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A new subfamily of starlike functions of complex order using Srivastava-Owa fractional operator

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Abstract. This work is about investigation of a certain new subfamily of starlike functions using the Srivastava-Owa fractional operator. For function in this new subfamily, a number of interesting problems, like coefficient bounds, distortion and radius bounds, which are best possible, are tackled. Various special cases deduced from the present results are also listed. This paper brings extension and refinement to earlier works of various authors.

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1. Introduction and Preliminaries

Let Q be the family of all functions $f(z)$, analytic in the unit open disk $Y = \{z : |z| < 1\}$, satisfying

$$f(0) = f'(0) - 1 = 0.$$

Similarly, Π represents the class of functions $J(z)$ analytic in Y with $J(0) = 1$. Let Λ, Υ are real numbers such that $-1 \leq \Upsilon < \Lambda \leq 1$, then $\Pi[\Lambda, \Upsilon]$ represents the Janowski class of functions $J(z)$ analytic in Y defined in [10] as:

$$J(z) = \frac{1 + \Lambda w(z)}{1 + \Upsilon w(z)}, \ (z \in Y)$$

where $w(z)$ is the Schwarz function for which

$$w(0) = 0, \quad |w(z)| < 1, \quad \forall z \in Y.$$

Note that $\Pi[1, -1] = \Pi$ is the Carathéodory class of functions, and $\Pi[1 - 2\delta, -1] = \Pi(\delta)$ is the Carathéodory class of functions with $\Re(J(z)) > \delta, \ (0 \leq \delta < 1)$. In addition to that, $\mathcal{C}[\Lambda, \Upsilon]$ and $\mathcal{S}^*[\Lambda, \Upsilon]$ are respectively, the classes of Janowki convex and starlike functions. Moreover, the convex and starlike classes of functions are obtained respectively, as

$$\mathcal{C}[1,-1]=\mathcal{C} \quad \text{and} \quad \mathcal{S}^*[1,-1]=\mathcal{S}^*.$$

Now, the following concepts of fractional calculus were introduced by Srivastava and Owa [12] (see also [9]). For some recent advances and applications of fractional calculus, see [1, 5, 6, 7, 8, 13].

Definition 1.1. *For $f \in Q$, let D_z^ν is the fractional derivative of order ν $(0 \leq \nu < 1)$ defined as*

$$D_z^\nu f(z)=\frac{d}{dz}\left(D_z^{\nu-1} f(z)\right)=\frac{1}{\Gamma(1-\nu)}\frac{d}{dz}\int_0^z \frac{f(\chi)}{(z-\chi)^\nu}d\chi,$$

where the multiplicity of $(z-\chi)^{-\nu}$ is removed by taking $\log(z-\chi)$ real.

In view of Definition 1.1, by induction we have

Definition 1.2. *For $f \in Q$, and $0 \leq \nu < 1,\ n \in \mathbb{N}_0=\mathbb{N}\cup\{0\}$*

$$D_z^{n+\nu} f(z)=\frac{d^n}{dz^n}\left(D_z^\nu f(z)\right)=\frac{1}{\Gamma(1-\nu)}\frac{d^{n+1}}{dz^{n+1}}\int_0^z \frac{f(\chi)}{(z-\chi)^\nu}d\chi.$$

where the multiplicity of $(z-\chi)^{-\nu}$ can be removed as before.

Thus for $0 \leq \nu < 1,\ m > 0,\ n \in \mathbb{N}_0,\ m-n \neq -1,-2,-3,\cdots$

$$D_z^{n+\nu} z^m=\frac{\Gamma(m+1)}{\Gamma(m+1-n-\nu)}z^{m-n-\nu},$$

and for any real ν $(\nu > 1)$

$$D_z^\nu z^m=\frac{\Gamma(m+1)}{\Gamma(m+1-\nu)}z^{m-\nu},\ (m-\nu \neq -1,-2,...).$$

Owa and Srivastava [12] introduced the fractional operator Φ^ν for $f \in Q$ as:

$$\Phi^\nu f(z)=\Gamma(2-\nu)z^\nu D_z^\nu f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(2-\nu)\Gamma(n+1)}{\Gamma(n+1-\nu)}a_n z^n. \tag{1}$$

