Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(9)(2019) pp. 33-44

An optimized single-step method for integrating Cauchy problems

Zaibunnisa Memon[a*], Sania Qureshi[a], Bisharat Rasool Memon[b] and Muhammad Saleem Chandio[c]
[a]Department of Basic Sciences & Related Studies, Mehran University of Engineering & Technology, Pakistan
[b]Institute of Information & Communication Technology, University of Sindh, Pakistan
[c]Institute of Mathematics & Computer Science,University of Sindh, Pakistan
*Email: zaib.memon@faculty.muet.edu.pk

Received: 26 October, 2018 / Accepted: 03 April, 2019 / Published online: 01 August, 2019

Abstract. This work involves development of an optimum third order single-step explicit method for Cauchy problems. The proposed method is analyzed for consistency, stability, local and global error bounds, and convergence. Further, numerical investigation is carried out to assess effectiveness of the method in comparison to existing numerical schemes, including Modified Improved Modified Euler (MIME) method, Third order Euler method (TOEM) and classical Runge-Kutta method of order three (RK3). The testing factors are error and CPU time which have been computed using Matlab R2014b. It is observed that the proposed method possesses minimum error bounds; and is also favourable in terms of both accuracy and computational cost.

AMS (MOS) Subject Classification Codes: 65-02; 65L-05; 65L-12; 65L-20; 65L-70
Key Words: Consistency, Stability, Convergence, Error bounds.

1. Introduction

Mathematical modeling of many real world phenomena often leads to ordinary or partial differential equations. Most of these equations are highly nonlinear and the exact solutions are not always possible. Therefore, researchers in engineering, applied mathematics and other scientific fields often resort to numerical techniques which provide approximate solutions. Numerical methods for the solution of nonlinear Cauchy problems are efficient tools that have been gradually developed by many mathematicians since the eighteenth century. Later on, with the invention of high speed digital computers in the late twentieth century, the study of various phenomena in engineering and science has been made much easier and efficient by the use of numerical techniques. These techniques have proved their efficacy in a number of physical problems, for example, chemical kinetics [8, 14, 15], fluid dynamics [11, 25, 30, 32], electrical circuits [5, 13, 27], computer virology [3, 21], image processing [26] and epidemiology [4, 10, 17, 19, 28, 29]. There has been extensive study of single-step explicit methods for the numerical integration of Cauchy problems [1, 2, 7, 12, 18, 20, 22, 23]. This further stimulates the interest in their theoretical and numerical investigation while carrying out computational studies for the real world application problems. In this work, an explicit single-step method of third order is developed for initial value problems of ordinary differential equations also known as Cauchy problems. Although, there exist higher order methods but with an increased number of slope evaluations per step. For example, the fourth order RK method requires four and the fifth order method involves six slope evaluations; thereby increasing the time for numerical simulation. This is the major concern in the present study - to develop a third order method which offers significant accuracy while consuming less CPU time. The proposed method is analyzed for consistency, stability and convergence. We investigate performance of the method on the basis of stability interval, local and global error bounds, accuracy and computational time in comparison to the following methods:

- **Modified Improved Modified Euler (MIME) method** [20]: It is a second order accurate method involving three function evaluations per integration step. The iterative scheme for MIME method is given as:

$$\begin{aligned} u_r &= u_{r-1} + d_3 \\ d_1 &= \Delta t\, g\left(t_{r-1}, u_{r-1}\right) \\ d_2 &= \Delta t\, g\left(t_{r-1}, u_{r-1} + \frac{d_1}{2}\right) \\ d_3 &= \Delta t\, g\left(t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_2}{2}\right) \end{aligned} \tag{1. 1}$$

- **Third Order Euler Method (TOEM)** [1]: It involves three function evaluations per step and is given by the following iterative scheme:

$$\begin{aligned} u_r &= u_{r-1} + d_3 \\ d_1 &= \Delta t\, g\left(t_{r-1}, u_{r-1}\right) \\ d_2 &= \Delta t\, g\left(t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_1}{3}\right) \\ d_3 &= \Delta t\, g\left(t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_2}{2}\right) \end{aligned} \tag{1. 2}$$

- **Classical Runge-Kutta method of third order (RK3)** [24]: Proposed by Runge in 1895, it involves three function evaluations per integration step as given in (1. 3):

$$\begin{aligned} u_r &= u_{r-1} + \frac{1}{6}\left(d_1 + 4d_2 + d_3\right) \\ d_1 &= \Delta t\, g\left(t_{r-1}, u_{r-1}\right) \\ d_2 &= \Delta t\, g\left(t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_1}{2}\right) \\ d_3 &= \Delta t\, g\left(t_{r-1} + \Delta t, u_{r-1} - d_1 + 2d_2\right) \end{aligned} \tag{1. 3}$$

It is found that TOEM, RK3 and proposed method all have the same interval of stability which is larger than that of MIME. The proposed method possesses the minimum error bounds followed by RK3, TOEM and MIME, and also outperforms these methods with respect to accuracy and CPU time.

