, \beta] \times [0, 1] \to \mathbb{R}$ is defined by

$$K_n(z,\lambda):=\begin{cases}\frac{\left(\lambda-\frac{\beta-\phi(z)}{\beta-\alpha}\right)^n}{n!}, & \textit{if} \quad \lambda\in\left[0,\frac{\beta-z}{\beta-\alpha}\right]\\ \frac{\left(\lambda-\frac{\beta-\psi(z)}{\beta-\alpha}\right)^n}{n!}, & \textit{if} \quad \lambda\in\left(\frac{\beta-z}{\beta-\alpha},1\right]\end{cases}$$

Moreover,

$$\begin{aligned}R_{\bar{m}}(z)&:=(\phi(z)-z)^{\bar{m}}+(-1)^{\bar{m}-1}(z-\psi(z))^{\bar{m}},\\ S_{\bar{m}}(z)&:=(\psi(z)-\alpha)^{\bar{m}}\omega^{(\bar{m}-1)}(\alpha)+(-1)^{\bar{m}-1}(\beta-\phi(z))^{\bar{m}}\omega^{(\bar{m}-1)}(\beta).\end{aligned}$$

Theorem 3.2. *Let $\omega : J \subseteq [0, \infty) \to (0, \infty)$ be an $n-$times differentiable function on J^0 and integrable on $[\alpha, \beta]$ for $\alpha, \beta \in J$ and $n \in \mathbb{N}$; If $|\omega^{(n)}|^q$ is s-logarithmically convex on $[\alpha, \beta]$ for $0 < s \le 1$ and $q \ge 1$, then*

$$\begin{aligned}&\left|\int_\alpha^\beta \omega(t)dt-\sum_{\bar{m}=1}^{n}\frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z)+S_{\bar{m}}(z)\right]\right|\\&\le\frac{(\beta-\alpha)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!}M_n(s,q,\eta_n^{s,q},\psi(z),\phi(z)),\end{aligned}$$

provided that: $M_n(s, q, \eta_n^{s,q}, \psi(z), \phi(z))$

$$:=\begin{cases}\left|\omega^{(n)}(\beta)\right|^{sq}G_n(\eta_n^{s,q},q,\psi(z),\phi(z)), & \textit{if} \quad 0<|\omega^{(n)}(\alpha)|,|\omega^{(n)}(\beta)|\le 1,\\ \left|\omega^{(n)}(\alpha)\right|^{q(1-s)}\left|\omega^{(n)}(\beta)\right|^{q}G_n(\eta_n^{s,q},q,\psi(z),\phi(z)), & \textit{if} \quad 1\le|\omega^{(n)}(\alpha)|,\omega^{(n)}(\beta)|,\\ \left|\omega^{(n)}(\beta)\right|^{q}G_n(\eta_n^{s,q},q,\psi(z),\phi(z)), & \textit{if} \quad 0<|\omega^{(n)}(\alpha)|\le 1<|\omega^{(n)}(\beta)|,\\ \left|\omega^{(n)}(\alpha)\right|^{q(1-s)}\left|\omega^{(n)}(\beta)\right|^{sq}G_n(\eta_n^{s,q},q,\psi(z),\phi(z)), & \textit{if} \quad 0<|\omega^{(n)}(\beta)|\le 1<|\omega^{(n)}(\alpha)|.\end{cases}$$