Note that

$$\Phi^0 f(z)=f(z) \quad \text{and} \quad \Phi^1 f(z)=zf'(z).$$

Moreover, it is noted that

$$\Phi^1\left(\Phi^\nu f(z)\right)=z\left(\Phi^\nu f(z)\right)'=\Gamma(2-\nu)z^\nu\left[\nu D_z^\nu f(z)+zD_z^{\nu+1}f(z)\right].$$

In what follows the following facts will be required. For $f, g \in Q$ one can say f is subordinated to g, $f \prec g$, if there is a function $w(z)$ for which

$$f(z)=g(w(z)),\ (z \in Y).$$

Additionally, if g is univalent in Y, then $f \prec g$ can be put in the form

$$f(0)=g(0) \quad \text{and} \quad f(Y)\subset g(Y),\ (z \in Y).$$

Çağlar et al. [3] introduced the class $\mathcal{S}_{\nu}^{*}[\Lambda,\Upsilon]$, $(-1\leq\Upsilon<\Lambda\leq 1)$, of Janowski starlike functions by using the Srivastava-Owa fractional operator as follow:

$$\mathcal{S}_{\nu}^{*}\left[\Lambda,\Upsilon\right]=\left\{f\in Q:\frac{z\left(\Phi^{\nu}f\left(z\right)\right)'}{\Phi^{\nu}f\left(z\right)}=J\left(z\right)\in\Pi\left[\Lambda,\Upsilon\right]\right\},$$

where $\nu\neq 2,3,4\cdots$. Coefficient bounds, distortion inequalities and some other interesting inequalities were obtained for this function class in [3]. Recently, in [9] the author provided some generalization to their work by investigating spiral-like functions class of complex order.

As motivation from the cited works [9, 3], the class $\mathcal{S}_{\Delta}^{*}\left(\nu,c\right)$ of starlike functions of complex order $(c\neq 0)$ is introduced by means of Srivastava-Owa fractional operator as follow: Let $f\in Q$, then by $\mathcal{S}_{\Delta}^{*}\left(\nu,c\right)$ we denote the family of all functions given as

$$\mathcal{S}_{\Delta}^{*}\left(\nu,b\right)=\left\{f\in Q:1+\frac{1}{c}\left(\frac{z\left(\Phi^{\nu}f\left(z\right)\right)'}{\Phi^{\nu}f\left(z\right)}-1\right)\in\Delta\left(Y\right)\right\},$$

where $\nu\neq 2,3,4,\cdots$, $\Delta\in\mathcal{C}$ with $\Delta\left(0\right)=1$, $\Re\left(\Delta(z)\right)>0$ for $z\in Y$, and $c\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$.

Various well-known classes appear as a special case of this new class. Indeed, if $\Delta(z)=\frac{1+\Lambda z}{1+\Upsilon z}$, $(-1\leq\Upsilon<\Lambda\leq 1)$, then $\mathcal{S}_{\Delta}^{*}\left(\nu,c\right)=\mathcal{S}^{*}\left(\nu,c,\Lambda,\Upsilon\right)$, and $\mathcal{S}_{\Delta}^{*}\left(\nu,1\right)=\mathcal{S}_{\nu}^{*}\left[\Lambda,\Upsilon\right]$ which was studied in [3]. For $\Delta(z)=\frac{1+z}{1-z}$, $\mathcal{S}_{\Delta}^{*}\left(0,c\right)=\mathcal{S}^{*}\left(c\right)$ (see [11]), and $\mathcal{S}_{\Delta}^{*}\left(1,c\right)=\mathcal{C}\left(c\right)$ (see [15]). Similarly, $\mathcal{S}_{\Delta}^{*}\left(0,1-\beta\right)=\mathcal{S}^{*}\left(\beta\right)$ and $\mathcal{S}_{\Delta}^{*}\left(1,1-\beta\right)=\mathcal{C}\left(\beta\right)$, which are the families of all starlike and convex functions of order β, $(0\leq\beta<1)$ respectively (see [2, 4]).

Next, we define the class $\mathcal{SQ}_{\Delta}\left(\nu,c,\lambda\right)$ by means of Cauchy-Euler type differential equation as follow:

Let $f\in Q$, then $f\in\mathcal{SQ}_{\Delta}\left(\nu,c,\lambda\right)$ if for some $h\in\mathcal{S}_{\Delta}^{*}\left(\nu,c\right)$, the following nonhomogenous differential equation holds true

$$z^{2}\frac{d^{2}u}{dz^{2}}+2(1+\lambda)z\frac{du}{dz}+\lambda(1+\lambda)u=(1+\lambda)(2+\lambda)h(z),\tag{2}$$

where $u=\Phi^{\nu}f\in Q$, and $\lambda\in\mathbb{R}-(-\infty,-1]$.

