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Solving fractional Bratu's equations using a semi-analytical technique

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Abstract. Finding the solution of the fractional Bratu's differential equations (FBDEs) in this paper is based on a semi-analytical iterative approach. Temimi and Ansari introduced this method and called it TAM. Three examples, with their approximate solutions, are presented in this way to show its suitability, convenience, simplicity and efficiency. The results demonstrate that the advantage of this method to other methods is that there are no limiting conditions for nonlinear fractional differential equations with initial conditions or boundary conditions. Regarding the help of the software $Mathematica$, all the results have been obtained and the calculations have been done.

AMS (MOS) Subject Classification Codes: 26A33; 43A08; 35R11.
Key Words: Semi-analytical method; Analytic solution; Bratu's differential equations; Caputo derivative.

1. Introduction

A problem of the non-linear eigenvalue problem in n dimensions is the Bratu differential equations (BDEs) as follows [25]

$$\Delta\Phi(t) + \lambda \exp(\Phi(t)) = 0, \tag{1. 1}$$

in which $t = (t_1, t_2, \cdots, t_n)$, Δ denotes the n-dimensional Laplace operator and $|t_i| \leq 1$ for $i = 1, 2, \ldots, n$, with the following initial conditions for $|t_i| = 1$

$$\Phi(t) = 0. \tag{1. 2}$$

In this paper, we consider one-dimensional (*1D*) BDEs

$$u''(t) + \lambda \exp(u(t)) = 0, \quad 0 < t \leq T, \tag{1. 3}$$

$$u(0) = u_0, \;\; u_t(0) = u'_0. \tag{1. 4}$$

where $\lambda > 0$ and $t \in \mathbb{R}$ are constant functions, and the analytic solution is presented as follows:

$$u(t) = log\left(\frac{cosh\left(\frac{\phi}{2}(t-\frac{1}{2})\right)}{cosh\left(\frac{\phi}{4}\right)}\right)^{-2},$$

in which ϕ is the solution of $\phi = \sqrt{2\lambda}\, cosh\left(\frac{\phi}{4}\right)$ [24, 42]. Whereas $\lambda_\epsilon = 3.513830719$, the BDEs has

- one solutions when $\lambda = \lambda_\epsilon$,
- two solutions if $\lambda < \lambda_\epsilon$,
- no solution when $\lambda > \lambda_\epsilon$.

The Bratu's problem has a long history and it was introduced by Bratu in 1914 [8]. The Bratu problem appears in a large variety of application areas such as the fuel ignition model of thermal combustion, radiative heat transfer, thermal reaction, the Chandrasekhar model of the expansion of the universe, chemical reactor theory and nanotechnology [21, 41, 23, 32]. In [21] a summary of the history of the problem is given.

On the motivation and significance of BDEs, it should be noted that it has a key role in many of the physical phenomena, chemical models and other sciences. Such applications include the model of thermal reaction process, the fuel ignition model of the thermal combustion theory, the Chandrasekhar model of the expansion of the universe, the radiative heat transfer nanotechnology and the chemical reaction theory [21, 42, 15, 32, 24].

As another instance, mathematical modeling in chemistry for the electro-spinning process is related to BDEs via thermo-electro-hydrodynamics balance equations. Colantoni and his co-author [10] represented a model that is the mono-dimensional Bratu equation as follows:

$$u''(t) + \lambda \exp(u(t)) = 0, \tag{1. 5}$$

featuring $\lambda = -\frac{18\,E^2(I - r^2 k\,E)^2}{\rho^2\,r^4}$, in which

- r is the radius of the jet at axial coordinate X in the Fig.1,
- I is the electrical current intensity,
- E is the electric area in the axial direction,
- ρ is the material density,
- k is a fixed value which is only dependent on temperature with regard to incompressible polymer.