$$G_n(\eta_n^{s,q}, q, \psi(z), \phi(z))$$

$$:= \begin{cases} \left((\beta-\phi(z))^{n+1} + (\phi(z)-z)^{n+1} + (z-\psi(z))^{n+1} + (\psi(z)-\alpha)^{n+1}\right)^{1-\frac{1}{q}} \\ \times\Bigg(\frac{\eta_n^{s,q\,\frac{\beta-\phi(z)}{\beta-\alpha}}}{\ln(\eta_n^{s,q})^{n+1}}\left(n!(-1)^{n+1} + \gamma\left(n+1, \ln\eta_n^{s,q\,\frac{\beta-\phi(z)}{\beta-\alpha}}\right)\right) \\ +\eta_n^{s,q\,\frac{\beta-z}{\beta-\alpha}} \sum_{\bar{m}=1}^{n+1}\left(\frac{(-1)^{\bar{m}-1}\left(\frac{\phi(z)-z}{\beta-\alpha}\right)^{n-\bar{m}+1} + (-1)^{2\bar{m}+1}\left(\frac{z-\psi(z)}{\beta-\alpha}\right)^{n-\bar{m}+1}}{(n-\bar{m}+1)!\ln(\eta_n^{s,q})^{\bar{m}}}\right) \\ +\frac{\eta_n^{s,q\,\frac{\beta-\psi(z)}{\beta-\alpha}}}{\ln(\eta_n^{s,q})^{n+1}}\left(n! + (-1)^{n+1}\gamma\left(n+1, \ln\eta_n^{s,q-\left(\frac{\psi(z)-\alpha}{\beta-\alpha}\right)}\right)\right)\Bigg)^{\frac{1}{q}}, & \textit{for} \quad 0<\eta_n^{s,q}<1 \\ \frac{(\beta-\phi(z))^{n+1}+(\phi(z)-z)^{n+1}+(z-\psi(z))^{n+1}+(\psi(z)-\alpha)^{n+1}}{((n+1)(\beta-\alpha)^{n+1})^{\frac{1}{q}}}, & \textit{for} \quad \eta_n^{s,q}=1 \end{cases}$$

where, $R_{\bar{m}}(z)$ and $S_{\bar{m}}(z)$ are defined as in Lemma 3.1 and the lower incomplete gamma function is defined as:

$$\gamma(\bar{x}, z) = \int_0^z t^{\bar{x}-1}e^{-t}dt; \quad \eta_n^{s,q} = \left|\frac{\omega^{(n)}(\alpha)}{\omega^{(n)}(\beta)}\right|^{sq}$$

Proof. Taking the absolute value on both sides of the equation (3.5). Applications of Hölder inequality and $|\omega^{(n)}|^q$ as an s-logarithmically convex on $[\alpha, \beta]$ yield the following inequalities:

$$\left|\int_\alpha^\beta \omega(t)dt - \sum_{\bar{m}=1}^{n}\frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z) + S_{\bar{m}}(z)\right]\right|$$

$$\leq (\beta-\alpha)^{n+1}\left(\int_0^1 |K_n(z,\lambda)|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 |K_n(z,\lambda)||\omega^{(n)}(\lambda\alpha+(1-\lambda)\beta|^q d\lambda\right)^{\frac{1}{q}}$$

$$\leq (\beta-\alpha)^{n+1}\left(\int_0^1 |K_n(z,\lambda)|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 |K_n(z,\lambda)|\left|\omega^{(n)}(\alpha)\right|^{q\lambda^s}\left|\omega^{(n)}(\beta)\right|^{q(1-\lambda)^s}d\lambda\right)^{\frac{1}{q}},$$

provided that:

$$\int_0^1 |K_n(z,\lambda)|d\lambda = \frac{(\beta-\phi(z))^{n+1} + (\phi(z)-z)^{n+1} + (z-\psi(z))^{n+1} + (\psi(z)-\alpha)^{n+1}}{(\beta-\alpha)^{n+1}(n+1)!}. \tag{3. 6}$$

$$\Rightarrow \int_0^1 |K_n(z,\lambda)|\eta_n^{s,q\lambda}d\lambda = \frac{1}{n!}\left(\frac{\eta_n^{s,q\,\frac{\beta-\phi(z)}{\beta-\alpha}}}{\ln(\eta_n^{s,q\,s,q}{}_n)^{n+1}}\left(n!(-1)^{n+1} + \gamma\left(n+1, \ln\eta_n^{s,q\,\frac{\beta-\phi(z)}{\beta-\alpha}}\right)\right)\right.$$
$$+\,\eta_n^{s,q\,\frac{\beta-z}{\beta-\alpha}}\sum_{\bar{m}=1}^{n+1}\left(\frac{(-1)^{\bar{m}-1}\left(\frac{\psi(z)-z}{\beta-\alpha}\right)^{n-\bar{m}+1} + (-1)^{2\bar{m}+1}\left(\frac{z-\psi(z)}{\beta-\alpha}\right)^{n-\bar{m}+1}}{(n-\bar{m}+1)!\ln(\eta_n^{s,q})^{\bar{m}}}\right)$$
$$\left.+\frac{\eta_n^{s,q\,\frac{\beta-\psi(z)}{\beta-\alpha}}}{\ln(\eta_n^{s,q})^{n+1}}\left(n! + (-1)^{n+1}\gamma\left(n+1, \ln\eta_n^{s,q-\left(\frac{\psi(z)-\alpha}{\beta-\alpha}\right)}\right)\right)\right). \tag{3. 7}$$

