Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(10)(2019) pp. 107-123

A hybrid approach for systems of Volterra integral differential equations

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Received: 27 April, 2019 / Accepted: 26 June, 2019 / Published online: 01 September, 2019

Abstract. In this paper, a hybrid approach consisting of the third order Chebyshev polynomials and block-pulse functions is used for solving systems of Volterra integral differential equations. Applying this approach transforms the system of integral differential equations into a system of algebraic equations. Existence and uniqueness of the solution, for such a system are addressed. Two examples are provided to shows the efficiency and reliability of the utilized approach.

AMS (MOS) Subject Classification Codes : 65R20; 33F05; 41A10
Key Words: Systems of Volterra Integral differential equations, Hybrid methods, Block-pulse functions, Third order Chebyshev polynomials, Operational matrices.

1. Introduction

Mathematical models in some phenomena in engineering, physics, biology, chemistry, and other disciplines, lead to systems of integral differential equations (SIDE) [7][5][6]. During the last years, different orthogonal polynomials are applied to get an approximate solution of such systems [1][2][3][4][12]. Morever, the hybrid methods combined of block-pulse functions with many different polynomials such as Legendre, Bernstein, and Chebyshev polynomials are used to approximate solutions of (SIDE)[8][9][12][10]. One of the advantages of applying a polynomial basis is that our systems transform into a system of algebraic equationns, which its solution is straightforward. In this paper, a numerical ap-proach consists of the third order Chebyshev polynomials and the block-pulse functions as a hybrid approach is used,

$$y_i^{(m)}(\zeta) = f_i(\zeta) + p_i(\zeta, y_1(\zeta), y_1^{'}(\zeta), ..., y_1^{(m)}(\zeta), ..., y_n^{(m)}(\zeta))$$
$$+ \sum_{j=1}^{m1} \int_0^{\zeta} K_{ij}(\zeta, \eta) q_{ij}(y_1(\eta), ..., y_1^{(m)}(\eta), ..., y_n^{(m)}(\eta)) d\eta \qquad (1.\ 1)$$
$$i = 0, 1, 2, ..., n.$$

where $m, m_1$are positive integers, $f_i(\zeta), i = 0, 1, 2, \cdots, n$, are known function, $p_i(\zeta), i = 0, 1, 2, \cdots, n$ are linear or non-linear functions, $k_{ij}(\zeta, \eta) \in L^2([0, 1] \times [0, 1])$ are the kernels, and $y_i(\zeta),\ i = 1, 2, \cdots n$ are unknown functions [13]. This method that is used for solving initial value problems and Fredholm integral equations is also used to solve higher-order initial value problems [12][9]. This paper is organized as follows, in section 2, a hybrid method and its properties are explained. Section 3 is devoted to obtaining operational matrices. In section 4, the hybrid method is applied for approximating the solution of a system of Volterra integral equation. In section 5, the existence and uniqueness of the solutions of systems of Volterra integral differential equations are addressed. Two numerical examples are appeared in section 6. The last section is devoted to discussing the result of this study.

2. The Hybrid method properties

In this section, we review briefly the third order Chebyshev polynomials and the Block-pulse functions, then describe a hybrid method consisting of Block-pulse functions and third order Chebyshev polynomials, and expansions of functions [9].

2.1. The Block-pulse functions. A N-set of Block-pulse functions, $b_i(\zeta),\ i = 1, 2, ..., N$, where N is a positive integer are defined as follows,

$$b_i(\zeta) = \begin{cases} 1, & \frac{(i-1)T}{N} \le \zeta \le \frac{iT}{N}, \\ 0, & otherwise. \end{cases} \quad (2.\ 2)$$

These functions, are disjoint, orthogonal and complete. To get more familiar see a 4-set block-pulse functions in Fiqure 1.

2.2. Third order Chebyshev polynomials. Third order Chebyshev polynomials, $v_i(\zeta),\ i = 1, 2, \cdots, n$ are defined as follows,

$$v_i(\zeta) = \frac{cos(i + \frac{1}{2}(\theta)}{cos\frac{\theta}{2}} \qquad (2.\ 3)$$

where $\zeta = cos\theta$, these polynomials are orthogonal on the interval $[-1, 1]$ with respect to the weight function i.e. $w(\zeta) = \sqrt{\frac{1+\zeta}{1-\zeta}}$

$$\int_{-1}^{1} w(\zeta) v_i(\zeta) v_j(\zeta) d(\zeta) = \sqrt{\pi}\delta_{ij}$$

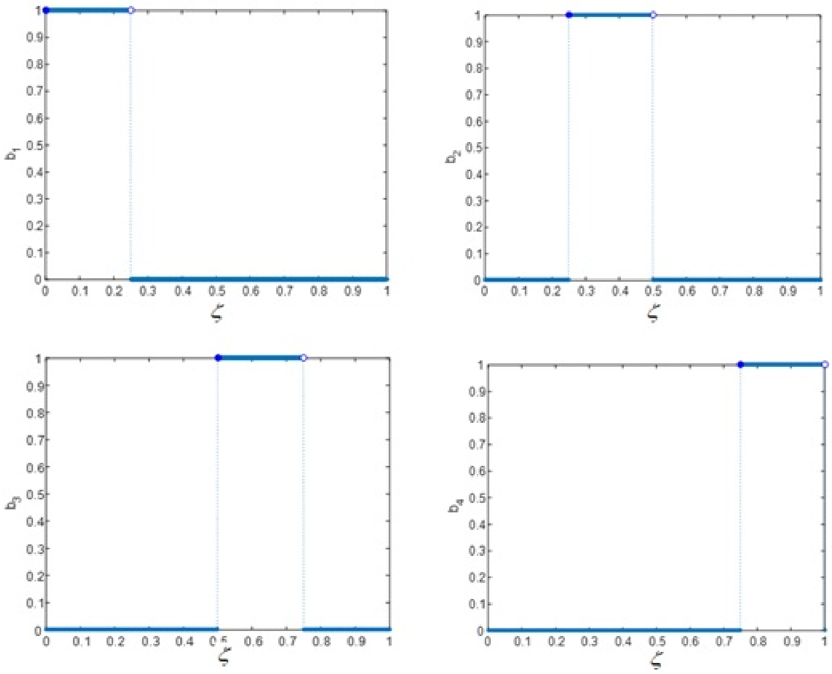

FIGURE 1. The block-pulse functions for $N = 4$.