Now, we assume $\nu\neq 2,3,4,...$, $n\in\mathbb{N}_{2}=\{2,3,4,..\}$, $c\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$, $\lambda\in\mathbb{R}\backslash(-\infty,-1]$ and $-1\leq\Upsilon<\Lambda\leq 1$, in rest of the discussion unless otherwise stated.

2. Coefficient inequalities for the functions classes $\mathcal{S}_{\Delta}^{*}\left(\nu,c\right)$ and $\mathcal{SQ}_{\Delta}\left(\nu,c,\lambda\right)$

The following result is needed to prove our main results in this section.

Lemma 2.1. [14, Rogosinski's Lemma] *Let* $g(z)=\sum\limits_{l=1}^{\infty}g_{l}z^{l}$ *be convex analytic and, let* $f(z)=\sum\limits_{l=1}^{\infty}a_{l}z^{l}$ *be analytic in* Y. *If* $f(z)\prec g(z),\ (z\in Y)$, *then*

$$|a_{l}|\leq|g_{1}|\,,\ l\in\mathbb{N}.$$

Now, for functions in class $\mathcal{S}_{\Delta}^{*}\left(\nu,c\right)$ we have the following bound of their coefficients.

Theorem 2.2. *Let $f \in Q$. If $f \in \mathcal{S}^{*}_{\Delta}(\nu, c)$, then*

$$|a_n| \leq \frac{|\Gamma(n+1-\nu)|}{\Gamma(n+1)|\Gamma(2-\nu)|} \frac{\prod_{m=0}^{n-2}(m+|c||\Delta'(0)|)}{(n-1)!}. \tag{3}$$

This result is sharp.

Proof. From the definition of class $\mathcal{S}^{*}_{\Delta}(\nu, c)$ we have

$$1+\frac{1}{c}\left(\frac{z(\Phi^{\nu}f(z))'}{\Phi^{\nu}f(z)}-1\right) \in \Delta(Y).$$

Or equivalently

$$1+\frac{1}{c}\left(\frac{z(\Phi^{\nu}f(z))'}{\Phi^{\nu}f(z)}-1\right) = J(z),$$

such that $J(0)=\Delta(0)$ and $J(z) \in \Delta(Y)$. Hence $J(z) \prec \Delta(z)$ for $z \in Y$. Thus we get

$$z(\Phi^{\nu}f(z))' = \Phi^{\nu}f(z)(c[J(z)-1]+1)$$

Using (1) with $J(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ in above equation, one gets

$$\begin{aligned} z\left(1+\sum_{n=2}^{\infty} n\phi_n z^{n-1}\right) &= \left(z+\sum_{n=2}^{\infty}\phi_n z^n\right)\left(1+c\sum_{n=1}^{\infty}p_n z^n\right), \\ &= z+\sum_{n=2}^{\infty}\phi_n z^n + \left(\sum_{n=1}^{\infty}\phi_n z^n\right)\left(c\sum_{n=1}^{\infty}p_n z^n\right), \end{aligned}$$

where $\phi_1 = 1$ and

$$\phi_n = \frac{\Gamma(2-\nu)\Gamma(n+1)}{\Gamma(n+1-\nu)}a_n. \tag{4}$$

Simplifying the above expression, we get

$$\sum_{n=2}^{\infty} n\phi_n z^n = \sum_{n=2}^{\infty}\phi_n z^n + \sum_{n=2}^{\infty}\left(c\sum_{m=1}^{n-1}\phi_{n-m}p_m\right)z^n,$$

equating the coefficients of z^n on both sides, and with little manipulation one can obtain

$$|\phi_n| \leq \frac{|c|}{n-1}\sum_{m=1}^{n-1}|\phi_{n-m}||p_m|; \text{ with } \phi_1 = 1.$$

Since from Rogosinski's Lemma we have

$$|p_m| = \left|\frac{J^{(m)}(0)}{m!}\right| \leq |\Delta'(0)|, \ (m \in \mathbb{N}_2),$$

then, we get

$$|\phi_n| \leq \frac{|c||\Delta'(0)|}{n-1}\sum_{m=1}^{n-1}|\phi_{n-m}|. \tag{5}$$