$K_n(z,\lambda)$ is defined as in Lemma 3.1. Let $0<\zeta\leq 1\leq \nu$, $0\leq t\leq 1$ and $0<s\leq 1$. Then

$$\zeta^{t^s}\leq \zeta^{st} \quad \text{and} \quad \nu^{t^s}\leq \nu^{st+1-s}. \tag{3. 8}$$

Case 1. Consider, $0<|\omega^{(n)}(\alpha)|,|\omega^{(n)}(\beta)|\leq 1$, then from $(3.7)-(3.8)$, we have

$$\begin{aligned}
&\left|\int_\alpha^\beta \omega(t)dt-\sum_{\bar{m}=1}^{n}\frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z)+S_{\bar{m}}(z)\right]\right| \\
&\leq \int_0^1 |K_n(z,\lambda)|\left|\omega^{(n)}(\alpha)\right|^{q\lambda^s}\left|\omega^{(n)}(\beta)\right|^{q(1-\lambda)^s}d\lambda \\
&\leq \int_0^1 |K_n(z,\lambda)|\left|\omega^{(n)}(\alpha)\right|^{sq\lambda}\left|\omega^{(n)}(\beta)\right|^{sq(1-\lambda)}d\lambda \\
&= \left|\omega^{(n)}(\beta)\right|^{sq}\int_0^1 |K_n(z,\lambda)|\eta_n^{s,q\lambda}d\lambda \\
&= \left|\omega^{(n)}(\beta)\right|^{sq} G_n(\eta_n^{s,q},q,\psi(z),\phi(z)),
\end{aligned} \tag{3. 9}$$

Case 2. Consider, $1\leq|\omega^{(n)}(\alpha)|,|\omega^{(n)}(\beta)|$, then from $(3.7)-(3.8)$, we have

$$\begin{aligned}
&\left|\int_\alpha^\beta \omega(t)dt-\sum_{\bar{m}=1}^{n}\frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z)+S_{\bar{m}}(z)\right]\right| \\
&\leq \int_0^1 |K_n(z,\lambda)|\left|\omega^{(n)}(\alpha)\right|^{q\lambda^s}\left|\omega^{(n)}(\beta)\right|^{q(1-\lambda)^s}d\lambda \\
&\leq \int_0^1 |K_n(z,\lambda)|\left|\omega^{(n)}(\alpha)\right|^{q(s\lambda+1-s)}\left|\omega^{(n)}(\beta)\right|^{q(s(1-\lambda)+1-s)}d\lambda \\
&= \left|\omega^{(n)}(\alpha)\right|^{q(1-s)}\left|\omega^{(n)}(\beta)\right|^{q}\int_0^1 |K_n(z,\lambda)|\eta_n^{s,q\lambda}d\lambda \\
&= \left|\omega^{(n)}(\alpha)\right|^{q(1-s)}\left|\omega^{(n)}(\beta)\right|^{q} G_n(\eta_n^{s,q},q,\psi(z),\phi(z)).
\end{aligned} \tag{3. 10}$$

Case 3. Consider, $0<|\omega^{(n)}(\alpha)|\leq 1\leq|\omega^{(n)}(\beta)|$, then from $(3.7)-(3.8)$, we have

$$\begin{aligned}
&\left|\int_\alpha^\beta \omega(t)dt-\sum_{\bar{m}=1}^{n}\frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z)+S_{\bar{m}}(z)\right]\right| \\
&\leq \int_0^1 |K_n(z,\lambda)|\left|\omega^{(n)}(\alpha)\right|^{q\lambda^s}\left|\omega^{(n)}(\beta)\right|^{q(1-\lambda)^s}d\lambda \\
&\leq \int_0^1 |K_n(z,\lambda)|\left|\omega^{(n)}(\alpha)\right|^{qs\lambda}\left|\omega^{(n)}(\beta)\right|^{q(s(1-\lambda)+1-s)}d\lambda \\
&= \left|\omega^{(n)}(\beta)\right|^{q}\int_0^1 |K_n(z,\lambda)|\eta_n^{s,q\lambda}d\lambda \\
&= \left|\omega^{(n)}(\beta)\right|^{q} G_n(\eta_n^{s,q},q,\psi(z),\phi(z)).
\end{aligned} \tag{3. 11}$$