Calculus and differential equations of non integer (or fractional differential equations (FDEs)) have many utilizations in the real world in different branches of sciences and topics of engineering. Some of these applications were offered by Sun et al. in [43]. These topics may be included in sciences such as physics, biology, environmental and disciplines of engineering such as control, signal processing, image processing, mechanics, dynamic systems.

We invite interested readers to check some informative books which have been written to get a better grasp of calculus with non integer derivative and non integer integral [6, 29, 35].

In this research work, we have, for the first time, shown that it is possible to use Temimi and Ansari method (TAM) to tackle with fractional Bratu's differential equations (FBDEs)

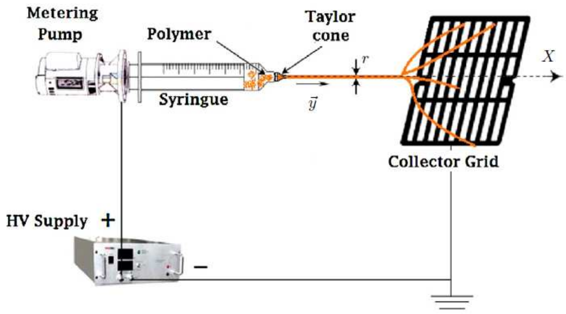

FIGURE 1. Electro-spinning process setup.

of the following form:

$$D^{\alpha}u(t) + \lambda \exp(u(t)) = 0, \;\; 1 < \alpha \leq 2, \;\; 0 < t \leq T, \tag{1. 6}$$
$$u(0) = u_0, \;\; u'(0) = u'_0. \tag{1. 7}$$

The operator D^{α} denotes the Caputo's derivative [29] of order α

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{\alpha-1} u^{(n)}(s) ds, \; t > a, \;\; n-1 < \alpha \leq n, \;\; n \in \mathbb{N}. \tag{1. 8}$$

Approximate methods have been introduced and used by many researchers to solve the BDEs and FBDEs. We will refer in the following part to the most widely used methods, including homotopic perturbation method [16, 17, 14], neural networks [37], finite difference method [33], differential transform method [20], optimal homotopy asymptotic method [11], wavelet method [28], Laplace transform decomposition method [27], B-splines method [9], variational iteration technique and modified variational iteration technique [12, 18], Adomian decomposition method [42, 19], differential quadrature method [36], Lie-group shooting method [1], reproducing kernel Hilbert space method [3, 5], pseudo-spectral collocation method [7], Picard's method [40], Sinc-Galerkin method [31], Taylor wavelets method [26], radial basis functions method [25] and etc [39, 23, 30]. We can solved BDEs and FBDEs with methods have been referred to in [2, 13, 4, 22].

As a preparation, in Section 2, we first elaborate on the methodology of TAM. In Section 3, convergence of this method and error analysis are verified. In Section 5, we provide the applications and results.

2. THE METHODOLOGY OF TAM

To explain the TAM, assume the nonlinear differential equation below featuring boundary assumptions

$$\begin{cases} L\left[u(t)\right] + N\left[u(t)\right] + G(t) = 0, \\ B\left(u, \frac{du}{dt}\right) = 0, \end{cases} \tag{2. 9}$$

in which t represents the independent variable, $u(t)$ is the unfamiliar function, $B(*)$ is a boundary operator, $G(t)$ is a given familiar function, $L(*)$ is the linear operator and $N(*)$

is the nonlinear operator. For Eq. (1. 6), we consider $L[u(t)] = D^{\alpha}u(t)$, $N[u(t)] = \lambda\exp(u(t))$ and $G(t) = 0$.

The TAM will start with an initial guess $u_0(t)$. To gain function $u(t)$ as a solution, we solve the following system of equations boundary conditions problems:

$$\begin{cases} L[u_0(t)] + G(t) = 0,\ B\left(u_0, \frac{du_0}{dt}\right) = 0, \\ L[u_1(t)] + N[u_0(t)] + G(t) = 0,\ B\left(u_1, \frac{du_1}{dt}\right) = 0, \\ L[u_2(t)] + N[u_1(t)] + G(t) = 0,\ B\left(u_2, \frac{du_2}{dt}\right) = 0, \\ \vdots \\ L[u_{n+1}(t)] + N[u_n(t)] + G(t) = 0,\ B\left(u_{n+1}, \frac{du_{n+1}}{dt}\right) = 0. \end{cases} \tag{2. 10}$$

Then, by $u = \lim_{n\to\infty} u_n$ the solution is given.