2. PROPOSED METHOD

Let

$$\begin{aligned} u_r &= u_{r-1} + c_1 d_1 + c_2 d_2 + c_3 d_3 \\ d_1 &= \Delta t\, g\left(t_{r-1}, u_{r-1}\right) = \Delta t\, g \\ d_2 &= \Delta t\, g\left(t_{r-1} + a_2 \Delta t, u_{r-1} + b_{21} d_1\right) \\ d_3 &= \Delta t\, g\left(t_{r-1} + a_3 \Delta t, u_{r-1} + b_{31} d_1 + b_{32} d_2\right) \end{aligned} \tag{2. 4}$$

Here $a_2,\, a_3,\, b_{21},\, b_{31},\, b_{32},\, c_1,\, c_2$ and c_3 are unknowns. Taylor series expansion of (2. 4) yields

$$\begin{aligned} u_r = {} & u_{r-1} + \left(c_1 + c_2 + c_3\right)\, \Delta t\, g + \left(a_2 c_2 + a_3 c_3\right)\, (\Delta t)^2\, g_t + \{b_{21} c_2 \\ & + \left(b_{31} + b_{32}\right)\, c_3\}\, (\Delta t)^2\, g g_u + \frac{1}{2}\left({a_2}^2 c_2 + {a_3}^2 c_3\right)\, (\Delta t)^3\, g_{tt} + \{a_2 b_{21} c_2 \\ & + \left(a_3 b_{31} + a_3 b_{32}\right) c_3\}\, (\Delta t)^3\, g g_{tu} + \frac{1}{2}\left\{{b_{21}}^2 c_2 + \left(b_{31} + b_{32}\right)^2 c_3\right\} (\Delta t)^3\, g^2 g_{uu} \\ & + a_2 b_{32} c_3\, (\Delta t)^3\, g_t g_u + b_{21} b_{32} c_3\, (\Delta t)^3\, g g_u^2 + O\left(\Delta t\right)^4 \end{aligned} \tag{2. 5}$$

The Taylor's expansion of u_r is given by

$$\begin{aligned} u_r = {} & u_{r-1} + \Delta t\, g + \frac{(\Delta t)^2}{2}\left(g_t + g g_u\right) + \frac{(\Delta t)^3}{6}\left(g_{tt} + 2 g g_{tu} + g^2 g_{uu} + g_t g_u + g g_u^2\right) \\ & + O\left(\Delta t\right)^4 \end{aligned} \tag{2. 6}$$

Comparing the coefficients of Δt in (2. 5) as far as $(\Delta t)^3$ with the Taylor's expansion (2. 6) yields the following order conditions:

$$\begin{aligned} & c_1 + c_2 + c_3 = 1 \\ & a_2 c_2 + a_3 c_3 = \frac{1}{2} \\ & b_{21} c_2 + \left(b_{31} + b_{32}\right) c_3 = \frac{1}{2} \\ & a_2^2 c_2 + a_3^2 c_3 = \frac{1}{3} \\ & a_2 b_{21} c_2 + \left(a_3 b_{31} + a_3 b_{32}\right) c_3 = \frac{1}{3} \\ & b_{21}^2 c_2 + \left(b_{31} + b_{32}\right)^2 c_3 = \frac{1}{3} \\ & a_2 b_{32} c_3 = \frac{1}{6} \\ & b_{21} b_{32} c_3 = \frac{1}{6} \end{aligned} \tag{2. 7}$$

An optimal solution of (2. 7) leads to following third order accurate method:

$$\begin{aligned} u_r &= u_{r-1} + \frac{1}{4}\left(d_1 + 3d_3\right) \\ d_1 &= \Delta t\, g\left(t_{r-1}, u_{r-1}\right) \\ d_2 &= \Delta t\, g\left(t_{r-1} + \frac{2}{3}\Delta t, u_{r-1} + \frac{2}{3} d_1\right) \\ d_3 &= \Delta t\, g\left(t_{r-1} + \frac{2}{3}\Delta t, u_{r-1} + \frac{1}{3}\left(d_1 + d_2\right)\right) \end{aligned} \tag{2. 8}$$

Here, by optimal solution we mean the values of the unknowns present in the nonlinear system (2. 7) which give rise to a numerical scheme with minimum error bounds and maximum accuracy in comparison to other methods under consideration. The subsequent sections have been accordingly expanded to delineate this notion of optimal solution, and hence an optimal method.

3. Analysis of Proposed Method

3.1. **Consistency.**

Definition 1. *A single-step method has the form* $u_r = u_{r-1} + \Delta t\, \psi(t_{r-1}, u_{r-1}; \Delta t)$ *where* $\psi(t_{r-1}, u_{r-1}; \Delta t)$ *is known as the increment function of the method.*

Definition 2. *Given an initial value problem* $\frac{du}{dt} = g(t, u); u(t_0) = u_0$*, a numerical method with an increment function* $\psi\left(t, u; \Delta t\right)$ *is said to be consistent if* $\lim\limits_{\Delta t \to 0} \psi\left(t, u; \Delta t\right) = g(t, u)$.