Case 4. Consider, $0 < |\omega^{(n)}(\beta)| \leq 1 \leq |\omega^{(n)}(\alpha)|$, then from $(3.7)-(3.8)$, we have

$$\begin{aligned} &\left|\int_{\alpha}^{\beta} \omega(t)dt - \sum_{\bar{m}=1}^{n} \frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z) + S_{\bar{m}}(z)\right]\right| \\ &\leq \int_0^1 |K_n(z,\lambda)| \left|\omega^{(n)}(\alpha)\right|^{q\lambda^s} \left|\omega^{(n)}(\beta)\right|^{q(1-\lambda)^s} d\lambda \\ &\leq \int_0^1 |K_n(z,\lambda)| \left|\omega^{(n)}(\alpha)\right|^{q(s\lambda+1-s)} \left|\omega^{(n)}(\beta)\right|^{sq(1-\lambda)} d\lambda \qquad (3.\ 12) \\ &= \left|\omega^{(n)}(\alpha)\right|^{q(1-s)} \left|\omega^{(n)}(\beta)\right|^{sq} \int_0^1 |K_n(z,\lambda)| \eta_n^{s,q\lambda} d\lambda \\ &= \left|\omega^{(n)}(\alpha)\right|^{q(1-s)} \left|\omega^{(n)}(\beta)\right|^{sq} G_n(\eta_n^{s,q}, q, \psi(z), \phi(z)). \end{aligned}$$

A combination of inequalities $(3.9)-(3.12)$, yields the desired result. □

Corollary 3.3. *Let the conditions of Theorem 3.2 be satisfied for* $s = 1$, *then the following inequality holds*

$$\begin{aligned} &\left|\int_{\alpha}^{\beta} \omega(t)dt - \sum_{\bar{m}=1}^{n} \frac{1}{\bar{m}!}\left[R_{\bar{m}}(z)\omega^{(\bar{m}-1)}(z) + S_{\bar{m}}(z)\right]\right| \\ &\leq \frac{(\beta-\alpha)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!} M_n(1, q, \eta_n^{s,q}, \psi(z), \phi(z)), \end{aligned}$$

provided that: $M_n(1, q, \eta_n^{s,q}, \psi(z), \phi(z))$

$$:= \begin{cases} \left|\omega^{(n)}(\beta)\right|^q G_n(\eta_n^{s,q}, q, \psi(z), \phi(z)), & \textit{if} \quad 0 < |\omega^{(n)}(\alpha)|, |\omega^{(n)}(\beta)| \leq 1, \\ \left|\omega^{(n)}(\beta)\right|^q G_n(\eta_n^{s,q}, q, \psi(z), \phi(z)), & \textit{if} \quad 1 \leq |\omega^{(n)}(\alpha)|, \omega^{(n)}(\beta)|, \\ \left|\omega^{(n)}(\beta)\right|^q G_n(\eta_n^{s,q}, q, \psi(z), \phi(z)), & \textit{if} \quad 0 < |\omega^{(n)}(\alpha)| \leq 1 < |\omega^{(n)}(\beta)|, \\ \left|\omega^{(n)}(\beta)\right|^q G_n(\eta_n^{s,q}, q, \psi(z), \phi(z)), & \textit{if} \quad 0 < |\omega^{(n)}(\beta)| \leq 1 < |\omega^{(n)}(\alpha)|, \end{cases}$$

Corollary 3.4. *Let the conditions of Theorem 3.2 be satisfied for* $\psi(z) = \bar{r}z + (1-\bar{r})\alpha$ *and* $\phi(z) = \bar{r}z + (1-\bar{r})\beta$, *then the following inequality holds*