3. Convergence of TAM and Error analysis

3.1. **Convergence of TAM.** The following topic and theorem are provided for convergence of the TAM.

Consider problem 2. 9 . Thus we have

$$\begin{aligned} y_0 &= u_0(t) \\ y_1 &= \mathfrak{K}(y_0) \\ y_2 &= \mathfrak{K}(y_0 + y_1) \\ &\dots \\ y_{n+1} &= \mathfrak{K}(y_0 + y_1 + \ldots + y_{n+1}), \end{aligned} \tag{3. 11}$$

in which operator $\mathfrak{K} = L^{-1}$ is found as follows for $m \geq 1$

$$\mathfrak{K}(y_m) = T_m - \sum_{i=0}^{m-1} y_i(t). \tag{3. 12}$$

The T_m in Eq. (3. 12) is the obtained solution by TAM

$$L[y_m(t)] + N\left[\sum_{i=0}^{m-1} y_i(t)\right] + G(t) = 0, \tag{3. 13}$$

in which $m \geq 1$. By an iterative process, we can get the solution as follows:

$$u(t) = \lim_{n\to\infty} u_n(t) = \sum_{i=0}^{\infty} y_i.$$

The solution is in the form of the series $u(t) = \sum_{i=0}^{\infty} y_i(t)$ by using Eq. (3. 12) and Eq. (3. 13).

Theorem 3.2. *Let $\mathfrak{K}$ defined in Eq.* (3. 12)*, be an operator from a Hilbert space H to H. The series solution $u_n(t) = \sum_{i=0}^{n} y_i(t)$ converges if there exists $\theta \in (0, 1)$ such that*

$$\mathfrak{K}(y_0 + y_1 + \ldots + y_{i+1}) \leq \theta\mathfrak{K}(y_0 + y_1 + \ldots + y_i),$$

(such that $y_{i+1} \leq \theta y_i$) for all $i = 0, 1, 2, \ldots$.

This theorem is a special case of Banach's fixed point theorem which is a sufficient condition to study the convergence.

Proof. See [34]. $\square$

Theorem 3.3. *Suppose operator $\mathfrak{K}$ considered in Eq. (3. 12) be an operator of a Hilbert space H to H. If there exists $\theta \in (0,\ 1)$ such that $\|y_{i+1}\| \leq \theta\ \|y_i\|$ for all $i \geq i_0$ for some $i_0 \in \mathbb{N}$, then the series solution $\sum\limits_{i=0}^{m-1} y_i$ is convergent.*

Proof. Suppose the sequences $\{V_p\}_{p=0}^{\infty}$ specified with

$$\begin{aligned} V_0 =& y_0 \\ V_1 =& y_0 + y_1, \\ V_2 =& y_0 + y_1 + y_2, \\ & \ldots \\ V_p =& y_0 + y_1 + y_2 + \ldots + y_p. \end{aligned} \tag{3. 14}$$

It is enough to show that in the Hilbert space $\mathbb{R}$ the sequence $\{V_p\}_{p=0}^{\infty}$ is a Cauchy sequence. For this target, suppose

$$\begin{aligned} \|V_{p+1} - V_p\| &= \|y_{p+1}\| \\ &\leq \theta \|y_p\| \\ &\leq \theta^2 \|y_{p-1}\| \\ &\vdots \\ &\leq \theta^{n-i_0+1} \|y_{i_0}\|. \end{aligned}$$