Applying these definitions to the increment function ψ of proposed method,

$\lim\limits_{\Delta t \to 0} \psi\ \left(t_{r-1},\ u_{r-1}\ ;\ \Delta t\right) = \lim\limits_{\Delta t \to 0} \frac{1}{4\Delta t}\left(d_1 + 3d_3\right) = g\left(t_{r-1},\ u_{r-1}\right)$.

Hence, the method is consistent.

Next, we state and prove two important theorems which are used to demonstrate stability and convergence of proposed method.

Theorem 1. *Let* $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n$ *be real numbers satisfying* $|\gamma_{i+1}| \leq (1 + \alpha)|\gamma_i| + \beta$*, with* $\alpha > 0, \beta \geq 0, i = 0, 1, 2, \ldots, n - 1$*, then* $|\gamma_n| \leq e^{n\alpha}|\gamma_0| + \frac{e^{n\alpha} - 1}{\alpha}\beta$.

Proof. From the assumptions, we get

$$\begin{aligned} &|\,\gamma_1\,| \leq (1 + \alpha)\ |\,\gamma_0\,| + \beta \\ &|\,\gamma_2\,| \leq (1 + \alpha)\ |\,\gamma_1\,| + \beta = (1 + \alpha)^2\ |\,\gamma_0\,| + (1 + \alpha)\,\beta + \beta \\ &\quad\vdots \\ &|\,\gamma_n\,| \leq (1 + \alpha)^n\ |\,\gamma_0\,| + \beta\left[\,1 + (1 + \alpha) + (1 + \alpha)^2 + \cdots + (1 + \alpha)^{n-1}\,\right] \\ &\qquad \leq e^{n\,\alpha}\,|\,\gamma_0\,| + \beta\frac{e^{n\,\alpha} - 1}{\alpha}\ \because 0 <\ 1 + \alpha \leq e^{\alpha} \text{ for } \alpha > -1. \end{aligned}$$

□

Theorem 2. *Suppose* (t_{r-1}, u_{r-1}) *and* $(t_{r-1}, \hat{u}_{r-1})$ *are any two points in the region T defined by* $T = \left\{ (t, u) \in R^2 \middle|\ t_0 \leq t \leq t_n, -\infty < u < \infty \right\}$*, and* g *is a Lipschitz continuous function on T such that* $\left|g\left(t_{r-1}, u_{r-1}\right) - g\left(t_{r-1}, \hat{u}_{r-1}\right)\right| \leq L\left|u_{r-1} - \hat{u}_{r-1}\right|$*, then the increment function* ψ *is Lipschitz continuous, and* $\left|\psi\left(t_{r-1}, u_{r-1}; \Delta t\right) - \psi\left(t_{r-1}, \hat{u}_{r-1}; \Delta t\right)\right| \leq \hat{L}\left|u_{r-1} - \hat{u}_{r-1}\right|$ *where* L *and* $\hat{L}$ *are, respectively, the Lipschitz constants for g and* ψ.

Proof. From the assumptions, we have

$$\left|d_1 - \hat{d}_1\right| = \Delta t\ \left|g\left(t_{r-1}, u_{r-1}\right) - g\left(t_{r-1}, \hat{u}_{r-1}\right)\right| \leq L\Delta t\ \left|u_{r-1} - \hat{u}_{r-1}\right|$$

$$\begin{aligned}\left|d_2 - \hat{d}_2\right| &= \Delta t\ \left|g\left(t_{r-1} + \tfrac{2}{3}\Delta t, u_{r-1} + \tfrac{2}{3}d_1\right) - g\left(t_{r-1} + \tfrac{2}{3}\Delta t, \hat{u}_{r-1} + \tfrac{2}{3}\hat{d}_1\right)\right| \\ &\leq L\Delta t\ \left|\left(u_{r-1} - \hat{u}_{r-1}\right)\right| + \tfrac{2}{3}L\Delta t\ \left|d_1 - \hat{d}_1\right| \\ &\leq \left(L\Delta t + \tfrac{2}{3}L^2\left(\Delta t\right)^2\right)\left|u_{r-1} - \hat{u}_{r-1}\right|\end{aligned}$$

$$\begin{aligned}&\left|d_3 - \hat{d}_3\right| \\ &= \Delta t \left|g\left(t_{r-1} + \tfrac{2}{3}\Delta t, u_{r-1} + \tfrac{1}{3}\left(d_1 + d_2\right)\right) - g\left(t_{r-1} + \tfrac{2}{3}\Delta t, \hat{u}_{r-1} + \tfrac{1}{3}\left(\hat{d}_1 + \hat{d}_2\right)\right)\right| \\ &\leq L\Delta t\ \left|u_{r-1} - \hat{u}_{r-1}\right| + \tfrac{L\Delta t}{3}\left|d_1 - \hat{d}_1\right| + \tfrac{L\Delta t}{3}\left|d_2 - \hat{d}_2\right| \\ &\leq \left(L\Delta t + \tfrac{2}{3}L^2\left(\Delta t\right)^2 + \tfrac{2}{9}L^3(\Delta t)^3\right)\left|u_{r-1} - \hat{u}_{r-1}\right|\end{aligned}$$