$$\begin{aligned} &\left|\int_{\alpha}^{\beta} \omega(t)dt - \sum_{\bar{m}=1}^{n} \frac{1}{\bar{m}!}\Big[\left((1-\bar{r})^{\bar{m}}(\beta-z)^{\bar{m}} + (-1)^{\bar{m}-1}(z-\alpha)^{\bar{m}}\right)\omega^{(\bar{m}-1)}(z)\right. \\ &\left. + r^{\bar{m}}\left((z-\alpha)^{\bar{m}}\omega^{(\bar{m}-1)}(\alpha) + (-1)^{\bar{m}-1}(\beta-z)^{\bar{m}}\omega^{(\bar{(m)}-1)}(\beta)\right)\Big]\right| \\ &\leq \frac{(\beta-\alpha)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!} M_n(s, q, \eta_n^{s,q}, z, \bar{r}), \end{aligned}$$

provided that: $M_n(s, q, \eta_n^{s,q}, z, \bar{r})$

$$:= \begin{cases} \left|\omega^{(n)}(\beta)\right|^{sq} G_n(\eta_n^{s,q}, q, z, \bar{r}), & \textit{if} \quad 0 < |\omega^{(n)}(\alpha)|, |\omega^{(n)}(\beta)| \leq 1, \\ \left|\omega^{(n)}(\alpha)\right|^{q(1-s)} \left|\omega^{(n)}(\beta)\right|^q G_n(\eta_n^{s,q}, q, z, \bar{r}), & \textit{if} \quad 1 \leq |\omega^{(n)}(\alpha)|, \omega^{(n)}(\beta)|, \\ \left|\omega^{(n)}(\beta)\right|^q G_n(\eta_n^{s,q}, q, z, \bar{r}), & \textit{if} \quad 0 < |\omega^{(n)}(\alpha)| \leq 1 < |\omega^{(n)}(\beta)|, \\ \left|\omega^{(n)}(\alpha)\right|^{q(1-s)} \left|\omega^{(n)}(\beta)\right|^{sq} G_n(\eta_n^{s,q}, q, z, \bar{r}), & \textit{if} \quad 0 < |\omega^{(n)}(\beta)| \leq 1 < |\omega^{(n)}(\alpha)|, \end{cases}$$

$G_n(\eta_n^{s,q}, q, z, \bar{r})$

$$:= \begin{cases} \left(\left(\bar{r}^{n+1} + (1-\bar{r}^{n+1})\right)\left((\beta - z)^{n+1} + (z-\alpha)^{n+1}\right)\right)^{1-\frac{1}{q}} \\ \times\left(\frac{\eta_n^{s,q\,\frac{\bar{r}(\beta-z)}{\beta-\alpha}}}{\ln(\eta_n^{s,q})^{n+1}}\left(n!(-1)^{n+1} + \gamma\big(n+1, \ln \eta_n^{s,q\,\frac{\bar{r}(\beta-z)}{\beta-\alpha}}\big)\right) + \eta_n^{s,q\,\frac{\beta-z}{\beta-\alpha}}\right. \\ \times \sum_{\bar{m}=1}^{n+1}\left(\frac{(-1)^{\bar{m}-1}\left(\frac{(1-\bar{r})(\beta-z)}{\beta-\alpha}\right)^{n-\bar{m}+1} + (-1)^{2\bar{m}+1}\left(\frac{(1-\bar{r})(z-\alpha)}{\beta-\alpha}\right)^{n-\bar{m}+1}}{(n-\bar{m}+1)!\ln(\eta_n^{s,q})^{\bar{m}}}\right) \\ \left.+\frac{\eta_n^{s,q\left(1-\frac{\bar{r}(z-\alpha)}{\beta-\alpha}\right)}}{\ln(\eta_n^{s,q})^{n+1}}\left(n! + (-1)^{n+1}\gamma\left(n+1, \ln \eta_n^{s,q - \frac{\bar{r}(z-\alpha)}{\beta-\alpha}}\right)\right)\right)^{\frac{1}{q}}, & \text{for } \quad 0 < \eta_n^{s,q} < 1 \\ \frac{\left(\bar{r}^{n+1} + (1-\bar{r}^{n+1})\right)\left((\beta-z)^{n+1} + (z-\alpha)^{n+1}\right)}{\left((n+1)(\beta-\alpha)^{n+1}\right)^{\frac{1}{q}}}, & \text{for } \quad \eta_n^{s,q} = 1. \end{cases}$$

Corollary 3.5. *Let the conditions of Corollary 3.4 for $n = 1$ be satisfied, then the following inequality holds*