Supposing that $p \geq q > i_0$ and for every $p, q \in \mathbb{N}$, we have

$$\begin{aligned} \|V_p - V_q\| &= \|(V_p - V_{p-1}) + (V_{p-1} - V_{p-2}) + \ldots + (V_q - V_{q-1})\| \\ &\leq \|(V_p - V_{p-1})\| + \|(V_{p-1} - V_{p-2})\| + \ldots + \|(V_q - V_{q-1})\| \\ &\leq \theta^{n-i_0} \|y_{i_0}\| + \theta^{p-i_0-1} \|y_{i_0}\| + \ldots + \theta^{m-i_0+1} \|y_{i_0}\| \\ &= \theta^{q-i_0+1} \left(\frac{1-\theta^{p-q}}{1-\theta} \right) \|y_{i_0}\|. \end{aligned}$$

It arrives at $\lim\limits_{\substack{p\to\infty \\ q\to\infty}} \|V_p - V_q\| = 0$, with regard to the $\theta \in (0,\ 1)$. So, in the Hilbert space $\mathbb{R}$, sequence $\{V_p\}_{p=0}^{\infty}$ is a Cauchy sequence and this implies that the series solution converges to series $\sum\limits_{i=0}^{\infty} y_i(t)$. $\square$

3.4. **Error analysis.** In this subsection, to provide an error analysis and the convergence criteria, we first recall the definition of L^2-norm on a certain domain ϕ for any continuous function h:

$$\|h\| = \sqrt{\int_\phi h^2 \, d\phi}.$$

In the following part, we present four convergence criteria in order to help them analyze the error analysis for the results of computations.

- The formula for calculation of the absolute error is given by

$$E_n = |u_n(t) - u_{Exact}(t)| .$$

- The formula for calculation of the consecutive error is given by

$$C_n = \|u_{n+1} - u_n\| .$$

- The formula for calculation of the L^2-norm reference error with respect to the exact solutions is given by

$$R_n = \|u_{Exact} - u_n\| .$$

- The formula for calculation of the residual error is given by

$$Res_n = \|L\,(u_n\,(t)) + N\,(u_n\,(t)) + G\,(t)\| .$$

4. Applications and results

Various examples in this section are now provided to help reader get familiar with the TAM for FBDEs. The software Mathematica in these examples has been utilized for computations and graphs.

Example 4.1. *We offer the FBDE equation for the first example:*

$$D^\alpha u(t) - 2\exp(u(t)) = 0, \quad 0 \le t \le 1, \; 1 < \alpha \le 2, \tag{4. 15}$$

with the exact solution $u(t) = u(t) = \log((\cos t)^{-2})$ *for* $\alpha = 1$ *and the initial conditions:*

$$u(0) = 0, \qquad u'(0) = 0. \tag{4. 16}$$

Following the TAM, according to what was formulated and presented in section 2 for Eqs.(4. 15)-(4. 16)*, we can calculate* $u_1, u_2, \ldots, u_n$ *and then gain the approximate solution* $u_n(t)$ *of* (4. 15).

The approximated solutions for $\alpha = 2$ *with* u_3*, which are obtained via various values of* t *in Table 1 is illustrated.*

We may see the exact and approximate solution via $\alpha = 2$*, in Figure 2.*

Example 4.2. *We offer the FBDE for the second example:*

$$D^\alpha u(t) + \pi^2 \exp(-u(t)) = 0, \tag{4. 17}$$
$$0 \le t \le 1, \; 1 < \alpha \le 2,$$

via the exact solution $u(t) = \log(1 + \sin(\pi t))$ *for* $\alpha = 2$ *and the initial conditions:*

$$u(0) = 0, \qquad u'(0) = \pi. \tag{4. 18}$$

TABLE 1. Comparative outcomes of Example 4.1.