To prove our claim, observe that for $n = 2, 3, 4, \cdots$, using (5) we get

$$|\phi_2| \leq |c||\Delta'(0)|, \ |\phi_1| = 1.$$

and

$$|\phi_3| \leq \frac{1}{2}|c|\left|\Delta'(0)\right|(|\phi_2|+|\phi_1|) = \frac{\prod_{m=0}^{1}\left(m+|c|\left|\Delta'(0)\right|\right)}{2!}$$

$$|\phi_4| \leq \frac{1}{3}|c|\left|\Delta'(0)\right|(|\phi_3|+|\phi_2|+|\phi_1|) = \frac{\prod_{m=0}^{2}\left(m+|c|\left|\Delta'(0)\right|\right)}{3!}.$$

Hence, by induction one gets

$$|\phi_n| \leq \frac{\prod_{m=0}^{n-2}\left(m+|c|\left|\Delta'(0)\right|\right)}{(n-1)!}.$$

Now using the relation (4) between ϕ_n and a_n, the claim is obtained. □

The followings are immediate consequences of our main result.

Corollary 2.3. *Let $\Delta(z) = \frac{1+\Lambda z}{1+\Upsilon z}$ and $f \in \mathcal{S}^*_\Delta(\nu, c)$, then*

$$|a_n| \leq \frac{|\Gamma(n+1-\nu)|}{\Gamma(n+1)\,|\Gamma(2-\nu)|}\frac{\prod_{m=0}^{n-2}\left(m+|c|\,|\Lambda-\Upsilon|\right)}{(n-1)!}.$$

Corollary 2.4. *Let $\Delta(z) = \frac{1+z}{1-z}$ and $f \in \mathcal{S}^*_\Delta(\nu, c)$, then*

$$|a_n| \leq \frac{|\Gamma(n+1-\nu)|}{\Gamma(n+1)\,|\Gamma(2-\nu)|}\frac{\prod_{m=0}^{n-2}\left(m+2\,|c|\right)}{(n-1)!}.$$

Remark 2.5. *Letting $c = 1$ in Corollary 2.3, we recover the coefficient estimate for $\mathcal{S}^*_\nu[\Lambda, \Upsilon]$ proved by Çağlar et al.* [3]. *Furthermore, assigning particular values to the parameters in Theorem 2.2 one can deduce coefficient bounds for the classes $\mathcal{C}[\Lambda, \Upsilon]$, $\mathcal{S}^*[\Lambda, \Upsilon]$, $\mathcal{S}^*(c)$, $\mathcal{C}(c)$, $\mathcal{C}(\beta)$, $\mathcal{S}^*(\beta)$ and various other classes. The details are left to the reader.*

Next, we consider bounds for coefficients of functions in $\mathcal{SQ}_\Delta(\nu, c, \lambda)$. This result is also a consequence of Theorem 2.2.

Theorem 2.6. *Let $f \in Q$. If $f \in \mathcal{SQ}_\Delta(\nu, c, \lambda)$, then*

$$|a_n| \leq \frac{(1+\lambda)(2+\lambda)}{(n+1+\lambda)(n+\lambda)}\left(\frac{|\Gamma(n+1-\nu)|}{\Gamma(n+1)\,|\Gamma(2-\nu)|}\right)^2\frac{\prod_{m=0}^{n-2}\left(m+|c|\left|\Delta'(0)\right|\right)}{(n-1)!}.$$

Proof. Let $f \in \mathcal{SQ}_\Delta(\nu, c, \lambda)$, then by definition there is a function

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n \in \mathcal{S}^*_\Delta(\nu, c),$$

such that (2) holds. Thus we have for $\lambda \in \mathbb{R}\backslash(-\infty,-1]$

$$a_n = \left[\frac{(1+\lambda)(2+\lambda)}{(n+\lambda)(n+1+\lambda)}\right]\left(\frac{|\Gamma(n+1-\nu)|}{\Gamma(n+1)|\Gamma(2-\nu)|}\right)h_n, \quad (n \in \mathbb{N}_2).$$

Now by application of Theorem 2.2, the desired result is straightforward. □

Remark 2.7. *One can deduce interesting results from Theorem 2.6 by taking $\Delta(z)$ as discussed in Corollaries 2.3 and 2.4. The details are left to the reader.*

3. Distortion and radius inequalities for $\mathcal{S}^*(\nu, c, \Lambda, \Upsilon)$

Now, we will consider distortion and radius inequalities for functions in $\mathcal{S}^*(\nu, c, \Lambda, \Upsilon)$.