These polynomials satisfy the following three-terms recurrence relation, that is arguably the most important property of such orthogonal polynomials,

$$v_i(\zeta) = 2_{i-1}(\zeta) - v_{i-2}(\zeta),\ i = 2, 3, \dots .$$
$$v_0(\zeta) = 1,\ v_1(\zeta) = 2\zeta - 1.$$

The shifted third order Chebyshev polynomials, on the interval $[a, b]$, are as following

$$v_i^*(\zeta) = v_i(\frac{2\zeta - a - b}{b - a})$$

These polynomials are orthogonal on$[a, b]$ with the weight function $w(\zeta) = \sqrt{\dfrac{\zeta - a}{b - \zeta}}$.

2.3. **The HBV functions.** The HBV functions on the interval $[0, T]$ are defined as follows,

$$H_{ij}(\zeta) = \begin{cases} \dfrac{2T}{N} v_i(\dfrac{2N\zeta}{T} - 2i + 1), & \dfrac{(i-1)T}{N} \le \zeta < \dfrac{iT}{N}, \\ 0,\ otherwise. \end{cases} \quad (2.\ 4)$$

with the weight function, $w_i(\zeta) = w(2N\zeta - 2i + 1),\ i = 1, 2, ..., N, j = 1, 2, .., M - 1,$ where N and $M$are the orders of the block-pulse function and the third order Chebyshev polynomial, respectively. $H_{ij}(\zeta)$is a combination of the orthogonal third order Chebyshev polynomial and the block-pulse functions, and generates a complete orthogonal system on

$L^{2,\lambda}[0,1)$, that is a suitable Morrey space. For example, see a plot of HBV functions in Figure 2.

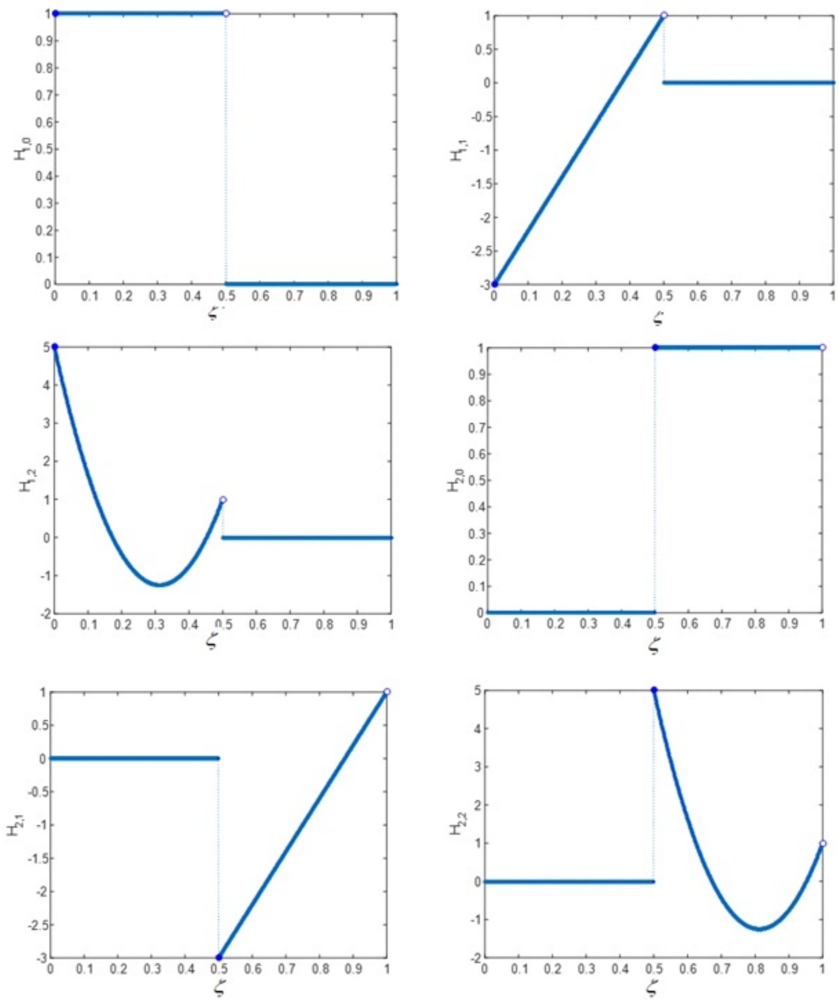

FIGURE 2. The HBV functions for $N = 2$, and $M = 3$.