Hence, for the increment function ψ of the proposed method:

$$\begin{aligned}&\left|\psi\left(t, u; \Delta t\right) - \psi\left(t, \hat{u}; \Delta t\right)\right| \\ &= \tfrac{1}{4\Delta t}\left|\left(d_1 + 3d_3\right) - \left(\hat{d}_1 + 3\hat{d}_3\right)\right| \\ &\leq \tfrac{1}{4\Delta t}\left|d_1 - \hat{d}_1\right| + \tfrac{3}{4\Delta t}\left|d_3 - \hat{d}_3\right| \\ &\leq \hat{L}\left|u_{r-1} - \hat{u}_{r-1}\right|; \qquad \hat{L} = L + \tfrac{1}{2}L^2\Delta t + \tfrac{1}{6}L^3(\Delta t)^2\end{aligned}$$

This implies that the increment function of proposed method is Lipschitz continuous. □

3.2. **Stability.**

Theorem 3. *Let u_r and $\hat{u}_r$ be two solutions to the differential equation $u' = g\left(t, u\right)$, generated by a numerical method, subject to the initial conditions $u\left(t_0\right) = u_0$ and, $\hat{u}\left(t_0\right) = \hat{u}_0$ respectively, such that $\left|u_0 - \hat{u}_0\right| < \varepsilon, \varepsilon > 0$. The condition $\left|u_r - \hat{u}_r\right| \leq K\left|u_0 - \hat{u}_0\right|, K > 0$ is the necessary and sufficient condition for the method to be stable.*

Applying Theorem 2 to the proposed method gives,

$$\begin{aligned}u_r &= u_{r-1} + \Delta t\ \psi\left(t_{r-1},\ u_{r-1};\ \Delta t\right) \\ \hat{u}_r &= \hat{u}_{r-1} + \Delta t\ \psi\left(t_{r-1},\ \hat{u}_{r-1};\ \Delta t\right)\end{aligned}$$

$$\begin{aligned}\therefore \left|\, u_r - \hat{u}_r\right| &\leq \left|\, u_{r-1} - \hat{u}_{r-1}\right| + \Delta t\ \left|\,\psi\ \left(t_{r-1},\ u_{r-1}\,;\ \Delta t\right) - \psi\ \left(t_{r-1},\ \hat{u}_{r-1}\,;\ \Delta t\right)\right| \\ &\leq \left(1 + \hat{L}\Delta t\right)\left|u_{r-1} - \hat{u}_{r-1}\right| \\ &= \left(1 + \hat{L}\Delta t\right)^r\ \left|u_0 - \hat{u}_0\right|\end{aligned}$$

Using Theorem 1 with $\alpha = \hat{L}\Delta t$ and $\beta = 0$ gives

$$\left|u_r - \hat{u}_r\right| \leq K\left|u_0 - \hat{u}_0\right|,\ K = e^{r\hat{L}\Delta t}.$$

This establishes that the proposed method is stable.

Applying proposed method to Dahlquist's model problem [9]

$$u\prime = \lambda u, \lambda \in \mathbb{C}, R_e(\lambda) < 0 \tag{3. 9}$$

gives

$$u_r = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)u_{r-1} \tag{3. 10}$$

where $z = \lambda\Delta t$.The ratio u_r/u_{r-1} is called stability function $\phi(z)$. Hence, stability function of the proposed method is

$$\phi(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} \tag{3. 11}$$

It is found that $|\phi(z)| \leq 1$ for $-2.51 \leq R_e(z) \leq 0$. Table 1 compares stability function

TABLE 1. Stability functions and intervals of stability.

Method	Linear stability function	Interval of linear stability
MIME	$1 + z + \frac{z^2}{2} + \frac{z^3}{4}$	(-2, 0)
TOEM	$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$	(-2.51, 0)
RK3	$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$	(-2.51, 0)
Proposed	$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$	(-2.51, 0)

and interval of stability of proposed method with those of MIME, TOEM and RK3 methods. While TOEM and proposed method each have same interval of stability as that of conventional RK3 method, MIME possesses a smaller stability interval.