Theorem 3.1. *Let $f \in \mathcal{S}^*(\nu, c, \Lambda, \Upsilon)$, then*

$$M_1(\Lambda, \Upsilon, c, r, \nu) \le |D_z^\nu f(z)| \le M_2(\Lambda, \Upsilon, c, r, \nu), \quad \Upsilon \neq 0,$$

$$\frac{r^{1-\nu}}{\Gamma(2-\nu)} e^{-r|c|\Lambda} \le |D_z^\nu f(z)| \le \frac{r^{1-\nu}}{\Gamma(2-\nu)} e^{r|c|\Lambda}, \quad \Upsilon = 0.$$

where

$$\begin{aligned} M_1(\Lambda, \Upsilon, c, r, \nu) &= \frac{r^{1-\nu}}{\Gamma(2-\nu)} \frac{(1-\Upsilon r)^{[|c|+\Re(c)]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)}}{(1+\Upsilon r)^{[|c|-\Re(c)]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)}}, \\ M_2(\Lambda, \Upsilon, c, r, \nu) &= \frac{r^{1-\nu}}{\Gamma(2-\nu)} \frac{(1+\Upsilon r)^{[|c|+\Re(c)]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)}}{(1-\Upsilon r)^{[|c|-\Re(c)]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)}}. \end{aligned}$$

This inequality is sharp.

Proof. For $J \in \Pi[\Lambda, \Upsilon]$, it has been proved that J verifies [10]

$$\begin{aligned} \left| J(z) - \frac{1-\Lambda\Upsilon r^2}{1-\Upsilon^2 r^2} \right| &\le \frac{(\Lambda-\Upsilon) r}{1-\Upsilon^2 r^2}, \quad \Upsilon \neq 0, \\ |J(z) - 1| &\le \Lambda r, \qquad \Upsilon = 0. \end{aligned}$$

As $f \in \mathcal{S}^*(\nu, c, \Lambda, \Upsilon)$, so for $\Upsilon \neq 0$, this gives us

$$\left| 1 + \frac{1}{c}\left(\frac{z(\Phi^\nu f(z))'}{\Phi^\nu f(z)} - 1\right) - \frac{1-\Lambda\Upsilon r^2}{1-\Upsilon^2 r^2} \right| \le \frac{(\Lambda-\Upsilon) r}{1-\Upsilon^2 r^2}.$$

After simplification

$$\left| \frac{z(\Phi^\nu f(z))'}{\Phi^\nu f(z)} - \frac{1-\Upsilon[\Upsilon + c(\Lambda-\Upsilon)] r^2}{1-\Upsilon^2 r^2} \right| \le \frac{|c|(\Lambda-\Upsilon) r}{1-\Upsilon^2 r^2},$$

and further manipulation implies that

$$m_1(\Lambda, \Upsilon, c, r) \le \Re\left(\frac{z(\Phi^\nu f(z))'}{\Phi^\nu f(z)}\right) \le m_2(\Lambda, \Upsilon, c, r),$$

where

$$m_1(\Lambda,\Upsilon,c,r) = \frac{1-\Upsilon\left[\Upsilon+\Re(c)\left(\Lambda-\Upsilon\right)\right]r^2-|c|\left(\Lambda-\Upsilon\right)r}{1-\Upsilon^2r^2},$$

$$m_2(\Lambda,\Upsilon,c,r) = \frac{1-\Upsilon\left[\Upsilon+\Re(c)\left(\Lambda-\Upsilon\right)\right]r^2+|c|\left(\Lambda-\Upsilon\right)r}{1-\Upsilon^2r^2}.$$

Since

$$\Re\left(\frac{z\left(\Phi^\nu f\left(z\right)\right)'}{\Phi^\nu f\left(z\right)}\right) = r\frac{\partial}{\partial r}\log\left|\Phi^\nu f\left(z\right)\right|,$$

we get

$$\frac{m_1(\Lambda,\Upsilon,c,r)}{r} \le \frac{\partial}{\partial r}\log\left|\Phi^\nu f\left(z\right)\right| \le \frac{m_2(\Lambda,\Upsilon,c,r,)}{r}.$$

The above inequality gives the desired result by integration from 0 to r and using (1). For $\Upsilon = 0$, the result is simple, and thus we complete the proof. □

