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The Cubic B-spline Operational Matrix Based on Haar Scaling Functions for Solving Varieties of the Fractional Integro-differential Equations

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Abstract. Nowadays, the operational matrix plays an important role in solving problems with partial, ordinary or fractional derivatives. In the current study, we construct the operational matrix of the fractional integral for cubic B-spline scaling function and wavelets and it applies to solve varieties of the fractional integro-differential equations. To do this, firstly, the operational matrix of fractional integral for Haar scaling functions is constructed by using the definition of the Riemann-Liouville fractional integral operator and the orthogonal projection of polynomial on space of Haar scaling functions. Afterward, we obtain the operational matrix of cubic B-spline functions from fractional order using approximation cubic B-spline functions with Haar scaling functions and collocation method. The principal characteristics of this method are as follows: The operational matrix of cubic B-spline functions is obtained simply because of the useful properties of the Haar scaling functions, reducing the time, the less occupation of the computer memory which converts to a system of linear and nonlinear equations. Finally, we will show the validity and efficiency of the new method by numerical examples and convergence analysis.

AMS (MOS) Subject Classification Codes: 45G99; 65D99; 65R20
Key Words: Fractional calculus, Fractional integro- differential equation, Haar scaling function, B-spline scaling function and wavelets, Operational matrix.

1. Introduction

Recently, investigations in engineering, science, and other fields illustrate that there are many considerable numbers of phenomena that can be modeled by fractional calculus, in particular, the material that have the property of memory and hereditary effects, heat conduction in fractal porous media, sliding mode control and viscoelastic damping. Also, many articles cover progression and growing of fractional calculus in various fields such as bioengineering [19], mechanics[2], system identification and controls[6], Signal Processing[32], economics[4] and etc.

The focus in the current study is on solving different varieties of linear and nonlinear fractional integro-differential equations with a numerical method. At the first place, the operational matrix of fractional integration is obtained for Haar scaling functions by using the definition of the Riemann-Liouville fractional integral operator and the orthogonal projection of polynomial on space of Haar scaling functions. The Haar scaling functions have useful properties, in particular, they are an orthonormal basis of v_{j_u} and $\{v_{j_u}\}_{j_u \in \mathbb{Z}}$ is a multiresolution analysis for $L^2(\mathbb{R})$ [22, 5]. Afterward, the operational matrix of fractional integration for cubic B-spline scaling functions and wavelets is presented by approximating them with the Haar scaling functions. Our purpose is the conversion of the problem under consideration into a system of linear or nonlinear equations. Because of the properties of Haar scaling function, the conversion is easily done in the nonlinear fractional integro-differential equations. The semi-orthogonal B-spline scaling functions and wavelets used in the present paper have the properties of compact support, vanishing moments, smoothness function and the representation by a closed-form expression[5]. With these assumptions, time is reduced, computer memory is less occupied and the operation matrix is always available.

The wavelets can be generated by transmission and dilation a function along the line real and they are an orthogonal basis for space $L^2(\mathbb{R})$. Their rich mathematical content has been led to the vast applications in a variety of fields especially in modeling multiscale phenomena, signal processing, solving of partial differential equations, integrodifferential equation, statistics, data and image compression, soft computing, and etc.

In recent years, various methods have been presented to solve fractional intgro-differential equations, fractional differential equations, and fractional partial differential equations [1, 3]. The method based on the Adomian decomposition method has been presented in[24, 25, 9, 23]. Jaradat in [13] proposed the homotopy analysis method. Also, there are methods based on the variational iteration method and fractional differential transform method in [26, 35]. An increasing number of methods are based on to convert the varieties of fractional equations into the algebraic system by using wavelets and wavelet operational matrices, we refer the interested reader to[21, 29, 17, 15, 16, 34, 37, 12, 33, 10, 14, 18, 11, 38, 36, 27, 28]. These articles use Legendre wavelets, Haar wavelets, Chebyshev wavelets, B-spline wavelets, and CAS wavelets for varieties of problems including the fractional derivative.

The structure of this paper is organized as follows: In Section 2, we introduce some basic definitions and preliminaries of Haar scaling functions, cubic B-spline scaling functions and wavelets and fractional calculus. In Section 3, we describe some properties and results of the Haar scaling functions and relationships between them and the cubic B-spline scaling functions and wavelets. In Section 4, the operational matrix of fractional integration for Haar scaling function is analytically obtained and then it used for construction of the fractional operational matrix of the cubic B-spline scaling functions and wavelets. The implementation of the proposed method and error analysis are given in Sections of 5 and 6. In Section 7, some numerical examples are presented to demonstrate the validity and accuracy of this method.

2. PRELIMINARIES

This section expresses some necessary definitions about Haar scaling functions, cubic B-spline scaling functions and wavelets and the fractional calculus operators which are used further in this study.

2.1. The Haar scaling functions.

The function ϕ which is defined as follows is the function of Haar[8]:

$$\phi(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & O.W \end{cases} \tag{2. 1}$$

we put $\phi_{j_u,n}(x) = (2^{j_u+1}+3)^{\frac{1}{2}}\phi((2^{j_u+1}+3)x-n)$ then

$$\phi_{j_u,n}(x) = \begin{cases} (2^{j_u+1}+3)^{\frac{1}{2}} & x \in [\frac{n}{2^{j_u+1}+3}, \frac{n+1}{2^{j_u+1}+3}) \\ 0 & O.W \end{cases} \tag{2. 2}$$

The subspace of v_{j_u} is considered as $\overline{span}\{\phi_{j_u,n}(x), n \in \mathbb{Z}\}$ is the space of Haar scaling functions. It has been proved [8] that the family of $\{\phi_{j_u,n}(x), n \in \mathbb{Z}\}$ forms an orthogonal basis in $L^2(\mathbb{R})$, also this family is an orthonormal basis for v_{j_u}. The sequence of vector spaces $\{v_{j_u}\}_{j_u \in \mathbb{Z}}$ is Multi Resolution Analysis(MRA) for $L^2(\mathbb{R})$. The properties of MRA gives

$$... \subset v_{j_u-1} \subset v_{j_u} \subset v_{j_u+1} \subset ..., \quad \overline{\bigcup_{j_u \in \mathbb{Z}} v_{j_u}} = L^2(\mathbb{R}) \tag{2. 3}$$

($v_{j_u} \subset v_{j_u+1}$ refers to v_{j_u} is a proper subset of v_{j_u+1}.)
Because the v_{j_u} makes up an increasing chain, then

$$\bigcup_{j_u \in \mathbb{Z}} v_{j_u} = \lim_{j_u \to \infty} v_{j_u} \tag{2. 4}$$

So it can be said v_{j_u} when j_u limits infinity is dense in $L^2(\mathbb{R})$, then any $f(x) \in L^2(\mathbb{R})$ may be approximated by linear combination as

$$f(x) \simeq \sum_{n \in \mathbb{Z}} c_n \phi_{j_u,n}(x), \qquad c_n = (2^{j_u+1}+3)^{\frac{1}{2}} \int_{\frac{n}{2^{j_u+1}+3}}^{\frac{n+1}{2^{j_u+1}+3}} f(x)dx \tag{2. 5}$$