2.4. **Expansions of functions.** A function $y(\zeta) \in L^2[0,1)$may be expanded as,

$$y(\zeta) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} H_{ij}(\zeta), \tag{2. 5}$$

where,

$$c_{ij} = \frac{(y(\zeta), H_{ij}(\zeta))}{(H_{ij}(\zeta), H_{ij}(\zeta))} = \frac{N^2}{\pi} \int_0^1 w(\zeta) H_{ij}(\zeta) y(\zeta) d\zeta, \tag{2. 6}$$

in which $(.,.)$ denotes an inner product on $L^2 \in [0,1]$ with the weight function $w_i(\zeta)$. In practice, infinite series (2.5) will be truncated into the following form

$$y(\zeta) \cong \sum_{i=1}^{N} \sum_{j=0}^{M-1} c_{ij} H_{ij}(\zeta) = C^T HBV(\zeta),$$

where the vectors C and $HBV(\zeta)$ are as the following,

$$\begin{aligned} C &= [c_{1,0}, ..., c_{1,M-1}, c_{2,0}, ..., c_{2,M-1}, ..., c_{N,0}, ..., c_{N,M-1}]^T \\ HBV(\zeta) &= [H_{1,0}, ..., H_{1,M-1}, H_{2,0}, ..., H_{2,M-1}, ..., H_{N,0}, ..., H_{N,M-1}]^T. \end{aligned} \quad (2.\ 7)$$

The kernel $k(\zeta, \eta) \in L^2[0,1] \times [0,1]$ can be separated,

$$k(\zeta, \eta) \approx HBV^T(\eta) K HBV(\zeta), \quad (2.\ 8)$$

where K is a $NM \times NM$ known matrix with the following entries

$$K_{ij} = \frac{(HBV_i(\zeta), (k(\zeta, \eta), HBV_j(\eta))}{(HBV_i(\zeta), HBV_i(\zeta))(HBV_j(\eta) HBV_j(\eta))}, \; i, j = 1, 2, ..., NM. \quad (2.\ 9)$$

Theorem 2.5. *Let $y(\zeta)$ be a second-order derivative square-integrable function defined on $[0,1]$ with bounded second-order derivative, say $|y^{''}(\zeta)| \leq A$, for some constant A, then*

(i) $y(\zeta)$ can be expanded as an infinite sum of the HBV and the series converges to $y(\zeta)$ uniformly, that is

$$y(\zeta) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} H_{ij}(\zeta),$$

where

$$C_{i,j} = \langle y(\zeta), H_{i,j}(\zeta) \rangle_{L_w^{2,\lambda}[0,1)}.$$

(ii)

$$\beta_{y,i,M} \leq \frac{\pi A^2}{8} \sum_{i=N+1}^{\infty} \sum_{j=M}^{\infty} \frac{1}{i^5 (j-1)^4},$$

where

$$\beta_{y,i,M} = [\int_0^1 |y(\zeta) - \sum_{i=1}^{N} \sum_{j=0}^{M-1} C_{i,j} H_{i,j}(\zeta)|^2 w_n(\zeta) d\zeta]^{\frac{1}{2}}.$$

Proof: see[13].

3. Operational Matrices

In this section, the operational matrices of integration and differentiation will be compute, also the product of two HBV functions, (2.7) will be determined.

3.1. **Operational Matrices.** For the sake of simplicity, computations performed only for $N = 2, M = 3$. The components of $HBV_6(\zeta)$ are as the following,

$$\begin{cases} H_{10} = 1 \\ H_{11} = 8\zeta - 3 \\ H_{12} = 64\zeta^2 - 40\zeta + 5 \end{cases}, 0 \le \zeta < \frac{1}{2}, \begin{cases} H_{20} = 1 \\ H_{21} = 8\zeta - 7 \\ H_{22} = 64\zeta^2 - 104x + 41 \end{cases}, \frac{1}{2} \le \zeta < 1 \quad (3.\ 10)$$

where $HBV_6(\zeta) = [H_{10}, H_{11}, H_{12}, H_{20}, H_{21}, H_{22}]$. Also, by integrating (3.10) and presenting in matrix form, we obtain the following approximations, that are applied for the third kind Chebyshev wavelets[15]. For the present method,

$$\int_0^\zeta H_{10}(t)dt = \begin{cases} \zeta,\ 0 \le \zeta < \frac{1}{2}, \\ \frac{1}{2},\ \frac{1}{2} \le \zeta < 1, \end{cases}$$
$$= \frac{3}{8}H_{10} + \frac{1}{8}H_{11} + \frac{1}{2}H_{20}$$
$$= \left[\frac{3}{8}\ \frac{1}{8}\ 0\ \frac{1}{2}\ 0\ 0\right] HBV_6(\zeta)$$

$$\int_0^\zeta H_{11}(t)dt = \begin{cases} 4\zeta^2 - 3\zeta,\ 0 \le \zeta < \frac{1}{2}, \\ \frac{-1}{2},\ \frac{1}{2} \le \zeta < 1, \end{cases}$$

$$= \frac{-1}{2}H_{10} + \frac{-1}{16}H_{11} + \frac{1}{16}H_{12} + \frac{-1}{2}H_{20}$$

also, we have

$$\int_0^\zeta H_{20}(t)dt = \left[0\ 0\ 0\ \frac{3}{8}\ \frac{1}{8}\ 0\right] HBV_6(\zeta)$$

$$\int_0^\zeta H_{21}(t)dt = \left[0\ 0\ 0\ \frac{-1}{2}\ \frac{-1}{16}\ \frac{1}{16}\right] HBV_6(\zeta)$$

$$\int_0^\zeta H_{22}(t)dt = \left[0\ 0\ 0\ \frac{5}{24}\ \frac{-1}{16}\ \frac{-1}{48}\right] HBV_6(\zeta) + \frac{1}{24}H_{23}(\zeta)$$