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(t)dt - (1-\bar{r})\omega(z) + \frac{\bar{r}\big((z-\alpha)\omega(\alpha) + (\beta - z)\omega(\beta)\big)}{\beta - \alpha}\right|$$
$$\leq \frac{(\beta-\alpha)^{\frac{2}{q}-1}}{2^{1-\frac{1}{q}}} M_1(\eta_1^{s,q}, q, z, \bar{r}),$$

provided that: $M_1(\eta_1^{s,q}, q, z, \bar{r})$

$$:= \begin{cases} \left|\omega'(\beta)\right|^{sq} G_1(\eta_1^{s,q}, q, z, \bar{r}), & \text{if } \quad 0 < |\omega'(\alpha)|, |\omega'(\beta)| \leq 1, \\ \left|\omega'(\alpha)\right|^{q(1-s)} \left|\omega^{(n)}(\beta)\right|^q G_1(\eta_1^{s,q}, q, z, \bar{r}), & \text{if } \quad 1 \leq |\omega'(\alpha)|, \omega'(\beta)|, \\ \left|\omega'(\beta)\right|^q G_1(\eta_1^{s,q}, q, z, \bar{r}), & \text{if } \quad 0 < |\omega'(\alpha)| \leq 1 < |\omega'(\beta)|, \\ \left|\omega'(\alpha)\right|^{q(1-s)} \left|\omega'(\beta)\right|^{sq} G_1(\eta_1^{s,q}, q, z, \bar{r}), & \text{if } \quad 0 < |\omega'(\beta)| \leq 1 < |\omega'(\alpha)|. \end{cases}$$

$G_1(\eta_1^{s,q}, q, z, \bar{r})$

$$:= \begin{cases} \left(\left(\bar{r}^2 + (1-\bar{r})^2\right)\left((\beta-z)^2 + (z-\alpha)^2\right)\right)^{1-\frac{1}{q}} \\ \times\left(\frac{\eta_1^{s,q\,\frac{\bar{r}(\beta-z)}{\beta-\alpha}}}{\ln(\eta_1^{s,q})^2}\left(1 + \eta_1^{s,q\,\frac{-\bar{r}(\beta-z)}{\beta-\alpha}}\left(\ln \eta_1^{s,q - \frac{\bar{r}(\beta-z)}{\beta-\alpha}} - 1 + \eta_1^{s,q\,\frac{\bar{r}(\beta-z)}{\beta-\alpha}}\right)\right)\right. \\ +\eta_1^{s,q\,\frac{\bar{r}(\beta-z)}{\beta-\alpha}}\left(\frac{(1-\bar{r})(\beta-z)}{(\beta-\alpha)\ln\eta_1^{s,q}} - \frac{(1-\bar{r})(z-\alpha)}{(\beta-\alpha)\ln\eta_1^{s,q}} - \frac{2}{\ln(\eta_1^{s,q})^2}\right) \\ \left.+\frac{\eta_1^{s,q\left(1-\frac{\bar{r}(z-\alpha)}{\beta-\alpha}\right)}}{\ln(\eta_1^{s,q})^2}\left(1 + \eta_1^{s,q\,\frac{\bar{r}(\beta-z)}{\beta-\alpha}}\left(\ln\eta_1^{s,q\,\frac{\bar{r}(z-\alpha)}{\beta-\alpha}} - 1 + \eta_1^{s,q\,\frac{-\bar{r}(z-\alpha)}{\beta-\alpha}}\right)\right)\right)^{\frac{1}{q}}, & \text{for } \quad 0 < \eta_1^{s,q} < 1 \\ \frac{\left(\bar{r}^2 + (1-\bar{r})^2\right)\left((\beta-z)^2 + (z-\alpha)^2\right)}{\left(2(\beta-\alpha)^2\right)^{\frac{1}{q}}}, & \text{for } \quad \eta_1^{s,q} = 1. \end{cases}$$

Remark 3.6. (1) *By setting* $\bar{r} \to 0$ *and* $z \to \frac{\alpha+\beta}{2}$ *in Corollary 3.5, we get the following midpoint type inequality:*