The condition of support $(\phi_{j_u,n}(x)) = [\frac{n}{2^{j_u+1}+3}, \frac{n+1}{2^{j_u+1}+3}]$, for $f(x) \in L^2[0,1]$ leads to

$$f(x) \simeq \sum_{n=0}^{2^{j_u+1}+2} c_n \phi_{j_u,n}(x) \tag{2. 6}$$

We define

$$c = [c_0, c_1, c_2, ..., c_{2^{j_u+1}+2}], \quad \gamma_{j_u}(x) = [\phi_{j_u,0}(x), \phi_{j_u,1}(x), ...\phi_{j_u,2^{j_u+1}+2}(x)]^T \tag{2. 7}$$

Therefore, (2. 6) in the form of a matrix can be expressed as follows:

$$f(x) \simeq c\gamma_{j_u}(x) \tag{2. 8}$$

2.2. **The cubic B-spline scaling functions and wavelets on [0,1].** The cubic B-spline scaling functions(Fourth-order B-spline) with knots sequence $X^{(j_0)}$ on [0,1] can be recursively defined by the following expansion [5, 7, 14]:

$$B_{X^{(j_0)},4;k}(x) = \frac{x - x_k}{x_{k+3} - x_k} B_{X^{(j_0)},3;k}(x) + \frac{x_{k+4} - x}{x_{k+4} - x_{k+1}} B_{X^{(j_0)},3;k+1}(x) \qquad (2.\ 9)$$

that

$$B_{X^{(j_0)},1;k}(x) = \chi[x_k, x_{k+1}](x), \qquad k = 0, 1, ..., 2^{j_0} - 1$$

$$X^{j_0} : x_{-3} = ... = x_0 = 0 < x_1 = \frac{1}{2^{j_0}} < x_2 = \frac{2}{2^{j_0}} < ... < x_k = \frac{k}{2^{j_0}} < \\ ... < 1 = x_{2^{j_0}} = ... = x_{2^{j_0}+3},$$

$$X^{j_0+1} : x_{-3} = ... = x_0 = 0 < x_1 = \frac{1}{2^{j_0+1}} < x_2 = \frac{2}{2^{j_0+1}} < ... < x_k = \frac{k}{2^{j_0+1}} \\ < ... < 1 = x_{2^{j_0+1}} = ... = x_{2^{j_0+1}+3.}$$

We put

$$\varphi_{j_0,k}(x) = B_{X^{(j_0)},4;k}(x), \qquad V_{j_0} = \overline{span}\{\varphi_{j_0,k}(x), k = -3, ..., 2^{j_0} - 1\} \qquad (2.\ 10)$$

support $(\varphi_{j_0,k}) = [2^{-j_0}k, 2^{-j_0}(k+4)] \bigcap [0,1]$[5].
Define the set of indices
$Q_{j0} = \{k \in \mathbb{Z} : [2^{-j_0}k, 2^{-j_0}(k+4)] \bigcap [0,1] \neq \varnothing\},\ j_0 \in \mathbb{Z}$
It is obvious that $\min Q_{j_0} = -3$ and $\max Q_{j_0} = 2^{j_0} - 1, j_0 \in \mathbb{Z}$.
Then $\varphi_{j_0,k}$ for $k = -3, -2, -1$ are left boundary, for $k = 2^{j_0} - 3, ..., 2^{j_0} - 1$ are right boundary and for $k = 0, ..., 2^{j_0} - 4$ are inner scaling function.

Also, the semi-orthogonal cubic B-spline wavelets corresponding with this scaling functions are described as follows. To facilitate our presentation, we define:

$$\begin{aligned} v_i &= \varphi_{j_0,i-4}(x) = B_{X^{(j_0)},4;i-4}(x), \quad i = 1, ..., 2^{j_0} + 3, \\ u_i &= \varphi_{j_0+1,i-4}(x) = B_{X^{(j_0+1)},4;i-4}(x), \quad i = 1, ..., 2^{j_0+1} + 3 \end{aligned} \qquad (2.\ 11)$$

It is obvious that V_{j_0} is a proper subspace of V_{j_0+1}, therefore, the orthogonal complementary subspace W_{j_0} of V_{j_0} relative to V_{j_0+1} according to the orthogonality conditions is expressed by the following equations [5].

$$W_{j_0} = \overline{span}\{\psi_{j_0,k}(x), k = -3, ..., 2^{j_0} - 4\}, \quad \psi_{j_0,i-4}(x) = w_i, \quad i = 1, ..., 2^{j_0} \quad (2.\ 12)$$

w_i is obtained by $w_i = \frac{1}{\|\Theta_i\|_2}\Theta_i$ that

$$\Theta_i = det\left(\begin{pmatrix} u_i & u_{i+1} & \cdots & u_{2i+6} \\ < u_i, v_1 > & < u_{i+1}, v_1 > & \cdots & < u_{2i+6}, v_1 > \\ \vdots & \vdots & \ddots & \vdots \\ < u_i, v_{i+6} > & < u_{i+1}, v_{i+6} > & \cdots & < u_{2i+6}, v_{i+6} > \end{pmatrix}\right), \\ i = 1, 2, 3$$

$$\Theta_i = det\left(\begin{pmatrix} u_{2i-4} & u_{2i-3} & \cdots & u_{2i+6} \\ < u_{2i-4}, v_{i-3} > & < u_{2i-3}, v_{i-3} > & \cdots & < u_{2i+6}, v_{i-3} > \\ \vdots & \vdots & \ddots & \vdots \\ < u_{2i-4}, v_{i+6} > & < u_{2i-3}, v_{i+6} > & \cdots & < u_{2i+6}, v_{i+6} > \end{pmatrix}\right), \quad i = 4, 5$$

($< u_i, v_j >, \ i = k, ..., 2k+6, \ j = 1, ..., k+6, \ k = 1, ..., 5$ refers to the inner product of functions u_i and v_j in the Hilbert space of $L^2(\mathbb{R})$.)
$w_i, i = 4, 5$ and $w_i, i = 1, 2, 3$ are interior B-spline wavelets and left boundary B-spline wavelets respectively, due to the symmetry property of B-spline wavelets for right boundary B-spline wavelets we will have:

$$w_i = \psi_{j_0, 5-i}(1-x), \qquad i = 6, 7, 8$$

Also $\psi_{j,k}(x)$ for $j \geqslant j_0$ is given by [20]:

$$\psi_{j,k}(x) = \begin{cases} \psi_{j_0,k}(2^{j-j_0}x) & k = -3, -2, -1 \\ \psi_{j_0, 2^j-7-k}(1 - 2^{j-j_0}x) & k = 2^j - 6, ..., 2^j - 4 \\ \psi_{j_0,0}(2^{j-j_0}x - \frac{k}{2^{j_0}}) & k = 0, ..., 2^j - 7 \end{cases} \tag{2. 13}$$