These approximations may be written in the matrix form as follows,

$$\int_0^\zeta HBV_6(t)dt = P_{6\times 6}HBV_6(\zeta) + H\tilde{B}V_6(\zeta), \quad (3.\ 11)$$

where,

$$P_{6\times 6}=\frac{1}{4}\begin{bmatrix}\frac{3}{2} & \frac{1}{2} & 0 & 2 & 0 & 0\\ -2 & \frac{-1}{4} & \frac{1}{4} & -2 & 0 & 0\\ \frac{5}{6} & \frac{-1}{4} & \frac{-1}{12} & \frac{2}{3} & 0 & 0\\ 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0 & -2 & \frac{-1}{4} & \frac{1}{4}\\ 0 & 0 & 0 & \frac{5}{6} & \frac{-1}{4} & \frac{-1}{12}\end{bmatrix}$$

and $H\tilde{B}V(\zeta)=\frac{1}{24}(0\ 0\ H_{13}(\zeta)\ 0\ 0\ H_{23}(\zeta))^T$. In fact, the matrix $P_{6\times 6}$ can be written as,

$$P_{6\times 6}=\frac{1}{4}\begin{bmatrix}L_{3\times 3} & J_{3\times 3}\\ 0_{3\times 3} & L_{3\times 3}\end{bmatrix}$$

where,

$$L_{3\times 3}=\frac{1}{4}\begin{bmatrix}\frac{3}{2} & \frac{1}{2} & 0\\ -2 & \frac{-1}{4} & \frac{1}{4}\\ \frac{5}{6} & \frac{-1}{4} & \frac{-1}{12}\end{bmatrix}, J_{3\times 3}=\frac{1}{4}\begin{bmatrix}2 & 0 & 0\\ -2 & 0 & 0\\ \frac{2}{3} & 0 & 0\end{bmatrix}$$

for $M\geq 4$,

$$P=\frac{1}{N^2}\begin{bmatrix}L & J & \cdots & J\\ 0 & L & \cdots & J\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & L\end{bmatrix}, \tag{3. 12}$$

where J and L are two $M\times M$ matrices as the following,
if M is even,

$$J=\begin{bmatrix}\tau_1 & 0 & \cdots & 0\\ -\tau_1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \tau_{\frac{M}{2}} & 0 & \cdots & 0\\ -\tau_{\frac{M}{2}} & 0 & \cdots & 0\end{bmatrix}, \tag{3. 13}$$

where $\tau_i=\frac{2}{2i-1}$, $i=1,2,...,\frac{M}{2}$,
if M is odd,

$$J = \begin{bmatrix} \tau_1 & 0 \cdots 0 \\ -\tau_1 & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ -\tau_{\frac{M+1}{2}-1} & 0 \cdots 0 \\ -\tau_{\frac{M+1}{2}-1} & 0 \cdots 0 \\ \tau_{\frac{M+1}{2}} & 0 \cdots 0 \end{bmatrix}, \qquad (3.\ 14)$$

where $\tau_i = \frac{2}{2i-1},\ i = 1, 2, ..., \frac{M+1}{2}$, and

$$H\tilde{B}V(\zeta) = \frac{1}{N^2}(\lambda_1\ \lambda_2\ \lambda_3\ ...\ \lambda_N)^T \qquad (3.\ 15)$$

where

$$\lambda_i = \frac{1}{2M}(0\ 0\ 0\ ...\ 0\ H_{iM}),\ i = 1, 2, ..., N. \qquad (3.\ 16)$$

$$L = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ -2 & \frac{-1}{4} & \frac{1}{4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{M-2}\frac{2M-3}{(M-1)(M-2)} & 0 & 0 & \cdots & \frac{-1}{2(M-1)(M-2)} & \frac{1}{2(M-1)} \\ (-1)^{M-1}\frac{2M-1}{M(M-1)} & 0 & 0 & \cdots & \frac{-1}{2(M-1)} & \frac{1}{-2M(M-1)} \end{bmatrix}. \qquad (3.\ 17)$$

In general, the integration of the vector $HBV(\zeta)$, defined in (2.7), can be presented as follows,

$$\int_0^{\zeta} HBV(t)dt = PHBV(\zeta) + H\tilde{B}V(\zeta) \qquad (3.\ 18)$$

3.2. **Operational matrix of derivative.** The derivative of the vector $HBV(\zeta)$, may be expressed as the following

$$\frac{d}{dx}(HBV(\zeta)) = DHBV(\zeta), \qquad (3.\ 19)$$

where D is the $NM \times NM$ matix of the derivative as the following [13],

$$D = \begin{bmatrix} d & o & \dots & 0 \\ 0 & d & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d \end{bmatrix}$$

and $d = [\alpha_{ij}]_{M\times M}$, whose entries are as the following,

$$\alpha_{ij} = \begin{cases} 2(i+j-1),\ i>j,\ (i+j)\ odd, \\ 2(i-j),\ i>j,\ (i+j)\ even, \\ 0,\ otherwise. \end{cases} \tag{3. 20}$$

As an example, the matrix d for $M = 5$ is as follows,

$$d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 \\ 8 & 4 & 12 & 0 & 0 \\ 8 & 12 & 4 & 16 & 0 \end{bmatrix}$$

3.3. **Product of HBV functions.** Let us define the product of two vectors $HBV(\zeta)$ which is useful in whatever coming up.

$$\int_0^\zeta HBV^T(t)HBV(t)Cdt = \tilde{C}HBV(\zeta) + H\tilde{B}V(\zeta), \tag{3. 21}$$

where $\tilde{C}$ is the product operational matrix, and $H\tilde{B}V(\zeta)$is introduced in (3.15). For $N = 2$ and $M = 3$, we have

$$\tilde{C} = \begin{bmatrix} \gamma_{10} & \gamma_{11} & \gamma_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_{11} & \gamma_{10}-\gamma_{11}+\gamma_{12} & \gamma_{11}-\gamma_{12} & \gamma_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_{12} & \gamma_{11}-\gamma_{12} & \gamma_{10}-\gamma_{11}+\gamma_{12} & \gamma_{11}-\gamma_{12} & \gamma_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{20} & \gamma_{21} & \gamma_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{21} & \gamma_{20}-\gamma_{21}+\gamma_{22} & \gamma_{21}-\gamma_{22} & \gamma_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{22} & \gamma_{21}-\gamma_{22} & \gamma_{20}-\gamma_{21}+\gamma_{22} & \gamma_{21}-\gamma_{22} & \gamma_{22} \end{bmatrix}$$