3.3. **Local Error Bounds.** The local truncation error for proposed method is

$$\begin{aligned}\tau_{r+1} = \frac{(\Delta t)^4}{216} \big(& g_{ttt} + 3gg_{ttu} + 3g^2 g_{tuu} + g^3 g_{uuu} + 3gg_t g_{uu} + 3g_t g_{tu} - 3gg_u g_{tu} \\ & -3g_{tt} g_u + 9g_t g_u^2 + 9gg_u^3 \big) + O\left(\Delta t\right)^5\end{aligned} \tag{3. 12}$$

Thus, the principal error function is given as

$$\begin{aligned}\Phi\left(t_r, u_r\right) = \frac{1}{216} \big(& g_{ttt} + 3gg_{ttu} + 3g^2 g_{tuu} + g^3 g_{uuu} + 3gg_t g_{uu} + 3g_t g_{tu} - 3gg_u g_{tu} \\ & -3g_{tt} g_u + 9g_t g_u^2 + 9gg_u^3 \big)\end{aligned} \tag{3. 13}$$

Lotkin [16] proposed following bounds for the function g and its partial derivatives for $t \in [a\,,\,b]$ and $u \in (-\infty\,,\,\infty)$:

$$|\,g\ (t\,,\,u)\,| < M, \left| \frac{\partial^{\mathrm{i+j}} g}{\partial t^i \partial u^j} \right| < \frac{N^{i+j}}{M^{j-1}}, \qquad i + j \leq p$$

where M and N are positive constants and p is order of accuracy. Using Lotkin's bounds,

$$|\,\Phi\ (t_r\,,\,u_r)\,| < \frac{19}{108} M N^3 \left(\Delta t\right)^4$$

Hence, the bound on local truncation error τ_{r+1} of the proposed method is obtained as

$$|\tau_{r+1}| < T, \qquad T = \frac{19}{108} M N^3 \left(\Delta t\right)^4$$

3.4. **Convergence.** Consider the proposed method:

$$u_{i+1} - u_i - \Delta t\, \psi\left(t_i, u_i; \Delta t\right) = 0 \tag{3. 14}$$

The quantity

$$\tau_i = u\left(t_{i+1}\right) - u\left(t_i\right) - \Delta t\, \psi\left(t_i, u\left(t_i\right); \Delta t\right) \tag{3. 15}$$

is local error of the proposed method. On subtracting (3. 14) from (3. 15), it is deduced that

$$|e_{i+1}| \leq |e_i| + \Delta t \ |\psi\left(t_i, u\left(t_i\right); \Delta t\right) - \psi\left(t_i, u_i; \Delta t\right)| + |\tau_i| \tag{3. 16}$$

where e_i is global error. Using Theorems 1 and 2, (3. 16) yields

$$|e_{i+1}| \leq \left(1 + \Delta t\, \hat{L}\right) |e_i| + |\tau_i| \tag{3. 17}$$

Let $T = \max |\tau_i|\,, i = 0,\ 1,\ 2, \ldots, k-1$, then Theorem 1 implies that

$$|e_r| \leq e^{r\hat{L}\Delta t}\,|e_0| + \frac{T}{\hat{L}\Delta t}\left[e^{r\hat{L}\Delta t} - 1\right] \tag{3. 18}$$

Finally, the global error bound for the proposed method is obtained as

$$|e_r| \leq \frac{T}{\hat{L}\Delta t}\left[e^{\hat{L}(t_r - t_0)} - 1\right] \tag{3. 19}$$

where T is its local error bound. As $\lim\limits_{\Delta t \to 0} |e_r| = 0$, this implies convergence of our method. Table 2 provides comparison of local and global error bounds of proposed method

TABLE 2. Comparison of error bounds.

Method	Local Error Bound	Global Error Bound
MIME	$\frac{1}{3}MN^2\left(\Delta t\right)^3$	$\frac{1}{3}MN^2\left(\Delta t\right)^2\left[\frac{e^{\hat{L}_{MIME}(t_r-t_0)}-1}{\hat{L}_{MIME}}\right]$
TOEM	$\frac{53}{144}MN^3\left(\Delta t\right)^4$	$\frac{53}{144}MN^3\left(\Delta t\right)^3\left[\frac{e^{\hat{L}_{TOEM}(t_r-t_0)}-1}{\hat{L}_{TOEM}}\right]$
RK3	$\frac{1}{4}MN^3\left(\Delta t\right)^4$	$\frac{1}{4}MN^3\left(\Delta t\right)^3\left[\frac{e^{\hat{L}_{RK3}(t_r-t_0)}-1}{\hat{L}_{RK3}}\right]$
Proposed	$\frac{19}{108}MN^3\left(\Delta t\right)^4$	$\frac{19}{108}MN^3\left(\Delta t\right)^3\left[\frac{e^{\hat{L}_{Proposed}(t_r-t_0)}-1}{\hat{L}_{Proposed}}\right]$

[1] $\hat{L}_{\text{MIME}}$, $\hat{L}_{\text{TOEM}}$, $\hat{L}_{\text{RK3}}$ and $\hat{L}_{\text{Proposed}}$ respectively denote the Lipschitz constant for the increment functions of MIME, TOEM, RK3 and proposed methods.