It can be proved that W_j for $j \in \mathbb{Z}$ is orthogonal complement V_j in V_{j+1} also $\{\psi_{j,k}, k, j \in \mathbb{Z}\}$ is Riesz basis for $L^2(\mathbb{R})$. Therefore

$$L^2(\mathbb{R}) = \sum_{j \in \mathbb{Z}} \oplus W_j = V_{j_0} \oplus \sum_{j=j_0}^{\infty} \oplus W_j = \overline{\bigcup_{j \in \mathbb{Z}} V_j} \tag{2. 14}$$

So according to (2. 14) the orthogonal projection any function $f(x) \in L^2([0,1])$, truncated in definite j_u and the lowest level $j_0 = 3$ can be expanded as follows:

$$f(x) \simeq \sum_{k=-3}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{j_u} \sum_{k=-3}^{2^j-4} d_{j,k}\psi_{j,k}(x) \tag{2. 15}$$

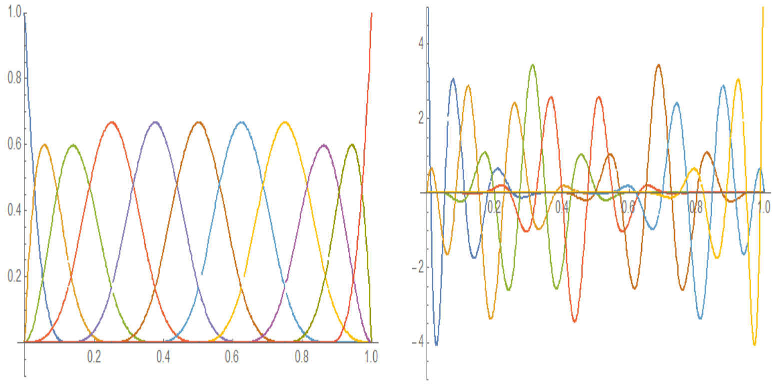
With assumption

$$\beta_{j_u}(x) = [\varphi_{j_0,-3}(x), ..., \varphi_{j_0,2^{j_0}-1}(x), \psi_{j_0,-3}(x), ..., \psi_{j_0,2^{j_0}-4}(x), \psi_{j_0+1,-3}(x), ..., \tag{2. 16}$$

$$\psi_{j_u,2^{j_u}-4}(x)]^T, \quad \rho = [c_{j_0,-3}, ..., c_{j_0,2_0^j-1}, d_{j_0,-3}, ..., d_{j_0,2^{j_0}-4}, d_{j_0+1,-3}, ..., d_{j_u,2^{j_u}-4}]$$

(2. 15) can be rewritten in the matrix form as

$$f(x) \simeq \rho\beta_{j_u}(x) \tag{2. 17}$$



(a) (b)

FIGURE 1. a. Scaling function of cubic B-spline in V_3 b. Wavelet cubic B-spline in W_3

2.3. **The fractional calculus.** In this part, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory such as Riemann-Liouville and Caputo fractional derivative[2, 6].
The Riemann-Liouville fractional integral of α order is defined as

$$ {}_aI_x^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt,\ x > a,\ \Re(\alpha) > 0. \qquad (2.\ 18)$$

The Caputo fractional derivative is expressed by:

$$ {}_{C_a}D_t^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{\varphi^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\ t \in \mathbb{R}^+,\ \alpha \notin \mathbb{N}. \qquad (2.\ 19)$$

In addition, the relationship between the Riemann-Liouvill integral and Caputo derivative for $n-1 < \alpha \leq n$ and $\varphi(t) \in C^{\lceil \alpha \rceil}[0,1]$ is described by:

$$ {}_0I_t^{\alpha}\ {}_{C_0}D_t^{\alpha}\varphi(t) = \varphi(t) - \sum_{j=0}^{\lceil \alpha \rceil - 1} \frac{t^j}{j!}(\frac{d^j}{dt^j}f)(0) \qquad (2.\ 20)$$

($\lceil \alpha \rceil$ referred to the smallest integer greater than or equal to α)

3. THE PROPERTIES OF HAAR SCALING FUNCTIONS AND THE IMPORTANT RELATIONSHIPS BETWEEN THEM AND CUBIC B-SPLINES

In this section, some specifications and results of Haar scaling functions in v_{j_u} space used for the easy accomplishment of the nonlinear fractional integro- differential equation are presented, subsequently, we describe the important relationships between the Haar scaling functions of v_{j_u} space and cubic B-spline scaling functions and wavelets.

Lemma 3.1. *Suppose that $f(x) \simeq c\gamma_{j_u}(x)$ is the approximation of the function $f(x) \in L^2[0,1]$ then we have for any integer number $k \geq 2$*

$$[f(x)]^k \simeq \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[((2^{j_u+1}+3)^{\frac{1}{2}}c_0)^k, ((2^{j_u+1}+3)^{\frac{1}{2}}c_1)^k, ..., ((2^{j_u+1}+3)^{\frac{1}{2}}c_{2^{j_u+1}+2})^k]\gamma_{j_u}(x) \quad (3.\ 21)$$

Proof. At first we prove that it is true for $k = 2$. By using equations (2. 2) and (2. 5) we have:

$$\gamma_{j_u}(x) = \begin{pmatrix} \phi_{j_u,0}(x) & 0 & \cdots & 0 \\ 0 & \phi_{j_u,1}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{j_u,2^{j_u+1}+2}(x) \end{pmatrix}$$

then

$$[f(x)]^2 \simeq f(x)f(x)^T = c\gamma_{j_u}(x)\gamma_{j_u}^T(x)c^T = [c_0, c_1, c_2, ..., c_{2^{j_u+1}+2}] \begin{pmatrix} \phi^2_{j_u,0}(x) & 0 & \cdots & 0 \\ 0 & \phi^2_{j_u,1}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi^2_{j_u,2^{j_u+1}+2}(x) \end{pmatrix} [c_0, c_1, c_2, ..., c_{2^{j_u+1}+2}]^T$$

we put

$$\begin{aligned} \eta_{j_u}(x) =& \gamma_{j_u}(x)\gamma_{j_u}^T(x) = [\phi^2_{j_u,0}(x), \phi^2_{j_u,1}(x), ..., \phi^2_{j_u,2^{j_u+1}+2}(x)]^T \\ =& (2^{j_u+1}+3)^{\frac{1}{2}}\gamma_{j_u}(x) \end{aligned} \quad (3.\ 22)$$

therefore

$$[f(x)]^2 \simeq [c_0^2, c_1^2, ..., c^2_{2^{j_u+1}+2}]\eta_{j_u}(x) = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[((2^{j_u+1}+3)^{\frac{1}{2}}c_0^2)^2, ((2^{j_u+1}+3)^{\frac{1}{2}}c_1^2)^2, ..., ((2^{j_u+1}+3)^{\frac{1}{2}}c^2_{2^{j_u+1}+2})^2]\gamma_{j_u}(x)$$

then, for $k = 2$ it is true. Now, we prove that is true for $k + 1$.