So, the matrix $\tilde{C}_{6\times 6}$can be written as,

$$\tilde{C} = \begin{bmatrix} \beta_1 & \vartheta_1 & 0 \\ 0 & \beta_2 & \vartheta_2 \end{bmatrix},$$

where

$$\beta_i = \begin{bmatrix} \gamma_{i0} & \gamma_{i1} & \gamma_{i2} \\ \gamma_{i1} & \gamma_{i0}-\gamma_{i1}+\gamma_{i2} & \\ \gamma_{i2} & \gamma_{i1}-\gamma_{i2} & \gamma_{i0}-\gamma_{i1}+\gamma_{i2} \end{bmatrix}, \vartheta_i = \begin{bmatrix} 0 & 0 \\ \gamma_{i2} & 0 \\ \gamma_{i1}-\gamma_{i2} & \gamma_{i2} \end{bmatrix}.$$

4. THE SOLUTION OF THE SYSTEM (1.1)

Consider system (1.1), with the following initial conditions,

$$y_i^{(s)}(0) = a_{is},\ i = 1, 2, \cdots, n,\ s = 0, 1, \cdots, m-1. \tag{4. 22}$$

Let us, approximate the functions in (1.1) as the following,

$$\begin{aligned} &f_i(\zeta) \approx F_i^T HBV(\zeta), \\ &p_i(\zeta, y_1(\zeta), ..., y_1^{(m)}(\zeta), ..., y_n^{(m)}(\zeta) \approx P_i^T HBV(\zeta), \\ &k_{ij}(\zeta,\eta) \approx HBV^T(\zeta)K_{ij}HBV(\eta), \\ &y_i^{(m)}(\zeta) \approx C_i^T D^m HBV(\zeta),\ i = 1, 2, ..., n,\ j = 1, 2, ..., m_1. \end{aligned} \tag{4. 23}$$

By substitution of these approximations in (1.1), we have,

$$C_i^T D^m HBV(\zeta) = F_i^T HBV(\zeta) + P_i^T HBV(\zeta) + \sum_{j=1}^{m_1} \int_0^{\zeta} HBV^T(\zeta) K_{ij} HBV(\eta) HBV^T(\eta) Q_{ij} d\eta$$
$$= F_i^T HBV(\zeta) + P_i^T HBV(\zeta) + HBV^T(\zeta) \sum_{j=1}^{m_1} K_{ij} [\tilde{C}_{ij} HBV(\zeta) + H\tilde{B}V(\zeta)] \quad (4.\ 24)$$

Multiplying $w_n(\zeta)HBV^T(\zeta)$ into both sides of system (4.24) and applying $\int_0^1 (.)d\zeta$, the coefficients $C_i, i = 1, 2, ..., n$, will be obtained. Also, the error function $e(y_i(\zeta))$ is constructed as follows,

$$e(y_i(\zeta)) = |y_i(\zeta) - \sum_{i=1}^{N} \sum_{j=0}^{M-1} c_{ij}^T H_{ij}(\zeta)|. \quad (4.\ 25)$$

we set $\zeta = \zeta_j$, where ζ_j are eleven equally spaced collocation points in the interval$[0, 1]$. The error values at this point will be obtained.

5. Existence and uniqueness of solution of the systems of non-linear Volterra integral differential equations

In this section, we are going to show the existence and uniqueness solution of a system of Volterra integral differential equations. To proceed, let us state the following theorem,

Theorem 5.1. *Consider $y_i(\zeta)$ as a continuous and differentiable function on the closed interval $[0, T]$. Also, $f_i(\zeta)$ is continuous on $C[0, T]$, and*

i. $k_{ij}(\zeta, \eta, y(\eta))$ is a continuous function on $\Omega = \{y \in C^k[0, T]; \|y\|_{c^k[0,T]} \leq L\}$ where L, is a positive real number.

ii. $k_{ij}(\zeta, \eta, y(\eta)))$ satisfies the Lipschitz condition, with respect to the third component

$$| K(\zeta, \eta, Y) - K(\zeta, \eta, Z) | < P | Y - Z |$$

where P is a positive real number which doesn't dependent on ζ, η, Y,and Z.

Then the contraction map $W : \Omega \longrightarrow \Omega$, has a fixed point on the sub-space $C^k[0, T]$ and system (1.1) has a unique continuous solution on Ω.

Proof. To make the analysis as simple as possible we assume that

$$\acute{Y}(\zeta) = F(\zeta) + \int_0^{\zeta} K(\zeta, \eta, y(\eta))d\eta, \quad (5.\ 26)$$

where,

$$\acute{Y}(\zeta) = [\acute{y}_1(\zeta), \acute{y}_2(\zeta), ..., \acute{y}_n(\zeta)]^T,$$
$$F(\zeta) = [f_1(\zeta), f_2(\zeta), ..., f_n(\zeta)]^T,$$
$$K(\zeta, \eta, Y(\eta)) = [k_{ij}(\zeta, \eta, y_j(\eta)],\ i, j = 1, 2, ..., n.$$

with initial conditions $y_i(0) = 0,\ i = 1, 2, ..., n.$

By integrating of both sides of system (5.26), we get

$$Y(\zeta) = \int_0^{\zeta} F(\eta)d\eta + \int_0^{\zeta}\int_0^{\zeta} K(\zeta,\eta,Y(\eta))d\eta\, d\eta. \tag{5. 27}$$