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\omega(t)dt-\omega\left(\frac{\alpha+\beta}{2}\right)\right| \leq \left(\frac{\beta-\alpha}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} M_1\left(s,q,\eta_1^{s,q},\frac{\alpha+\beta}{2},0\right),$$

$$M_1\left(s,q,\eta_1^{s,q},\tfrac{\alpha+\beta}{2},0\right)$$

$$:=\begin{cases}\left|\omega^{'}(\beta)\right|^{sq} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},0\right), & \textit{if} \quad 0<|\omega^{'}(\alpha)|,|\omega^{'}(\beta)|\leq 1,\\ \left|\omega^{'}(\alpha)\right|^{q(1-s)}\left|\omega^{(n)}(\beta)\right|^{q} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},0\right), & \textit{if} \quad 1\leq|\omega^{'}(\alpha)|,\omega^{'}(\beta)|,\\ \left|\omega^{'}(\beta)\right|^{q} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},0\right) & \textit{if} \quad 0<|\omega^{'}(\alpha)|\leq 1<|\omega^{'}(\beta)|,\\ \left|\omega^{'}(\alpha)\right|^{q(1-s)}\left|\omega^{'}(\beta)\right|^{sq} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},0\right), & \textit{if} \quad 0<|\omega^{'}(\beta)|\leq 1<|\omega^{'}(\alpha)|,\end{cases}$$

$$G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},0\right):=\begin{cases}\left(\frac{2+2\eta-4\eta^{\frac{1}{2}}}{\ln(\eta_1^{s,q})^2}\right)^{\frac{1}{q}} & \textit{for} \quad 0<\eta_1^{s,q}<1\\ \left(\frac{1}{2}\right)^{\frac{1}{q}}, & \textit{for} \quad \eta_1^{s,q}=1.\end{cases}$$

(2) *By setting* $\bar{r} \to 1$ *and* $z \to \frac{\alpha+\beta}{2}$ *in Corollary 3.5, we get the following trapezoidal type inequality:*

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\omega(t)dt+\frac{\omega(\alpha)+\omega(\beta)}{2}\right| \leq \left(\frac{\beta-\alpha}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} M_1\left(s,q,\eta_1^{s,q},\frac{\alpha+\beta}{2},1\right),$$

$$M_1\left(s,q,\eta_1^{s,q},\tfrac{\alpha+\beta}{2},1\right)$$

$$:=\begin{cases}\left|\omega^{'}(\beta)\right|^{sq} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},1\right), & \textit{if} \quad 0<|\omega^{'}(\alpha)|,|\omega^{'}(\beta)|\leq 1,\\ \left|\omega^{'}(\alpha)\right|^{q(1-s)}\left|\omega^{(n)}(\beta)\right|^{q} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},1\right), & \textit{if} \quad 1\leq|\omega^{'}(\alpha)|,\omega^{'}(\beta)|,\\ \left|\omega^{'}(\beta)\right|^{q} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},1\right) & \textit{if} \quad 0<|\omega^{'}(\alpha)|\leq 1<|\omega^{'}(\beta)|,\\ \left|\omega^{'}(\alpha)\right|^{q(1-s)}\left|\omega^{'}(\beta)\right|^{sq} G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},1\right), & \textit{if} \quad 0<|\omega^{'}(\beta)|\leq 1<|\omega^{'}(\alpha)|.\end{cases}$$

$$G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},1\right):=\begin{cases}\left(\frac{(\eta-1)\ln\eta_1^{s,q}-2(\eta^{\frac{1}{2}}-1)^2}{\ln(\eta_1^{s,q})^2}\right)^{\frac{1}{q}}, & \textit{for} \quad 0<\eta_1^{s,q}<1\\ \left(\frac{1}{2}\right)^{\frac{1}{q}}, & \textit{for} \quad \eta_1^{s,q}=1.\end{cases}$$

(3) *By setting* $\bar{r} \to \frac{1}{2}$ *and* $z \to \frac{\alpha+\beta}{2}$ *in Corollary 3.5, we get the following average of midpoint and trapezoid type inequality:*