$$\begin{aligned} [f(x)]^{k+1} &\simeq [f(x)]^k f(x)^T \\ &= \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[((2^{j_u+1}+3)^{\frac{1}{2}}c_0)^k, ((2^{j_u+1}+3)^{\frac{1}{2}}c_1)^k, ..., \\ &\quad ((2^{j_u+1}+3)^{\frac{1}{2}}c_{2^{j_u+1}+2})^k]\gamma_{j_u}(x)\gamma_{j_u}^T(x)[c_0, c_1, c_2, ..., c_{2^{j_u+1}+2}]^T \end{aligned}$$

According to (3. 22) are obtained:

$$[f(x)]^{k+1} \simeq \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[((2^{j_u+1}+3)^{\frac{1}{2}}c_0)^{k+1}, ((2^{j_u+1}+3)^{\frac{1}{2}}c_1)^{k+1}, ..., ((2^{j_u+1}+3)^{\frac{1}{2}}c_{2^{j_u+1}+2})^{k+1}]\gamma_{j_u}(x)$$

so using mathematical induction it can be shown that (3. 21) is established. □

Theorem 3.2. *Suppose that $h(x)$ is an analytic function, prove that the function $h(f(x))$ can be expanded in the space of v_{j_u} as*

$$h(f(x)) \simeq \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[h((2^{j_u+1}+3)^{\frac{1}{2}}c_0), h((2^{j_u+1}+3)^{\frac{1}{2}}c_1), ..., h((2^{j_u+1}+3)^{\frac{1}{2}}c_{2^{j_u+1}+2})]\gamma_{j_u}(x) \tag{3. 23}$$

Proof. Because $h(x)$ is an analytic function, then Maclaurin's series expansion is available in the form of

$$h(x) = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}x^k$$

According to Lemma 3.1 and equation (3. 21) we have
$h(f(x)) \simeq \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}(f(x))^k = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}[((2^{j_u+1}+3)^{\frac{1}{2}}c_0)^k,$
$((2^{j_u+1}+3)^{\frac{1}{2}}c_1)^k, ..., ((2^{j_u+1}+3)^{\frac{1}{2}}c_{2^{j_u+1}+2})^k]\gamma_{j_u}(x) = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}\Big[\sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}$
$((2^{j_u+1}+3)^{\frac{1}{2}}c_0)^k, \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}((2^{j_u+1}+3)^{\frac{1}{2}}c_1)^k, ..., \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}((2^{j_u+1}+3)^{\frac{1}{2}}$
$c_{2^{j_u+1}+2})^k\Big]\gamma_{j_u}(x) = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[h((2^{j_u+1}+3)^{\frac{1}{2}}c_0), h((2^{j_u+1}+3)^{\frac{1}{2}}c_1), ...,$
$h((2^{j_u+1}+3)^{\frac{1}{2}}c_{2^{j_u+1}+2})]\gamma_{j_u}(x)$ □

Now, by using the collocation points, the vector of $\beta_{j_u}(x)$ (cubic B-spline scaling functions and wavelets) may be approximated by the vector of $\gamma_{j_u}(x)$ (Haar scaling functions) and conversely. For this approximation, we consider the collocation points as follows:

$$x_i = \frac{2i-1}{2(2^{j_u}+3)}, \qquad (i = 1, 2, ..., 2^{j_u}+3) \tag{3. 24}$$

The cubic B-spline scaling fuctions and wavelets Δ_{j_u} can be expressed by:

$$\Delta_{j_u} = \Big[\beta_{j_u}(x_1), \beta_{j_u}(x_2), ..., \beta_{j_u}(x_{j_u})\Big] \tag{3. 25}$$

We put

$$\beta_{j_u}(x) = A\gamma_{j_u}(x)$$

With respect to the Haar scaling functions in space of v_{j_u} are incompatible and each of these collocation points are located in an interval, with some simple calculations it can be concluded that

$$A = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}\Delta_{j_u}$$

Therefore

$$\beta_{j_u}(x) = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}\Delta_{j_u}\gamma_{j_u}(x) \tag{3. 26}$$

We have to solve the system (3. 26) by using the Gauss elimination method, consequently, the matrix $\Delta_{j_u}^{-1}$ of the inverse matrix Δ_{j_u} are obtained. Therefore,

$$\gamma_{j_u}(x) = (2^{j_u+1}+3)^{\frac{1}{2}}\Delta_{j_u}^{-1}\beta_{j_u}(x) \tag{3. 27}$$

Now we apply the equations of (3. 21)-(3. 27) to describe the useful properties of the cubic scaling fuctions and Wavelets that will be useful in solving the linear and nonlinear fractional equations. By assuming $C = [c_0, c_1, c_2, ..., c_{2^{j_u+1}+2}]$, $D = [d_0, d_1, d_2, ..., d_{2^{j_u+1}+2}]$ and F is an analytic function, we define

$$CD = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[(2^{j_u+1}+3)c_0d_0, (2^{j_u+1}+3)c_1d_1, ..., (2^{j_u+1}+3)c_{2^{j_u+1}+2}d_{2^{j_u+1}+2}] \quad (3.\ 28)$$

$$F(C) = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}[F((2^{j_u+1}+3)^{\frac{1}{2}}c_0), F((2^{j_u+1}+3)^{\frac{1}{2}}c_1), ..., F((2^{j_u+1}+3)^{\frac{1}{2}}c_{2^{j_u+1}+2})] \quad (3.\ 29)$$

We assume $f(x) \simeq C\beta_{j_u}(x)$ is the expansion of $f(x)$, by using cubic B-spline scaling fuctions and wavelets and with respect to the equations of (3. 23), (3. 26), (3. 27) and (3. 29), we obtain:

$$F(f(x)) \simeq F(C\beta_{j_u}(x)) \simeq F(\frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}C\Delta_{j_u}\gamma_{j_u}(x)) \simeq F(\bar{C})\gamma_{j_u}(x)) \simeq (2^{j_u+1}+3)^{\frac{1}{2}}F(\bar{C})\Delta_{j_u}^{-1}\beta_{j_u}(x),$$

$$that\ \bar{C} = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}C\Delta_{j_u}$$

Therefore

$$F(f(x)) \simeq (2^{j_u+1}+3)^{\frac{1}{2}}F(\bar{C})\Delta_{j_u}^{-1}\beta_{j_u}(x) \quad (3.\ 30)$$