According to the continuity of the given functions, on a closed interval, they are bounded, i.e.

$$\|F(\zeta)\| = max|f_i(\zeta)| < F,$$

$$\|K(\zeta,\eta)\| = max_{1\le i\le n}\sum_{j=1}^{n}|k_{ij}(\zeta,\eta)| < K,$$

where K and F are positive real constants. So,

$$\begin{aligned}&\textstyle\int_0^{\zeta}\int_0^{\zeta} K(\zeta,\eta,y(\eta))d\eta d\eta < \int_0^{\zeta}\int_0^{\zeta} K d\eta < \frac{KT^2}{2},\\ &\textstyle\int_0^{\zeta} F(\eta)d\eta < \int_0^{\zeta} F d\eta < FT.\end{aligned} \tag{5. 28}$$

From the definition of $Y_k(\zeta)$ and inequalities (5.28), we can state the following iteration method

$$Y_k(\zeta) = \int_0^{\zeta} F(\eta)d\eta + \int_0^{\zeta}\int_0^{\zeta} K(\zeta,\eta,Y_{k-1}(\eta))d\eta\, d\eta, \tag{5. 29}$$

Therefore,

$$\begin{aligned}\|Y_k(\zeta)\| &= \|\int_0^{\zeta} F(\eta)d\eta + \int_0^{\zeta}\int_0^{\zeta} K(\zeta,\eta,Y_{k-1}(\eta))d\eta\, d\eta\|,\\ &\le \|\int_0^{\zeta} F d\eta\| + \|\int_0^{\zeta} K d\eta d\eta\| \le \frac{KT^2}{2} + FT \le \beta,\end{aligned}$$

where $0 < \beta < T$. These bounds state that

$$\|Y_k(\zeta)\| = \sum_{i=1}^{n}\|Y_i(\zeta) - Y_{i-1}(\zeta)\|$$

is convergent. Now, we consider the series $\sum_{i=0}^{\infty}(Y_k(\zeta) - Y_{k-1}(\zeta))$, given that $\|Y\| \le L$ where L is a positive constant. Now, we can prove the following inequality

$$\|Y_k(\zeta) - Y_{k-1}(\zeta)\| \le \frac{LP^{k-1}(\zeta)^k}{k!}. \tag{5. 30}$$

To proceed, by induction inequality (5.30) holds for the first two values of k,

$$\begin{aligned}\|Y_1(\zeta) - Y_0(\zeta)\| &\le L\\ \|Y_2(\zeta) - Y_1(\zeta)\| &\le \textstyle\int_0^{\zeta}\int_0^{\zeta}\|K(\zeta,\eta,Y_1(\eta) - K(\zeta,\eta,Y_0(\eta))d\eta d\eta\|\\ &\le P\textstyle\int_0^{\zeta}\int_0^{\zeta}\|Y_1(\eta) - Y_0(\eta)\|d\eta d\eta \le LP\frac{\zeta^2}{2!}.\end{aligned}$$

Suppose inequality (5.30) holds for k. i.e.

$$\|Y_k(\zeta)-Y_{k-1}(\zeta)\| \leq LP^{k-1}\frac{\zeta^k}{k!},$$

One can easily show that

$$\|Y_{k+1}(\zeta)-Y_k(\zeta)\| \leq LP^k\frac{\zeta^{k+1}}{k+1!}.$$

Moreover, regarding (5.30), we can conclude that

$$\|\sum_{k=0}^{\infty} Y_{k+1}(\zeta)-Y_k(\zeta)\| \leq L\sum_{k=0}^{\infty} P^k\frac{\zeta^{k+1}}{k+1!} \leq \frac{L}{1-P}e^{\zeta}$$

Therefore, $Y_k(\zeta)$ converges uniformly to the following function,

$$Y(\zeta)=\sum_{k=0}^{\infty}(Y_{k+1}(\zeta)-Y_k(\zeta)), \tag{5. 31}$$

on the interval $[0,T]$.

Now, we can show that the function $Y(\zeta)$ satisfies (5.26). To go forward, for any ε there exists a k, such that

$$\|Y_k(\zeta)-Y(\zeta)\| < \varepsilon$$

therefore,

$$\int_0^{\zeta}\int_0^{\zeta}\|K(\zeta,\eta,Y_k(\eta)-K(\zeta,\eta,Y(\eta)\|d\eta d\eta \leq P\int_0^{\zeta}\int_0^{\zeta}\|Y_k(\eta)-Y(\eta)\|d\eta d\eta \leq P\int_0^{\zeta}\int_0^{\zeta}\varepsilon d\eta d\eta \leq P\varepsilon\frac{\zeta^2}{2}.$$

So, $Y(\zeta)$ is a solution of system (5.26) as$\varepsilon \longrightarrow 0$.

Let us prove the uniqueness of the solution $Y(\zeta)$ on the interval $[0,T]$. We assume that there is another continuous solution, say $Z(\zeta)$, such that $Z(0)=0$. Set $C=max\|Y(\zeta)-Z(\zeta)\|$.