Remark 3.2. *To verify that the result is sharp, one may use as extremal function*

$$\Phi^\nu f\left(z\right) = \begin{cases} z\left(1+\Upsilon z\right)^{\frac{(\Lambda-\Upsilon)c}{\Upsilon}}, & \Upsilon \neq 0 \\ ze^{c\Lambda z}, & \Upsilon = 0. \end{cases}$$

or

$$D_z^\nu f\left(z\right) = \begin{cases} \frac{1}{\Gamma(2-\nu)}z^{1-\nu}\left(1+\Upsilon z\right)^{\frac{(\Lambda-\Upsilon)c}{\Upsilon}}, & \Upsilon \neq 0 \\ \frac{1}{\Gamma(2-\nu)}z^{1-\nu}e^{c\Lambda z}, & \Upsilon = 0. \end{cases}$$

Remark 3.3. *For $c = 1$, we obtain immediately the distortion inequalities of Çağlar et al.* [3]*. Also, on letting c real such that $0 \le c < 1$, then we obtain, for $\Upsilon \neq 0$*

$$\frac{r^{1-\nu}\left(\frac{1-\Upsilon r}{1+\Upsilon r}\right)^{\frac{(1-c)(\Lambda-\Upsilon)}{\Upsilon}}}{\Gamma\left(2-\nu\right)} \le \left|D_z^\nu f\left(z\right)\right| \le \frac{r^{1-\nu}\left(\frac{1+\Upsilon r}{1-\Upsilon r}\right)^{\frac{(1-c)(\Lambda-\Upsilon)}{\Upsilon}}}{\Gamma\left(2-\nu\right)},$$

and $\Upsilon = 0$ gives

$$\frac{r^{1-\nu}e^{-(1-c)\Lambda}}{\Gamma\left(2-\nu\right)} \le \left|D_z^\nu f\left(z\right)\right| \le \frac{r^{1-\nu}e^{(1-c)\Lambda}}{\Gamma\left(2-\nu\right)}.$$

The next theorem presents the radius of largest disk in which $f\left(z\right)$ is starlike.

Theorem 3.4. *The radius of starlikeness $r_{\mathcal{S}^*}$ for $|z| = r < r_{\mathcal{S}^*}$ $(0 < r < 1)$ is given by*

$$r_{\mathcal{S}^*} = \frac{|\Upsilon|\left(\Lambda-\Upsilon\right)-\sqrt{|c|^2\left(\Lambda-\Upsilon\right)^2+4\Upsilon\left[\Upsilon+\Re(c)\left(\Lambda-\Upsilon\right)\right]}}{-2\Upsilon\left[\Upsilon+\Re(c)\left(\Lambda-\Upsilon\right)\right]}.$$

This result is sharp.

Proof. From Theorem 3.1 we can write

$$\Re\left(\frac{z\left(\Phi^\nu f\left(z\right)\right)'}{\Phi^\nu f\left(z\right)}\right) \ge \frac{1-\Upsilon\left[\Upsilon+\Re(c)\left(\Lambda-\Upsilon\right)\right]r^2-|c|\left(\Lambda-\Upsilon\right)r}{1-\Upsilon^2r^2}.$$

This result is true for $r < r_{\mathcal{S}^*}$, if

$$r_{\mathcal{S}^*} = \frac{|\Upsilon|\left(\Lambda-\Upsilon\right)-\sqrt{|c|^2\left(\Lambda-\Upsilon\right)^2+4\Upsilon\left[\Upsilon+\Re(c)\left(\Lambda-\Upsilon\right)\right]}}{-2\Upsilon\left[\Upsilon+\Re(c)\left(\Lambda-\Upsilon\right)\right]}.$$

□

Remark 3.5. *The sharpness can be check by using the function given as*

$$\Phi^{\nu} f\left(z\right) = z\left(1+\Upsilon z\right)^{\frac{c(\Lambda-\Upsilon)}{\Upsilon}}.$$

Remark 3.6. *On letting* $c = \gamma e^{-i\theta}\cos\theta, \quad (|\theta| < \pi/2,\ \gamma \in \mathcal{C}^*)$*, all of the presented results coincides with those reported in* [9].

4. Conclusion

In this paper, a new subfamily of complex order has been investigated defined by using Srivastava-Owa fractional operator. Coefficient bounds, radius and distortion inequalities have been obtained which were best possible. Several links were also highlighted to the earlier reported results. The contents of this paper brought unification to those results obtained by some researchers.

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