Moreover, if $g(x) \simeq D\beta_{j_u}(x)$ is expansion of $g(x)$, then, according to the equations of (3. 21), (3. 26), (3. 27) and (3. 28), we have

$$f(x)g(x) \simeq C\beta_{j_u}(x)D\beta_{j_u}(x) \simeq \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}C\Delta_{j_u}\gamma_{j_u}(x)\frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}D\Delta_{j_u}\gamma_{j_u}(x) \simeq \bar{C}\gamma_{j_u}(x)\bar{D}\gamma_{j_u}(x) \simeq \bar{C}\bar{D}\gamma_{j_u}(x) \simeq (2^{j_u+1}+3)^{\frac{1}{2}}\bar{C}\bar{D}\Delta_{j_u}^{-1}\beta_{j_u}(x),$$

$$that\ \bar{C} = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}C\Delta_{j_u}\ and\ \bar{D} = \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}D\Delta_{j_u}$$

Thus

$$f(x)g(x) \simeq (2^{j_u+1}+3)^{\frac{1}{2}}\bar{C}\bar{D}\Delta_{j_u}^{-1}\beta_{j_u}(x) \quad (3.\ 31)$$

Also, from relations (3. 30) and (3. 31), with the assumption of the analyticity of G, it can be concluded that:

$$F(f(x))G(g(x)) \simeq (2^{j_u+1}+3)^{\frac{1}{2}}F(\bar{C})G(\bar{D})\Delta_{j_u}^{-1}\beta_{j_u}(x)$$

4. THE OPERATIONAL MATRIX OF THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL FOR CUBIC B-SPLINE SCALING FUNCTIONS AND WAVELETS

In this section, at first the operational matrix of the Riemann-Liouville fractional integral for the Haar scaling functions ($\gamma_{j_u}(x)$), that produce space of v_{j_u}, are obtained in a way that will be mentioned, then, by using relationships between Haar scaling functions and cubic B-spline scaling functions and wavelets derive the operational matrix of Riemann-Liouville fractional integral for cubic B-spline scaling functions and wavelets. The following theorem gives us the operational matrix of the Haar scaling functions from α order.

Theorem 4.1. *Suppose ${}_aI_x^\alpha$ is operator of the Riemann-Liouville fractional integral from α order and $\gamma_{j_u}(x)$ is the vector of the Haar scaling functions in space of v_{j_u} also ${}_aI_x^\alpha\gamma_{j_u}(x) = G_\alpha\gamma_{j_u}(x)$ then G_α is called the operational matrix of the Riemann-Liouville fractional integral for the Haar scaling functions from α order and it is obtained from the following relation.*

$$G_\alpha \simeq \frac{1}{\Gamma(\alpha+2)(2^{j_u+1}+3)^\alpha}\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 & \cdots & \omega_{2^{j_u+1}+3} \\ 0 & \omega_1 & \omega_2 & \cdots & \omega_{2^{j_u+1}+2} \\ 0 & 0 & \omega_1 & \cdots & \omega_{2^{j_u+1}+1} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & \omega_1 \end{pmatrix} \tag{4. 32}$$

that

$$\omega_1 = 1,\ \omega_s = s^{\alpha+1} - 2(s-1)^{\alpha+1} + (s-2)^{\alpha+1},$$
$$(s = 2, 3, ..., 2^{j_u+1} + 3 - t),\ (t = 0, 1, ..., 2^{j_u+1} + 2)$$

Proof. By substituting $\gamma_{j_u}(x)$ instead $\varphi(x)$ in equation (2. 18), we have:

$${}_0I_x^\alpha\gamma_{j_u}(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\gamma_{j_u}(t)dt = G_\alpha\gamma_{j_u}(x),\ 0 \le x < 1 \tag{4. 33}$$

By using the definition of convolution, from the equation of (4. 33) it can be concluded:

$${}_0I_x^\alpha\gamma_{j_u}(x) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1} * \gamma_{j_u}(x) \tag{4. 34}$$

The right side of equation (4. 34) is a matrix from order of $1 \times 2^{j_u+1} + 3$ that each entry of matrix is as follows:

$$\begin{aligned} {}_0I_x^\alpha\phi_{j_u,t}(x) =& \frac{1}{\Gamma(\alpha)}x^{\alpha-1} * \phi_{j_u,t}(x) \\ =& \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\Gamma(\alpha)}x^{\alpha-1} * (u_{\frac{t}{2^{j_u+1}+3}}(x) - u_{\frac{t+1}{2^{j_u+1}+3}}(x)) \end{aligned} \tag{4. 35}$$

that $u_a(x)$ is a unit step function and $t = 0, 1, ..., 2^{j_u+1} + 2$
by taking Laplace transform on both sides of the equation (4. 35), we have

$$\begin{aligned} L[{}_0I_x^\alpha \phi_{j_u,t}(x)] =& \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\Gamma(\alpha)} L[x^{\alpha-1}]] L[(u_{\frac{t}{2^{j_u+1}+3}}(x) - u_{\frac{t+1}{2^{j_u+1}+3}}(x))] \\ =& \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{s^\alpha} \frac{1}{s} [e^{\frac{-t}{2^{j_u+1}+3}s} - e^{-\frac{t+1}{2^{j_u+1}+3}s}] \\ =& \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{s^{\alpha+1}} e^{\frac{-t}{2^{j_u+1}+3}s} - \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{s^{\alpha+1}} e^{-\frac{t+1}{2^{j_u+1}+3}s} \end{aligned} \tag{4. 36}$$

If we get the inverse Laplace transform from equation (4. 36), then the following equation will be obtained.

$$\begin{aligned} {}_0I_x^\alpha \phi_{j_u,t}(x) = \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\Gamma(\alpha+1)} \Big[& (x - \frac{t}{2^{j_u+1}+3})^\alpha u_{\frac{t}{2^{j_u+1}+3}}(x) - \\ & (x - \frac{t+1}{2^{j_u+1}+3})^\alpha u_{\frac{t+1}{2^{j_u+1}+3}}(x) \Big] \end{aligned} \tag{4. 37}$$

According to the space of v_{j_u}, we expand the right side of equation (4. 37), therefore

$$(x - \frac{t}{2^{j_u+1}+3})^\alpha u_{\frac{t}{2^{j_u+1}+3}}(x) \simeq \Pi \gamma_{j_u}(x),\ \Pi = [\pi_0, \pi_1, ..., \pi_{2^{j_u+1}+2}] \tag{4. 38}$$

It can be derived from equations of (2. 5) and (4. 38) that

$$\begin{aligned} \pi_k &= \int_{\frac{k}{2^{j_u+1}+3}}^{\frac{k+1}{2^{j_u+1}+3}} (x - \frac{t}{2^{j_u+1}+3}) u_{\frac{t}{2^{j_u+1}+3}}(x) \varphi_{j_u,k}(x) dx \\ &= (2^{j_u+1}+3)^{\frac{1}{2}} \int_{\frac{k}{2^{j_u+1}+3}}^{\frac{k+1}{2^{j_u+1}+3}} (x - \frac{t}{2^{j_u+1}+3}) u_{\frac{t}{2^{j_u+1}+3}}(x) dx \\ &= \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\alpha+1} \Big((\frac{k+1-t}{2^{j_u+1}+3})^{\alpha+1} - (\frac{k-t}{2^{j_u+1}+3})^{\alpha+1} \Big) \end{aligned}$$

for $k \geq t$ and according to definition of $(u_{\frac{t}{2^{j_u+1}+3}}(x))$ for $k < t$ we have: $\pi_k = 0$.Thus

$$\pi_k = \begin{cases} \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\alpha+1} \Big((\frac{k+1-t}{2^{j_u+1}+3})^{\alpha+1} - (\frac{k-t}{2^{j_u+1}+3})^{\alpha+1} \Big) & k \geq t \\ 0 & k < t \end{cases}$$