Therefore

$$\|Y(\zeta)-Z(\zeta)\| \leq \int_0^{\zeta}\int_0^{\zeta}K(\zeta,\eta,Y(\eta)-K(\zeta,\eta,Z(\eta))d\eta d\eta \leq P\int_0^{\zeta}\int_0^{\zeta}Y(\eta)-Z(\eta)d\eta d\eta$$
$$\leq P\int_0^{\zeta}\int_0^{\zeta}Cd\eta d\eta \leq CP\frac{\zeta^2}{2!},$$

It will be obtaining for any k,

$$\|Y(\zeta)-Z(\zeta)\| \leq CP^{k-1}\frac{\zeta^k}{k!},$$

obviously $Y(\zeta)=Z(\zeta),\ 0\leq\zeta\leq T$ as $k \longrightarrow \infty$.■

6. NUMERICAL EXAMPLE

For showing the efficiency and reliability of the utilized numerical method, parameters N and M are considered to be 1 and 4, respectively. Let us provide the following examples:

Example 6.1. *In this example we study solution of the following system oF linear Volter integral equations* [14]*:*

$$\begin{cases} y_1^{''}(\zeta) = -\zeta^3 - \zeta^4 + \int_0^{\zeta}(3y_2(\eta) + 4y_2(\eta))d\eta, \ y_1(0) = 0, \ y_1^{'}(0) = 1, \\ y_2^{''}(\zeta) = 2 + \zeta^2 - \zeta^4 + \int_0^{\zeta}(4y_3(\eta) - 2y_1(\eta))d\eta, \ y_2(0) = 0, \ y_2^{'}(0) = 0, \\ y_3^{''}(\zeta) = 6\zeta - \zeta^2 + \zeta^3 + \int_0^{\zeta}(2y_1(\eta) - 3y_2(\eta))d\eta, \ y_3(0) = 0, \ y_3^{'}(0) = 0. \end{cases}$$

The exact solutions are $y_1(\zeta) = \zeta, y_2(\zeta) = \zeta^2$, and $y_3(\zeta) = \zeta^3$. In this example, lets take the following approximations,

$$-\zeta^3 - \zeta^4 = F_1^T HBV(\zeta),$$
$$2 + \zeta^2 - \zeta^4 = F_2^T HBV(\zeta),$$
$$6\zeta - \zeta^2 + \zeta^3 = F_3^T HBV(\zeta)$$
$$y_i(\zeta) = C_i^T HBV^T(x), \ y_i^{''}(\zeta) = C_i^T D^2 HBV^T(x), \ i = 1, 2, 3$$

Results are shown in Table1 and the exact and approximate solutions are present in Figure3

TABLE 1. The exact, approximate, and absolute errors of the solutions, Example1.

x	$y_1 - exact$	$y_1 - estimated$	Error-y_1	$y_2 - exact$	$y_2 - estimated$
0.1	0.1	0.099827	0.00017268	0.01	0.0098361
0.2	0.2	0.1994	0.0006028	0.04	0.03943
0.3	0.3	0.29884	0.0011588	0.09	0.088909
0.4	0.4	0.39829	0.0017086	0.16	0.1584
0.5	0.5	0.49788	0.0021204	0.25	0.24803
0.6	0.6	0.59774	0.0022626	0.36	0.35794
0.7	0.7	0.698	0.0020033	0.49	0.48824
0.8	0.8	0.79879	0.0012107	0.64	0.63906
0.9	0.9	0.90025	0.00024701	0.81	0.81054
1	1	1.0025	0.0025016	1	1.0028

Example 6.2. *In this example, we solve following non-linear system of Volterra integral differential equations* [11]*:*

$$\begin{cases} y_1^{'}(\zeta) = -\zeta^3 - 6\zeta - 1 + y_1(\zeta) + (7 - 2\zeta)y_2(\zeta) + \int_0^{\zeta}((\zeta + \eta)y_1(\eta) + (\eta - \zeta)^3 y_2(\eta)d\eta, \ y_1(0) = 1, \\ y_2^{'}(\zeta) = -3\zeta^2 + \zeta - 6 + (7 - 2\zeta)y_1(\zeta) + y_2(\zeta) + \int_0^{\zeta}((\zeta + \eta)^3 y_1(\eta) + (\eta + \zeta)^3 y_2(\eta)d\eta, \ y_2(0) = 0, \end{cases}$$

With the exact solutions $y_1(\zeta) = cosh(\zeta), \ y_2(\zeta) = sinh(\zeta)$. The numerical results and

$Error - Y_2$	$y_3 - exact$	$y_3 - estimated$	Error-y_3
0.00016386	0.001	0.0010282	$2.8211e - 05$
0.00057019	0.008	0.0082636	0.00026361
0.0010911	0.027	0.027932	0.00093235
0.0015988	0.064	0.066261	0.0022606
0.0019653	0.125	0.12947	0.0044745
0.0020628	0.216	0.2238	0.0078001
0.0017634	0.343	0.35546	0.012464
0.00093923	0.512	0.53069	0.018691
0.00053755	0.729	0.75571	0.026709
0.0027948	1	1.0367	0.036744

absolute errors are appeared in Table2 and the exact and approximate solutions are depicted in Figure4

TABLE 2. The exact, approximate, and absolute errors of the solutions, Example2.

x	$y_1 - exact$	$y_1 - estimated$	*Error*-y_1	$y_2 - exact$	$y_2 - estimated$	*Error*-y_2
0.1	1.005	1.005	$4.689e - 05$	0.10017	0.10012	$4.738e - 05$
0.2	1.0201	1.02	$9.5129e - 05$	0.20134	0.20127	$6.7719e - 05$
0.3	1.0453	1.0453	$8.1691e - 05$	0.30452	0.30463	0.00011266
0.4	1.0811	1.081	$4.5731e - 05$	0.41075	0.4114	0.00064725
0.5	1.1276	1.1275	0.00013111	0.5211	0.52275	0.0016591
0.6	1.1855	1.1849	0.00058998	0.63665	0.63988	0.00323
0.7	1.2552	1.2534	0.0017874	0.75858	0.76397	0.0053896
0.8	1.3374	1.3332	0.0042073	0.88811	0.89621	0.0081038
0.9	1.4331	1.4246	0.0084592	1.0265	1.0378	0.011263
1	1.5431	1.5278	0.015287	1.1752	1.1899	0.014667

7. CONCLUSION

In this paper, the existence and uniqueness of the solutions to a system of Volterra integral differential equation are addressed and a hybrid method employed to solve such systems. The operational matrices of the integration and product were determined based, on the hybrid basis black-pulse functions and third order Chebyshev polynomials which utilized to solve two examples to show the efficiency used method. As one can learn from the Tables, as much as the values of the variable increase, the accuracy decreases. This method is more accurate for linear systems than non-linear ones.