t	TAM	$Exact$	*Absolute error*
0.0	*0.0*	*0.0*	*0.0*
0.2	*0.0402695*	*0.0402695*	*1.05948* $\times 10^{-9}$
0.4	*0.164458*	*0.164458*	*308.693*$\times 10^{-9}$
0.6	*0.38392*	*0.38393*	*9.94015* $\times 10^{-9}$
0.8	*0.72264*	*0.722781*	*141.182* $\times 10^{-6}$
1.0	*1.22983*	*1.23125*	*1.41966*$\times 10^{-3}$

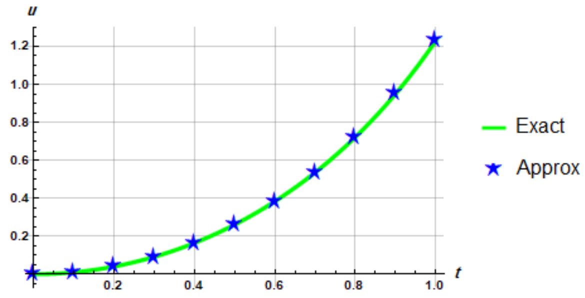

FIGURE 2. Agreement *TAM* for Eq.(4. 15) and exact solution.

The unknown coefficients u_i, $i = 1, 2, \ldots, n$ *with the TAM, matching to section 2 for Eq.* (4. 17) *are determined.*

In Figure 2 and in Table 2, the exact and third approximate answers featuring different values α *through applying TAM can be seen.*

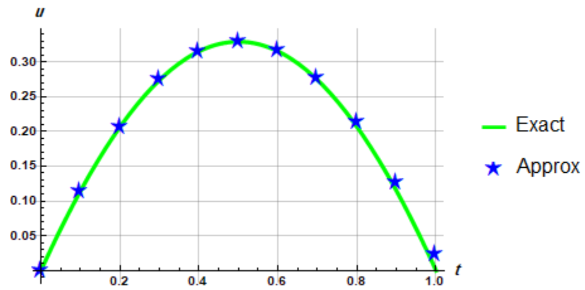

FIGURE 3. Agreement TAM for Eq.(4. 17) and exact solution.

Example 4.3. *We offer the FBDE for the third example:*

$$D^\alpha u(t) + 2\exp(u(t)) = 0, \quad 0 \le t \le 1, \ 1 < \alpha \le 2, \tag{4. 19}$$

including the following initial conditions and the exact solution $u(t) = -2\log(\cosh(t))$*:*

$$u(0) = 0, \qquad u'(0) = 0. \tag{4. 20}$$

TABLE 2. Approximate result of example 4.2 with various values of α.

t	TAM $\alpha = 1.7$	TAM $\alpha = 1.8$	TAM $\alpha = 1.9$	TAM $\alpha = 2$	*Exact*
0.0	*0.0*	*0.0*	*0.0*	*0.0*	*0.0*
0.2	*0.413879*	*0.419304*	*0.408313*	*0.462341*	*0.46234*
0.4	*0.537005*	*0.551575*	*0.522075*	*0.668414*	*0.668371*
0.6	*0.426693*	*0.453927*	*0.398671*	*0.66901*	*0.668371*
0.8	*0.0628188*	*0.109603*	*0.0142946*	*0.467205*	*0.46234*

In Figure 4 and in Table 3, the exact and third approximate solutions featuring various values α through applying TAM can be seen.

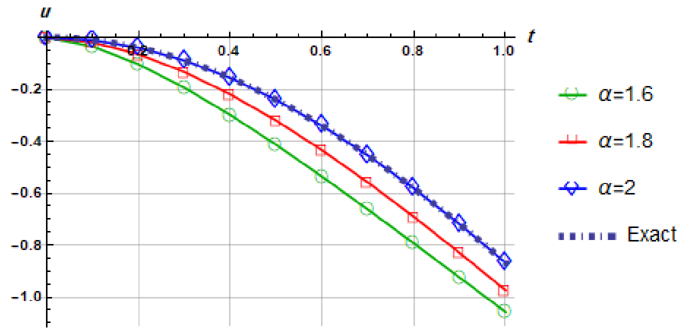

FIGURE 4. Comparative outcomes via TAM of Eq.(4. 19) for various values of t and α.