Similarly, if we put

$$(x - \frac{t+1}{2^{j_u+1}+3})^\alpha u_{\frac{t+1}{2^{j_u+1}+3}}(x) \simeq \Pi' \gamma_{j_u}(x),\ \Pi' = [\pi_0', \pi_1', ..., \pi_{2^{j_u+1}+2}'] \tag{4. 39}$$

where

$$\pi_k' = \begin{cases} \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\alpha+1} \Big((\frac{k-t}{2^{j_u+1}+3})^{\alpha+1} - (\frac{k-t-1}{2^{j_u+1}+3})^{\alpha+1} \Big) & k \geq t+1 \\ 0 & k < t+1 \end{cases}$$

Therefore, it is obvious that the vectors Π and $\Pi^{'}$ in equations of (4. 38) and (4. 39) are as follows:

$$\Pi = [0, 0, ..., \nu_1, \nu_2, ..., \nu_{2^{j_u+1}+3-t}],\ \Pi^{'} = [0, 0, ..., \nu_1, \nu_2, ..., \nu_{2^{j_u+1}+2-t}] \tag{4. 40}$$

$$that\ \nu_1 = \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{(\alpha+1)(2^{j_u+1}+3)^{\alpha+1}}\ and\ \nu_s = \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{(\alpha+1)(2^{j_u+1}+3)^{\alpha+1}}(s^{\alpha+1}-(s-1)^{\alpha+1})$$

The equations (4. 37) and (4. 40) conclude that

$$\begin{aligned} {}_0I_x^{\alpha}\phi_{j_u,t}(x) &\simeq \frac{(2^{j_u+1}+3)^{\frac{1}{2}}}{\Gamma(\alpha+1)}[0, 0, ..., 0, \nu_1, \nu_2-\nu_1, ..., \nu_{2^{j_u+1}+3-t}-\nu_{2^{j_u+1}+2-t}] \qquad (4.\ 41) \\ &\simeq \frac{1}{\Gamma(\alpha+2)(2^{j_u+1}+3)^{\alpha}}[0, 0, ..., 0, \omega_1, \omega_2, ..., \omega_{2^{j_u+1}+3-t}] \end{aligned}$$

According to the above concepts, the matrix G_α is represented by equation (4. 32). □

Now, by from the operational matrix of the Riemann-Liouville fractional integral for the Haar scaling functions, which is represented by equation (4. 32), we obtain the operational matrix of the Riemann-Liouville fractional integral for cubic B-spline scaling functions and wavelets. For this purpose, we assume G_α^β is the operational matrix of the Riemann-Liouville fractional integral from α order for cubic B-spline scaling functions and wavelets, then

$${}_0I_x^{\alpha}\beta_{j_u}(x) \simeq G_\alpha^\beta \beta_{j_u}(x) \tag{4. 42}$$

Employing equations (3. 26), (3. 27) and (4. 32), we get

$$\begin{aligned} {}_0I_x^{\alpha}\beta_{j_u}(x) &\simeq {}_0I_x^{\alpha}\frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}\Delta_{j_u}\gamma_{j_u}(x) \simeq \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}\Delta_{j_u}\ {}_0I_x^{\alpha}\gamma_{j_u}(x) \qquad (4.\ 43) \\ &\simeq \frac{1}{(2^{j_u+1}+3)^{\frac{1}{2}}}\Delta_{j_u}G_\alpha\gamma_{j_u}(x) \simeq \Delta_{j_u}G_\alpha\Delta_{j_u}^{-1}\beta_{j_u}(x) \end{aligned}$$

Thus, from equations (4. 42) and (4. 43), we have

$$G_\alpha^\beta \simeq \Delta_{j_u}G_\alpha\Delta_{j_u}^{-1} \tag{4. 44}$$

Therefor

$${}_0I_x^{\alpha}\beta_{j_u}(x) \simeq G_\alpha^\beta\beta_{j_u}(x) \simeq \Delta_{j_u}G_\alpha\Delta_{j_u}^{-1}\beta_{j_u}(x) \tag{4. 45}$$

5. Solving the fractional integro-differential equations by the proposed method

In this section, we use the cubic B-spline operational matrix of fractional order, described in the previous section, for solving a generalized form from the nonlinear fractional Volterra integro-differential equation. Also, in this method, we apply some relationships between the Haar scaling functions and cubic B-spline scaling functions and wavelets. The proposed method can be similarly used for the varieties of fractional order equations.

We consider a generalized form from the nonlinear fractional Volterra integro-differential equation with initial conditions as:

$$ {}_{C_0}D_x^{\alpha}y(x) = p(x)F(y(x)) + f(x) + \lambda\int_0^x k(x,t)G(y(t))dt, \quad (5.\ 46)$$
$$ y(0) = y_0, y^{'}(0) = y_1, ..., y^{\lceil \alpha-1 \rceil}(0) = y_{\lceil \alpha-1 \rceil},$$

Without the loss of generality, we can assume $0 \leq x \leq 1$, the functions of $f(x),\ p(x)$ and $k(x,t)$ are continuous. Also F and G are analytic functions.

By using the fractional operational matrix, we transform the equation (5. 46) into a system of nonlinear algebraic equations. So, ${}_{C_0}D_x^{\alpha}y(x)$ is approximated by $\beta_{j_u}(x)$ as follows:

$$ {}_{C_0}D_x^{\alpha}y(x) = S\beta_{j_u}(x) \quad (5.\ 47)$$

where the vector of S is unknown coefficients and it must be obtained. We apply ${}_aI_x^{\alpha}$ integral operation on the both sides (5. 47) and by using the equations (2. 20) and (4. 42), we will have:

$$ y(x) - \sum_{k=0}^{\lceil \alpha \rceil -1} \frac{y^{(k)}(0)}{k!}x^k \simeq SG_{\alpha}^{\beta}\beta_{j_u}(x) \quad (5.\ 48)$$

we put $h(x) = -\sum_{k=0}^{\lceil \alpha \rceil -1} \frac{y^{(k)}(0)}{k!}x^k$. For simplicity, we approximate the function $h(x)$ with $\beta_{j_u}(x)$ as:

$$ h(x) \simeq \kappa\beta_{j_u}(x),\ where\ \kappa \simeq H\Delta_{j_u}^{-1}\ and\ H = [h(x_1), h(x_2), ..., h(x_{2^{j_u+1}+3})] \quad (5.\ 49)$$