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\omega(t)dt-\frac{1}{4}\left[\omega(\alpha)+2\omega\left(\frac{\alpha+\beta}{2}\right)+\omega(\beta)\right]\right|$$
$$\leq\left(\frac{\beta-\alpha}{4}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} M_1\left(s,q,\eta_1^{s,q},\frac{\alpha+\beta}{2},\frac{1}{2}\right),$$

$$M_1\Big(s,q,\eta_1^{s,q},\tfrac{\alpha+\beta}{2},\tfrac{1}{2}\Big)$$

$$:=\begin{cases} \left|\omega'(\beta)\right|^{sq} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{2}\Big), & \textit{if} \quad 0<|\omega'(\alpha)|,|\omega'(\beta)|\le 1,\\ \left|\omega'(\alpha)\right|^{q(1-s)} \left|\omega^{(n)}(\beta)\right|^{q} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{2}\Big), & \textit{if} \quad 1\le|\omega'(\alpha)|,\omega'(\beta)|,\\ \left|\omega'(\beta)\right|^{q} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{2}\Big) & \textit{if} \quad 0<|\omega'(\alpha)|\le 1<|\omega'(\beta)|,\\ \left|\omega'(\alpha)\right|^{q(1-s)} \left|\omega'(\beta)\right|^{sq} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{2}\Big), & \textit{if} \quad 0<|\omega'(\beta)|\le 1<|\omega'(\alpha)|, \end{cases}$$

$$G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},\frac{1}{2}\right):=\begin{cases} \left(\frac{-4\eta+8\eta^{\frac{3}{4}}-8\eta_1^{s,q\,\frac{1}{2}}+8\eta_1^{s,q\,\frac{1}{4}}+4(\eta_1^{s,q}-1)\ln\eta_1^{s,q\,\frac{1}{4}}-4}{\ln(\eta_1^{s,q})^2}\right)^{\frac{1}{q}}, & \textit{for} \quad 0<\eta<1,\\ \left(\frac{1}{2}\right)^{\frac{1}{q}}, & \textit{for} \quad \eta=1. \end{cases}$$

(4) *By setting $\bar{r}\to\frac{1}{3}$ and $z\to\frac{\alpha+\beta}{2}$ in Corollary 3.5, we get the following Simpson type inequality:*

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(t)dt-\frac{1}{6}\left[\omega(\alpha)+4\omega\left(\frac{\alpha+\beta}{2}\right)+\omega(\beta)\right]\right|$$
$$\le\frac{5(\beta-\alpha)}{18}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} M_1\left(s,q,\eta_1^{s,q},\frac{\alpha+\beta}{2},\frac{1}{3}\right),$$

$$M_1\Big(s,q,\eta,\tfrac{\alpha+\beta}{2},\tfrac{1}{2}\Big)$$

$$:=\begin{cases} \left|\omega'(\beta)\right|^{sq} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{3}\Big), & \textit{if} \quad 0<|\omega'(\alpha)|,|\omega'(\beta)|\le 1,\\ \left|\omega'(\alpha)\right|^{q(1-s)} \left|\omega^{(n)}(\beta)\right|^{q} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{3}\Big), & \textit{if} \quad 1\le|\omega'(\alpha)|,\omega'(\beta)|,\\ \left|\omega'(\beta)\right|^{q} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{3}\Big) & \textit{if} \quad 0<|\omega'(\alpha)|\le 1<|\omega'(\beta)|,\\ \left|\omega'(\alpha)\right|^{q(1-s)} \left|\omega'(\beta)\right|^{sq} G_1\Big(\eta_1^{s,q},q,\tfrac{\alpha+\beta}{2},\tfrac{1}{3}\Big), & \textit{if} \quad 0<|\omega'(\beta)|\le 1<|\omega'(\alpha)|, \end{cases}$$

$$G_1\left(\eta_1^{s,q},q,\frac{\alpha+\beta}{2},\frac{1}{3}\right):=\begin{cases} \left(\frac{9(-2\eta_1^{s,q}+4\eta_1^{s,q\,\frac{5}{6}}-4\eta_1^{s,q\,\frac{1}{2}}+4\eta_1^{s,q\,\frac{1}{6}}+2(\eta_1^{s,q}-1)\ln\eta_1^{s,q\,\frac{1}{6}}-2)}{5\ln(\eta_1^{s,q})^2}\right)^{\frac{1}{q}}, & \textit{for} \quad 0<\eta_1^{s,q}<1,\\ \left(\frac{1}{2}\right)^{\frac{1}{q}}, & \textit{for} \quad \eta_1^{s,q}=1. \end{cases}$$

4. Conclusions

Some new results of Hermite-Hadamard type inequality for n-times differentiable s-logrithamically convex functions have been established. Special cases were also discussed. It is expected that the results of the paper will inspire interested readers.

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