TABLE 3. Comparative outcomes of Example 4.3

t	TAM	$Exact$	*Absolute error*
0.0	*0.0*	*0.0*	*0.0*
0.2	*-0.0397361*	*-0.0397361*	*974.547* $\times 10^{-12}$
0.4	*-0.155907*	*-0.155907*	*220.863*$\times 10^{-9}$
0.6	*-0.340275*	*-0.340271*	*4.65491* $\times 10^{-6}$
0.8	*-0.581543*	*-0.581507*	*35.7172* $\times 10^{-6}$
1.0	*-0.867714*	*-0.867562*	*152.138*$\times 10^{-6}$

Fig.5 shows the absolute error for various values of $0 \leq t \leq 1$ *for* $\alpha = 2$. *Table 4 illustrates an absolute error comparison of the TAM and approximate methods: Block Nyström method (BNM)* [24], *Non-polynomial spline (NPS)* [23], *Laplace transform method (LTM)* [32], *Decomposition method (DM)* [30], *B-splines method (BSM)* [9], *Lie-group shooting method (LGSM)* [1] *and Sinc-collocation method (SCM)* [38].

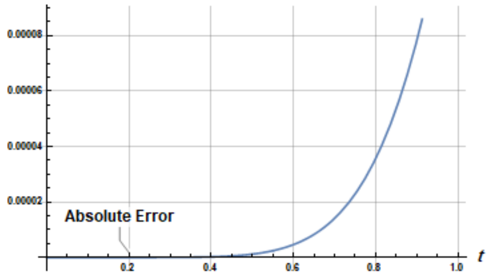

FIGURE 5. Absolute error for test example 4.3.

TABLE 4. Absolute error comparison of example 4.3.

t	BNM	NPS	LTM	DM	BSM	LGSM	SCM	TAM
0.1	1.91×10^{-14}	9.71×10^{-9}	2.13×10^{-3}	1.52×10^{-2}	1.72×10^{-5}	4.03416×10^{-6}	6.88×10^{-4}	0
0.3	1.17×10^{-13}	1.98×10^{-8}	6.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}	5.22122×10^{-6}	8.21×10^{-4}	955.649×10^{-9}
0.5	1.88×10^{-13}	2.60×10^{-8}	9.60×10^{-3}	6.98×10^{-3}	5.56×10^{-5}	1.4554×10^{-8}	8.60×10^{-4}	322.33×10^{-6}
0.7	1.16×10^{-13}	1.98×10^{-8}	1.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}	5.19455×10^{-6}	8.21×10^{-4}	2.72804×10^{-3}
0.9	1.90×10^{-14}	9.71×10^{-9}	1.09×10^{-3}	1.52×10^{-3}	1.72×10^{-5}	4.01345×10^{-6}	6.88×10^{-4}	12.6937×10^{-3}

5. CONCLUSION

We have efficiently utilized TAM to acquire approximate solution of the fractional Bratu differential equations (FBDEs). The results demonstrate that via few iterations of TAM, we can achieve useful approximate solutions.

Finally, it should be noted that the suggested technique can be utilized for solving fractional integral equations *FIEs*, fractional integro partial differential equations *FIPDEs*, fractional differential equations *FDEs*, fractional partial differential equations *FPDEs*, fractional differential system equations *FDSEs* and fractional partial differential system equations *FPDSEs*.