$(x_i,\ i = 1, 2, ..., 2^{j_u+1}+3)$ are collocation points expressed by equation (3. 24). Expressions(5. 48) and (5. 49) yield

$$ y(x) \simeq \Sigma\beta_{j_u}(x),\ where\ \Sigma \simeq SG_{\alpha}^{\beta} + H\Delta_{j_u}^{-1} \quad (5.\ 50)$$

From equation (3. 30), we get

$$ G(y(x)) \simeq (2^{j_u+1}+3)^{\frac{1}{2}}G(\bar{\Sigma})\Delta_{j_u}^{-1}\beta_{j_u}(x), \quad (5.\ 51)$$
$$ F(y(x)) \simeq (2^{j_u+1}+3)^{\frac{1}{2}}F(\bar{\Sigma})\Delta_{j_u}^{-1}\beta_{j_u}(x)$$

Also, by substituting the equations (5. 47), (5. 50) and (5. 51) into equation (5. 46), we get

$$ S\beta_{j_u}(x) = f(x) + (2^{j_u+1}+3)^{\frac{1}{2}}F(\bar{\Sigma})\Delta_{j_u}^{-1}p(x)\beta_{j_u}(x) + \quad (5.\ 52)$$
$$ \lambda(2^{j_u+1}+3)^{\frac{1}{2}}G(\bar{\Sigma})\Delta_{j_u}^{-1}\int_0^x k(x,t)\beta_{j_u}(t)dt$$

Equation (5. 52) is a nonlinear system. We put collocation points, that were defined by equation (3. 24) instead of x into above equation, hence it converts to a nonlinear system of algebraic equations with $2^{j_u+1}+3$ unknowns. By solving this nonlinear system with the Newton iteration method, the unknown coefficients S are obtained and $y(x)$ is eventually derived from the equation (3. 24). On the other hand, by rewriting $g(x) = \int_0^x k(x,t)\beta_{j_u}(t)dt$ in the form $\kappa\beta_{j_u}(x)$, where $\kappa \simeq T\Delta_{j_u}^{-1}$ and

$T = [g(x_1), g(x_2), ..., g(x_{2^{j_u+1}+3})]$, and by assuming $p(x) = 1$, the equation (5. 46) will be

$$\Big(S - (2^{j_u+1}+3)^{\frac{1}{2}} F(\bar{\Sigma})\Delta_{j_u}^{-1} - \lambda(2^{j_u+1}+3)^{\frac{1}{2}} G(\bar{\Sigma})\Delta_{j_u}^{-1} T \Delta_{j_u}^{-1}\Big)\beta_{j_u}(x) = f(x) \quad (5.\ 53)$$

By multiplying the dual of cubic B-spine scaling functions and wavelets on both sides equation (5. 53), it can be solved by the Galerkin method.

6. Convergence analysis

In this section, at first, the error upper bound for equation (5. 46) is investigated when $y(x)$ were approximated by cubic B-spline scaling functions and wavelets. Then, we show that this proposed method is convergence. At the end, the existence and uniqueness of the solution of equation (5. 46) are proved.

Theorem 6.1. *For the m-th order B-spline wavelet suppose y_{j_u} is the approximate solution of the exact solution y, then error upper bound of approximation is given by*

$$\|y - y_{j_u}\| \leq C_m 2^{-j_u m} \|y^{(m)}\|_2 \ , \ C_m = \sqrt{\frac{B_{2m}}{(2m)!}} \quad (6.\ 54)$$

Where B_{2m} is Bernoulli number of order $2m$ [20].

The following theorem investigates the error upper bound and convergence for the approximate solution.

Theorem 6.2. *Suppose that $M_1 = \max|F'(x)|$, $M_2 = \max|G'(x)|$, $M_3 = \max_{x\in[0,1]} |p(x)|$, $M_4 = \max_{(x,t)\in[0,1]\times[0,1]} |k(x,t)|$,F and G are analytic functions, $\tilde{y}_{j_u}(x)$ is approximate solution and $y(x)$ is exact solution then the error upper bound for $y(x)$, that applies in (5. 46) equation, can be given from the following equation.*

$$|y(x) - \tilde{y}_{j_u}(x)| \leq C_m 2^{-j_u m} \|y^{(m)}\|_2 \big(\frac{M_1 M_3}{\Gamma(\alpha+1)} + \frac{M_2 M_4}{\Gamma(\alpha+2)}\big) \quad (6.\ 55)$$

also we have $\lim_{j_u\to\infty} \tilde{y}_{j_u}(x) = y(x)$

Proof. Since fuctions F and G is analytic functions, then

$$\begin{aligned} |F(y(x)) - F(\tilde{y}_{j_u}(x))| &\leq M_1 |y(x) - \tilde{y}_{j_u}(x)| \\ |G(y(x)) - G(\tilde{y}_{j_u}(x))| &\leq M_2 |y(x) - \tilde{y}_{j_u}(x)| \end{aligned} \quad (6.\ 56)$$

By implementation the operation of ${}_aI_x^\alpha$ on both sides (5. 46) also by using the definition of fractional integration (2. 18) [30], equation (2. 20) and the mentioned concept in the previous section, we obtain:

$$\begin{aligned} y(x) =& h(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} p(t) F(y(t)) dt + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha} k(x,t) G(y(t)) dt, \end{aligned} \quad (6.\ 57)$$

that $h(x) = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{x^j}{j!}(\frac{d^j}{dt^j}y)(0)$.
Also the pproximate solution $\tilde{y}_{j_u}(x)$ establishes the following equation:

$$\begin{aligned}\tilde{y}_{j_u}(x) =& h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)F(\tilde{y}_{j_u}(t))dt \\ &+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt + \frac{1}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}k(x,t)G(\tilde{y}_{j_u}(t))dt,\end{aligned} \tag{6. 58}$$

We subtract equation (6. 58) from (6. 57) as:

$$\begin{aligned}y(x) - \tilde{y}_{j_u}(x) =& \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)\big(F(y(t)) - F(\tilde{y}_{j_u}(t))\big)dt \\ &+\frac{1}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}k(x,t)\big(G(y(t)) - G(\tilde{y}_{j_u}(t))\big)dt\end{aligned}$$

with respect to the equations (6. 54), (6. 56), $x \in [0,1]$ and the assumption of theorem we obtain:

$$\begin{aligned}|y(x) - \tilde{y}_{j_u}(x)| \leq& \frac{1}{\Gamma(\alpha)}C_m 2^{-j_u m}M_1M_3\|y^{(m)}\|_2\int_0^x (x-t)^{\alpha-1}dt \\ &+\frac{1}{\Gamma(\alpha+1)}C_m 2^{-j_u m}M_2M_4\|y^{(m)}\|_2\int_0^x (x-t)^{\alpha}dt \\ \leq& \frac{1}{\Gamma(\alpha)}C_m 2^{-j_u m}M_1M_3\|y^{(m)}\|_2\frac{1}{\alpha} \\ &+\frac{1}{\Gamma(\alpha+1)}C_m 2^{-j_u m}M_2M_4\|y^{(m)}\|_2\frac{1}{\alpha+1} \\ =& C_m 2^{-j_u m}\|y^{(m)}\|_2\big(\frac{M_1M_3}{\Gamma(\alpha+1)} + \frac{M_2M_4}{\Gamma(\alpha+2)}\big)\end{aligned}$$

and it is clear if $j_u \to \infty$ then $\tilde{y}_{j_u}(x) \to y(x)$. □

Theorem 6.3. *Suppose that the assumption of Theorem (6.2) are established. Then equation (5. 46) have the unique solution provided* $M_1M_3 + M_2M_4 < \Gamma(\alpha+1)$