[2] $\hat{L}_{\text{MIME}} = L + \frac{1}{2}L^2\Delta t + \frac{1}{4}L^3(\Delta t)^2$,
$\hat{L}_{\text{TOEM}} = \hat{L}_{\text{RK3}} = \hat{L}_{\text{Proposed}} = L + \frac{1}{2}L^2\Delta t + \frac{1}{6}L^3(\Delta t)^2$

with those of MIME, TOEM and RK3 methods. A simple analysis of these error bounds leads to the following observation:

For given values of *M*, *N* and Δt;
$(\text{Error Bound})_{\text{Proposed}} < (\text{Error Bound})_{\text{RK3}} < (\text{Error Bound})_{\text{TOEM}} < (\text{Error Bound})_{\text{MIME}}$
which holds for both local and global error bounds.

4. NUMERICAL EXAMPLES

4.1. **Example 1.** $\dfrac{du}{dt} = u - tu^2$ subject to $u\left(0\right) = 1$. The exact solution is $u\left(t\right) = \dfrac{1}{2e^{-t} + t - 1}$.

TABLE 3. Absolute errors vs. step size (Δt) at $t = 1$.

Step size	MIME	TOEM	RK3	Proposed Method
0.001	8.52E-08	1.03E-07	1.50E-10	5.64E-12
0.01	8.55E-06	1.04E-05	1.5E-07	4.93E-09
0.1	8.93E-04	1.12E-03	1.53E-04	3.23E-06
0.25	6.07E-03	8.07E-03	2.40E-03	3.28E-04

TABLE 4. CPU time (sec) vs. error tolerance at $t = 1$.

Error tolerance	MIME	TOEM	RK3	Proposed Method
E-03	1.16E-04	1.21E-04	1.52E-04	1.32E-04
E-04	1.52E-04	2.36E-04	1.63E-04	1.09E-04
E-05	4.21E-04	4.23E-04	2.08E-04	1.82E-04
E-06	8.58E-04	1.05E-03	2.54E-04	1.97E-04

For the nonlinear problem under consideration, we present a comparison of absolute errors yielded by MIME, TOEM, RK3 and proposed method at $t = 1$ corresponding to various step sizes in Table 3 , whereas, Table 4 presents the CPU time consumed by these methods against different values of error tolerance at the same point of domain. It can be clearly seen that the proposed method not only produces the smallest errors for each step size value, but also takes the least computational time to attain the desired accuracy.

4.2. **Example 2.** $\dfrac{du_1}{dt} = u_2^2 - 2u_1$, $\dfrac{du_2}{dt} = u_1 - u_2 - tu_2^2$ subject to $u_1(0) = 0$ and $u_2(0) = 1$. The exact solution is $u_1(t) = te^{-2t}$, $u_2(t) = e^{-t}$.

TABLE 5. L_2 error norm vs. step size (Δt) at $t = 2$.

Step size	MIME	TOEM	RK3	Proposed Method
0.001	3.92E-08	1.34E-08	9.92E-12	7.98E-12
0.01	3.93E-06	1.34E-06	9.98E-09	8.14E-09
0.1	3.99E-04	1.29E-04	1.1E-05	9.8E-06
0.25	2.41E-03	7.03E-04	2.04E-04	1.97E-04

[1] L_2 error norm $= \sqrt{\Sigma(\text{Exact solution - Approximate solution})^2}$ where the sum is taken over the number of components in the solution of test problem.

Table 5 shows L_2 error norm values given by MIME, TOEM, RK3 and proposed method at $t = 2$ for the nonlinear system under consideration. It can be seen that the minimum error norm values are obtained from the proposed method followed by RK3, TOEM and MIME methods. In terms of CPU time, proposed method is found to be efficient in contrast to both MIME and TOEM methods and is comparable to RK3 as illustrated from Table 6.

TABLE 6. CPU time (sec) vs. L_2 error norm tolerance at $t = 2$.

Error tolerance	MIME	TOEM	RK3	Proposed Method
E-03	6.18E-04	6.19E-04	6.24E-04	6.06E-04
E-04	2.08E-03	1.04E-03	8.72E-04	6.57E-04
E-05	8.24E-03	2.43E-03	1.10E-03	1.11E-03
E-06	1.74E-02	7.21E-03	2.43E-03	2.44E-03

4.3. **Example 3 - Application in population dynamics.**

$$\frac{du}{dt} = a\left(1 - \frac{u}{C}\right)u; \quad u(0) = u_0 \tag{4. 20}$$

The nonlinear autonomous Cauchy problem (4. 20) is the population growth model known as Verhulst or logistic equation [31]. The positive constants a and C are respectively known as intrinsic growth rate and saturation level for the given population. The analytical solution is given as:

$$u(t) = \frac{u_0 C}{u_0 + (C - u_0)e^{-at}} \tag{4. 21}$$

In [6, p. 80], logistic equation is applied on Pacific halibut growth where u denotes the biomass (in kilogram) of halibut fishery and t is time. The parameter values used are $a = 0.71$/year and $C = 8.05 \times 10^7$ kg with initial biomass $u_0 = 2.0125 \times 10^7$ kg. We integrate logistic equation from $t = 0$ to $t = 2$ subject to these conditions using MIME, TOEM, RK3 and proposed method. The results are presented in Tables 7 and 8.