REFERENCES

[1] S. Abbasbandy, M. S. Hashemi, C. S. Liu, *The Lie-group shooting method for solving the Bratu equation*, Communications in Nonlinear Science and Numerical Simulation. **16**(11), (2011) 4238-4249.
[2] M. Abbas, B. Zafar, *New Quartic B-Spline Approximation for Numerical Solution of Third Order Singular Boundary Value Problems*, Punjab Univ. j. math. **51**(5), (2019) 43-59.
[3] A. Akgul, F. Geng, *Reproducing kernel Hilbert space method for solving Bratu's problem*, Bulletin of the Malaysian Mathematical Sciences Society. **38**(1), (2015) 271-287.
[4] S. Akhtar, G. Mustafa, *Semi-Analytical Solutions of Multilayer Flow of Viscous Fluids in a Channel*, Punjab Univ. j. math. **51**(4), (2019) 103-112.
[5] E. Babolian, S. Javadi, E. Moradi, *RKM for solving Bratu type differential equations of fractional order*, Mathematical Methods in the Applied Sciences. **39**(6), (2016) 1548-1557.
[6] D. Baleanu, A. C. Luo, *Discontinuity and Complexity in Nonlinear Physical Systems*, J. T. Machado (Ed.). Springer, 2014.
[7] J. P. Boyd, *One-point pseudospectral collocation for the one-dimensional Bratu equation*, Applied Mathematics and Computation. **217**(12), (2011) 5553-5565.
[8] G. Bratu, *Sur les équations intégrales non linéaires*, Bull. Soc. Math. France. 42, (1914) 113-142.
[9] H. Caglar, N. Caglar, M. Ozer, A. Valaristos, A. N. Anagnostopoulos, *B-spline method for solving Bratu's problem*, International Journal of Computer Mathematics. **87**(8), (2010) 1885-1891.

[10] A. Colantoni, K. Boubaker, *Electro-spun organic nanofibers elaboration process investigations using comparative analytical solutions*, Carbohydrate polymers. **101**, (2014) 307-312.

[11] M. A., Darwish, B. S. Kashkari, *Numerical solutions of second order initial value problems of Bratu-type via optimal homotopy asymptotic method*, American Journal of Computational Mathematics. **4**(02), (2014) 47.

[12] N. Das, R. Singh, A. M. Wazwaz, J. Kumar, *An algorithm based on the variational iteration technique for the Bratu-type and the Lane-Emden problems*, Journal of Mathematical Chemistry. **54**(2), (2016) 527-551.

[13] M. Didgar, A. Vahidi, J. Biazar, *Application of Taylor Expansion for Fredholm Integral Equations of the First Kind*, Punjab Univ. j. math. **51**(5), (2019) 1-14.

[14] V. P., R. Kumar, D. Kumar, *Analytical study of fractional Bratu-type equation arising in electro-spun organic nanofibers elaboration*, Physica A: Statistical Mechanics and its Applications. 2019.

[15] D. A. Frank-Kamenetskii, *Diffusion and heat exchange in chemical kinetics*, Princeton University Press. 2015.

[16] X. Feng, Y. He, J. Meng, *Application of homotopy perturbation method to the Bratu-type equations*, Topological Methods in Nonlinear Analysis. **31**(2), (2008) 243-252.

[17] B. Ghazanfari, A. Sepahvandzadeh, *Homotopy perturbation method for solving fractional Bratu-type equation*, Journal of Mathematical Modeling. **2**(2), (2015) 143-155.

[18] B. Ghazanfari, A. Sepahvandzadeh,*Solving fractional Bratu-type equations by modified variational iteration method*, Selcuk Journal of Applied Mathematics. **15**(1), (2013).

[19] B. Ghazanfari, A. Sepahvandzadeh, *Adomian decomposition method for solving fractional Bratu-type equations*, J. Math. Comput. Sci. **8**(3), (2014) 236-244.

[20] M. Grover, A. K. Tomer, *Numerical approach to differential equations of fractional order Bratu-type equations by differential transform method*, Global Journal of Pure and Applied Mathematics. 13(9), (2017) 5813-5826.

[21] J. Jacobsen, K. Schmitt, *The Liouville-Bratu-Gelfand problem for radial operators*, Journal of Differential Equations. **184**(1), (2002) 283-298.