8. ACKNOWLEDGMENTS

Authors would like to express sincere appreciation to the Editor for reading and helpful remarks, and to anonymous referees for their very precise and useful comments that would improve the quality of our research.

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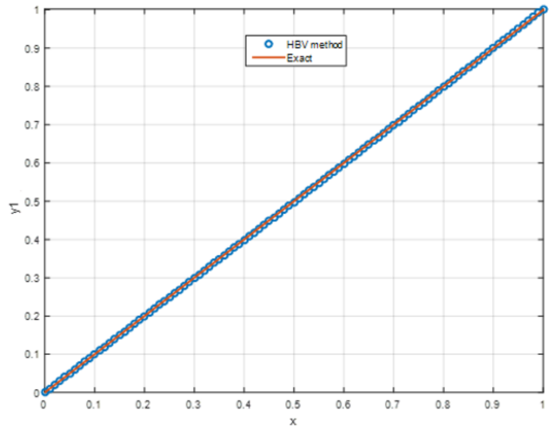

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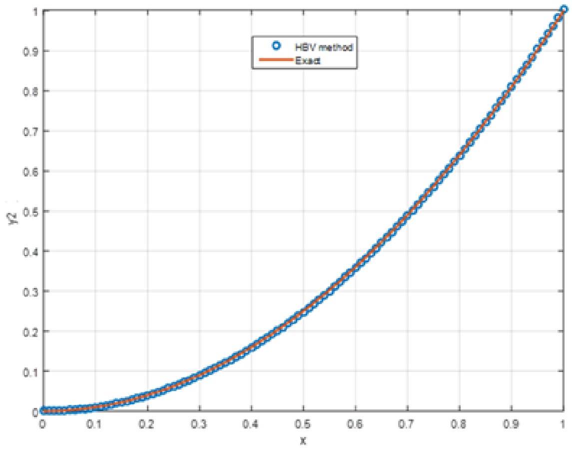

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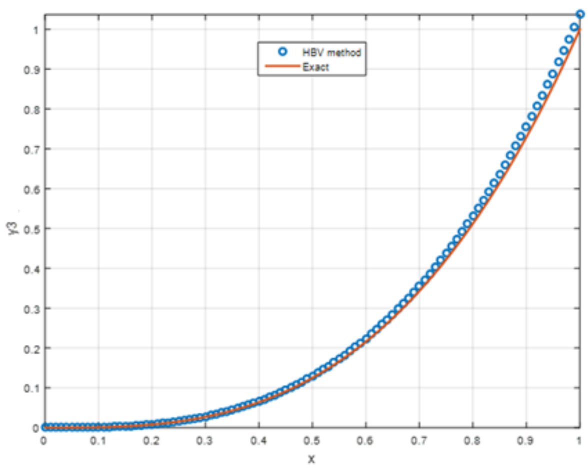
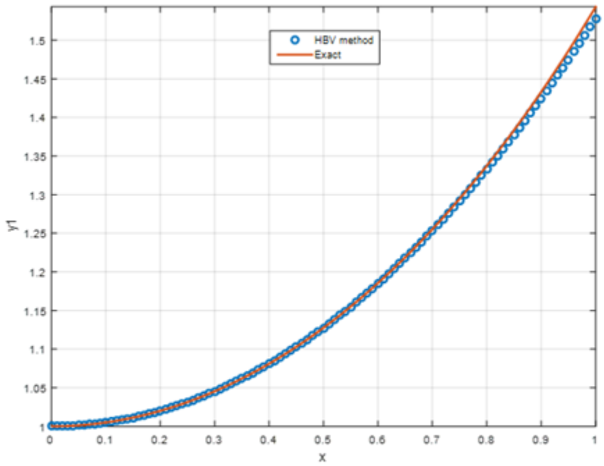


(c)

FIGURE 3. Plots of exact and HBV solutions (a):$y_1(\zeta), (b) : y_2(\zeta), and (c) : y_3(\zeta) of Example 1.$



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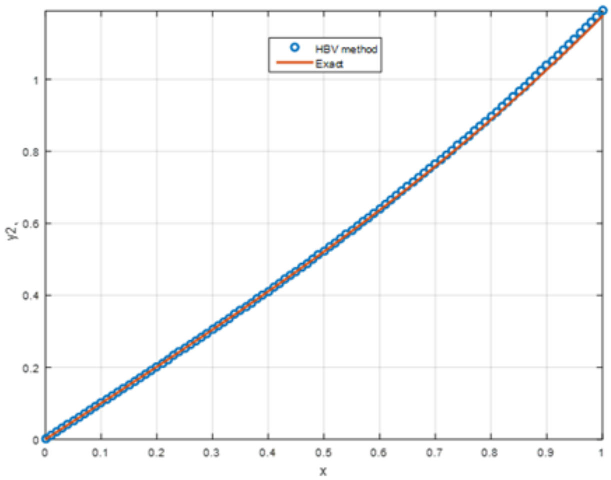

(b)

FIGURE 4. Plots of exact and HBV solutions: (a): $\mathrm{y}_1(x); (b) : y_2(x) of Example 2.$