TABLE 7. Absolute errors vs. step size (Δt) at $t = 2$.

Step size	MIME	TOEM	RK3	Proposed Method
0.001	3.55E-01	2.72E-01	4.22E-05	9.34E-06
0.01	3.55E+01	2.72E+01	4.39E-02	1.08E-02
0.05	8.84E+02	6.77E+02	5.56E+00	1.30E+00
0.1	3.52E+03	2.70E+03	4.53E+01	9.87E+00
0.25	2.18E+04	1.67E+04	7.41E+02	1.29E+02

TABLE 8. CPU time (sec) vs. error tolerance at $t = 2$.

Error tolerance	MIME	TOEM	RK3	Proposed Method
E-03	1.09E-01	1.11E-01	6.84E-03	2.04E-03
E-04	2.72E-01	2.63E-01	5.75E-03	3.77E-03
E-05	1.11E+00	1.10E+00	1.26E-02	7.60E-03
E-06	1.67E+00	1.74E+00	3.20E-02	1.59E-02

From the tables, it is evident that the performance of proposed method is far better than each of MIME, TOEM and RK3 when implemented to a physical problem. The proposed method leads to the minimal absolute errors, at the final time $t = 2$, corresponding to different step-size values. It also shows the fastest convergence by consuming the least amount of CPU time to achieve the specified accuracy levels. It is particularly noticeable here that both MIME and TOEM give poor accuracy as depicted from the high error values at even small step size of 0.01. Moreover, both these methods also prove to be computationally expensive.

5. Conclusion

In this work, an optimal third order accurate scheme has been developed for solving Cauchy problems. First, an abstract analysis of the proposed scheme is carried out for stability, consistency, convergence and error bounds. The proposed method is then implemented on three nonlinear problems for analyzing its performance in terms of accuracy and computational time in contrast to three existing methods – MIME, TOEM and Classical RK3. The observations drawn are as follows:

(i) TOEM, RK3 and proposed methods all have the same interval of stability which is larger than that of MIME.
(ii) Proposed method has the minimum local and global error bounds followed by RK3, TOEM and MIME methods.
(iii) In terms of accuracy, proposed method yields errors significantly smaller in magnitude than resulting from each of RK3, TOEM and MIME.
(iv) Overall, the proposed scheme proves to be computationally cost effective as it consumes considerably less CPU time while attaining the desired level of accuracy.

6. Authors' contribution

The idea of this work is given by first author. All authors have contributed equally towards preparation of this article.

7. Acknowledgement

The authors are thankful to the anonymous reviewers for their valuable comments and suggestions.

8. Declaration of interest

'Declarations of interest: none'

9. Role of the funding source

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

REFERENCES

[1] M. A. Akanbi, *Third order Euler method for numerical solution of ordinary differential equations*, ARPN Journal of Engineering and Applied Sciences. **5**, (2010) 42-49.

[2] M. A. Akanbi, *On 3-stage geometric explicit Runge–Kutta method for singular autonomous initial value problems in ordinary differential equations*, Computing. **92**, No. 3 (2011) 243-263.

[3] J. Ali, M. Saeed, M. Rafiq and S. Iqbal, *Numerical treatment of nonlinear model of virus propagation in computer networks: an innovative evolutionary Padé approximation scheme*, Advances in Difference Equations. **2018**, No. 1 (2018) 214.

[4] M. A. Ali, M. Rafiq and M. O. Ahmad, *Numerical Analysis of a Modified SIR Epidemic Model with the Effect of Time Delay*, Punjab Univ. j. math. **51**, No. 1 (2019) 79-90.

[5] B. Bilgehan, B. Eminağa and M. Riza, *New solution method for electrical systems represented by ordinary differential equation*, Journal of Circuits, Systems and Computers. **25**, No. 2 (2016) 1650011.

[6] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, New York, John Wiley and Sons Inc., 2001.

[7] V. Chauhan and P. K. Srivastava, *Trio-Geometric mean-based three-stage Runge–Kutta algorithm to solve initial value problem arising in autonomous systems*, International Journal of Modeling, Simulation, and Scientific Computing. **9**, No. 4 (2018) 1850026.

[8] B. Coto and I. Suárez, *Euler algorithm to solve reaction kinetic equations: Mathematical formulation, programing, and applications*, Computer Applications in Engineering Education. **26**, No. 1 (2018) 29-36.

[9] G. G. Dahlquist, *A special stability problem for linear multistep methods*, BIT Numerical Mathematics. **3**, No. 1 (1963) 27-43.

[10] S. Javeed, A. Ahmed, M. S. Khan and M. A. Javed, *Stability Analysis and Solutions of Dynamical Models for Dengue*, Punjab Univ. j. math. **50**, No. 2 (2018) 45-67.