[22] K. Jahangir, S. U. Rehman, F. Ahmad, A. Pervaiz, *Sixth-Order Stable Implicit Finite Difference Scheme for 2-D Heat Conduction Equation on Uniform Cartesian Grids with Dirichlet Boundaries*, Punjab Univ. j. math. **51**(5), (2019) 27-42.

[23] R. Jalilian, *Non-polynomial spline method for solving Bratu's problem*, Computer Physics Communications. **181**(11), (2010) 1868-1872.

[24] S. N. Jator, V. Manathunga, *Block Nystrom type integrator for Bratu's equation*, Journal of Computational and Applied Mathematics. **327**, (2018) 341-349.

[25] J. Karkowski, *Numerical experiments with the Bratu equation in one, two and three dimensions*, Computational and Applied Mathematics. **32**(2), (2013) 231-244.

[26] E. Keshavarz, Y. Ordokhani, M. Razzaghi, *The Taylor wavelets method for solving the initial and boundary value problems of Bratu-type equations*, Applied Numerical Mathematics. **128**, (2018) 205-216.

[27] S. A. Khuri, *A new approach to Bratu's problem*, Applied mathematics and computation. **147**(1), (2004) 131-136.

[28] Y. Liu, X., Zhou, X. Wang, J. Wang, *A wavelet method for solving a class of nonlinear boundary value problems*, Communications in Nonlinear Science and Numerical Simulation. **18**(8), (2013) 1939-1948.

[29] Kilbas, A. A., Srivastava, H. M., and Trujillo, J.J., (2006). *Theory and application of fractional differential equations*, Elsevier B.V, Netherlands.

[30] S. Liao, Y. Tan, *A general approach to obtain series solutions of nonlinear differential equations*, Studies in Applied Mathematics. **119**(4), (2007) 297-354.

[31] J., Maleknejad, K., N. Taheri, *Sinc-Galerkin method for numerical solution of the Bratu's problems*, Numerical Algorithms. **62**(1), (2013) 1-11.

[32] J. S. McGough, *Numerical continuation and the Gelfand problem*, Applied mathematics and computation. **89**(1-3), (1998) 225-239.

[33] A. Mohsen, *A simple solution of the Bratu problem*, Computers & Mathematics with Applications. **67**(1), (2014) 26-33.

[34] Z. M. Odibat, *A study on the convergence of variational iteration method*, Mathematical and Computer Modelling. **51**(9-10), (2010) 1181-1192.

[35] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Vol. **198**, Academic press 1998.

[36] O. Ragb, L. F. Seddek, Matbuly, M. S. *Iterative differential quadrature solutions for Bratu problem*, Computers & Mathematics with Applications. **74**(2), (2017) 249-257.

[37] M. A. Z. Raja, *Numerical treatment for solving one-dimensional Bratu problem using neural networks*, Neural Computing and Applications. **24**(3-4), (2014) 549-561.

[38] J. Rashidinia, K. Maleknejad, N. Taheri, *Sinc-Galerkin method for numerical solution of the Bratu's problems*, Numerical Algorithms. **62**(1), (2013) 1-11.

[39] H. Temimi, M. Ben-Romdhane, *An iterative finite difference method for solving Bratu's problem*, Journal of Computational and Applied Mathematics. **292**, (2016) 76-82.

[40] S. Tomar, R. K. Pandey, *An efficient iterative method for solving Bratu-type equations*, Journal of Computational and Applied Mathematics. **357**, (2019) 71-84.

[41] Y. Q. Wan, Q. Guo, N. Pan, *Thermo-electro-hydrodynamic model for electrospinning process*, International Journal of Nonlinear Sciences and Numerical Simulation. **5**(1), (2004) 5-8.

[42] A. M. Wazwaz, *Adomian decomposition method for a reliable treatment of the Bratu-type equations*, Applied Mathematics and Computation. **166**(3), (2005) 652-663.

[43] H. Sun, Y. Zhang, D. Baleanu, , W. Chen, Y. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Communications in Nonlinear Science and Numerical Simulation. **64**(2), (2018) 213-231.