[11] Y. D. Jikantoro, F. Ismail, N. Senu and Z. B. Ibrahim, *A New Integrator for Special Third Order Differential Equations With Application to Thin Film Flow Problem*, Indian Journal of Pure and Applied Mathematics.**49**, No. 1 (2018) 151-167.

[12] M. Kamruzzaman and M. M. Hasan, *A New Numerical Approach for Solving Initial Value Problems of Ordinary Differential Equations*, Annals of Pure and Applied Mathematics. **17**, No. 2 (2018) 157-162.

[13] T. A. Kee and R. Ranom, *Comparison of Numerical Techniques in Solving Transient Analysis of Electrical Circuits*, ARPN Journal of Engineering and Applied Sciences. **13**, No. 1 (2018) 314-320.

[14] M. M. Khader, *On the numerical solutions for chemical kinetics system using Picard–Padé technique*, Journal of King Saud University-Engineering Sciences. **25**, No. 2 (2013) 97-103.

[15] J. Li, H. Ren and J. Ning, *Additive Runge-Kutta methods for H_2/O_2/Ar detonation with a detailed elementary chemical reaction model*, Chinese Science Bulletin. **58**, No. 11 (2013) 1216-1227.

[16] M. Lotkin, *On the accuracy of Runge-Kutta's method*, Mathematical tables and other aids to computation. **5**, No. 35 (1951) 128-133.

[17] R. Memarbashi, F. Alipour and A. Ghasemabadi, *A Nonstandard Finite Difference Scheme for a SEI Epidemic Model*, Punjab Univ. j. math. **49**, No. 3 (2017) 133-147.

[18] Z. Memon, S. Qureshi, A. A. Shaikh and M. S. Chandio, *A Modified ODE Solver for Autonomous Initial Value Problems*, Mathematical Theory and Modeling. **4**, No. 3 (2014) 80-85.

[19] F. Nadeem, M. Zamir, A. Tridane and Y. Khan *Modeling and Control of Zoonotic Cutaneous Leishmaniasis*, Punjab Univ. j. math. **51** No. 2 (2019) 105-121.

[20] A. Ochoche, *Improving the Improved Modified Euler Method for Better Performance on Autonomous Initial Value Problems*, Leonardo Journal of Sciences. **7**, No. 12 (2008) 57-66.

[21] C. Onwubuoya, A. Samuel and O. I. Odabi, *Numerical Simulation of a Computer Virus Transmission Model using Euler Predictor-Corrector Method*, IDOSR Journal of Applied Sciences. **3**, No. 1 (2018) 16-28.

[22] S. Qureshi and Z. Memon and A. A. Shaikh and M. S. Chandio, *On the Construction and Comparison of an Explicit Iterative Algorithm with Nonstandard Finite Difference Schemes*, Mathematical Theory and Modeling. **3**, No. 12 (2013) 78-87.

[23] S. Qureshi and H. Ramos, *L-stable Explicit Nonlinear Method with Constant and Variable Step-size Formulation for Solving Initial Value Problems*, International Journal of Nonlinear Sciences and Numerical Simulation. **19**, No. 7-8 (2018) 741-751.

[24] C. Runge, *Über die numerische Auflösung von Differentialgleichungen*, Mathematische Annalen. **46**, No. 2 (1895) 167-178.

[25] B. Sanderse and B. Koren, *Accuracy analysis of explicit Runge–Kutta methods applied to the incompressible Navier–Stokes equations*, Journal of Computational Physics. **231**, No. 8 (2012) 3041-3063.

[26] S. Senthilkumar, *A Study on Third Order Runge-Kutta Techniques for Solving Practical Problems*, Walailak Journal of Science and Technology (WJST). **11**, No. 8 (2013) 679–686.

[27] A. Suhag, *Transient Analysis of Electrical Circuits Using Runge-Kutta Method and its Application*, International Journal of Scientific and Research Publications. **3**, No. 11 (2013) 1-5.

[28] M. Suleman and S. Riaz, *Unconditionally Stable Numerical Scheme to Study the Dynamics of Hepatitis B Disease*, Punjab Univ. j. math. **49**, No. 3 (2017) 99-118.

[29] M. Tahir, S. I. A. Shah, G. Zaman and T. Khan, *A Dynamic Compartmental Mathematical Model Describing The Transmissibility of MERS-CoV Virus In Public*, Punjab Univ. j. math. **51**, No. 4 (2019) 57-71.

[30] E. O. Tuck and L. W. Schwartz, *A numerical and asymptotic study of some third-order ordinary differential equations relevant to draining and coating flows*, SIAM review. **32**, No. 3 (1990) 453–469.

[31] P. F. Verhulst, *Notice sur la loi que la population poursuit dans son accroissement*, Correspondance mathmatique et physique. **10** (1838) 113-121.

[32] V. Vuorinen, J. P. Keskinen, C. Duwig and B. J. Boersma, *On the implementation of low-dissipative Runge–Kutta projection methods for time dependent flows using OpenFOAM®*, Computers & Fluids. **93** (2014) 153-163.