Proof. With the definition of integral operator $\mathcal{A} : C[0,1] \to C[0,1]$ as follows:

$$\begin{aligned}\mathcal{A}y(x) =& h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)F(y(t))dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt \\ &+\frac{1}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}k(x,t)G(y(t))dt,\end{aligned}$$

that $h(x) = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{x^j}{j!}(\frac{d^j}{dt^j}y)(0)$.
Suppose $\tilde{y}_{j_u}(x) \in C[0,1]$ is an approximation of $y \in C[0,1]$; therefore,

$$\begin{aligned}\mathcal{A}y(x) - \mathcal{A}\tilde{y}_{j_u}(x) =& \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)\big(F(y(t)) - F(\tilde{y}_{j_u}(t))\big)dt \\ &+\frac{1}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}k(x,t)\big(G(y(t)) - G(\tilde{y}_{j_u}(t))\big)dt\end{aligned}$$

with respect to the equation (6. 56), $x \in [0,1]$ and the assumption of theorem we obtain:

$$\begin{aligned}|\mathcal{A}y(x)-\mathcal{A}\tilde{y}_{j_u}(x)| \leq & \frac{1}{\Gamma(\alpha)}M_1M_3\|\tilde{y}_{j_u}-y\|_\infty \int_0^x (x-t)^{\alpha-1}dt \\ & + \frac{1}{\Gamma(\alpha+1)}M_2M_4\|\tilde{y}_{j_u}-y\|_\infty \int_0^x (x-t)^{\alpha}dt \\ \leq & \frac{1}{\Gamma(\alpha)}M_1M_3\|\tilde{y}_{j_u}-y\|_\infty \frac{1}{\alpha} \\ & + \frac{1}{\Gamma(\alpha+1)}M_2M_4\|\tilde{y}_{j_u}-y\|_\infty \frac{1}{\alpha+1} \\ = & \big(\frac{M_1M_3}{\Gamma(\alpha+1)}+\frac{M_2M_4}{\Gamma(\alpha+2)}\big)\|\tilde{y}_{j_u}-y\|_\infty \\ \leq & \big(M_1M_3+M_2M_4\big)\frac{1}{\Gamma(\alpha+1)}\|\tilde{y}_{j_u}-y\|_\infty\end{aligned}$$

We put $L_{p,K,M_1,M_2,\alpha} = \big(M_1M_3+M_2M_4\big)\frac{1}{\Gamma(\alpha+1)}$. Then, according to assumption $L_{p,K,M_1,M_2,\alpha} < 1$.
due to the Banach contraction mapping theorem the equation of (5. 46) has unique answer on $C[0,1]$. □

7. NUMERICAL EXAMPLES

In this section, to illustrate the validity and applicability of the proposed method in this study, we solve the different varieties of fractional integro-differential equations and compare proposed method with the other methods in various paper. Also, we use the root mean square error to show the accuracy of approximation as:

$$\begin{aligned}\| e_{j_u}(x) \|_2 &= \Big(\int_0^1 e_{j_u}^2(x)dx\Big)^{\frac{1}{2}} = \Big(\int_0^1 (y(x)-\tilde{y}_{j_u}(x))^2dx\Big)^{\frac{1}{2}} \\ &\simeq \left(\frac{1}{N}\sum_{i=0}^{N}(y(x_i)-\tilde{y}_{j_u}(x_i))^2\right)^{\frac{1}{2}}\end{aligned} \tag{7. 59}$$

To solve these examples, the mathematica10.4 software has been used.

Example 1. Consider the following nonlinear fractional Volterra integro-differential equation

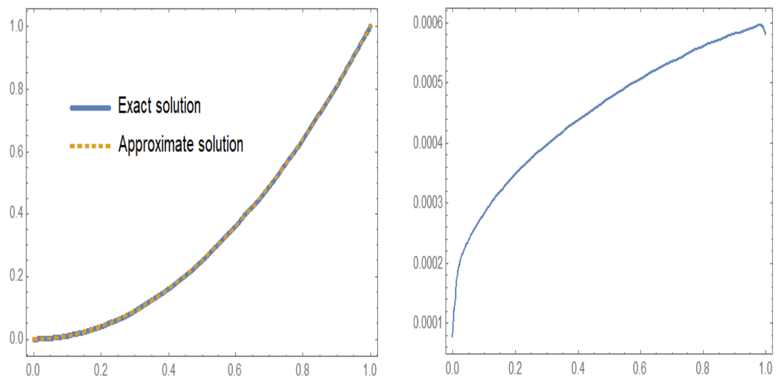
$$\begin{aligned}{}_{C_0}D_x^{\frac{3}{2}}y(x) =& 720(x-\sin(x)) - 120x^3 + 6x^5 + \frac{2}{\Gamma(1.5)}x^{\frac{1}{2}} \\ & - \int_0^x \cos(x-t)(y(t))^3dt,\end{aligned}$$

with initial conditions $y(0) = y^{'}(0) = 0$.
The exact solution of this problem is $y(x) = x^2$. We solve this example by the proposed method. The numerical results for $j_u = 4$ are shown in Fig. 2 and Table 1. The absolute error and root mean square error have been compared with the proposed method in [27], for the method of [27] $\| e(x) \|_2 = 5.97586 \times 10^{-4}$, but with this our method is $\| e_4(x) \|_2 = 4.72951 \times 10^{-4}$. These results illustrate that the accuracy of our numerical solutions are a

TABLE 1. The results for example 1 by cubic B-spline scaling functions and wavelets($j_u = 3, 4$).

x_i	Exact solution	Absolute error of this propsed method ($j_u = 3$)	Absolute error of this proposed method ($j_u = 4$)	Absolute error the method of [27]
0.1	0.001	9.90985×10^{-4}	2.85175×10^{-4}	2.7975×10^{-4}
0.2	0.004	1.24×10^{-3}	3.49151×10^{-4}	5.4594×10^{-4}
0.3	0.09	1.4268×10^{-3}	3.97966×10^{-4}	6.3356×10^{-4}
0.4	0.16	1.58273×10^{-3}	4.38991×10^{-4}	5.5630×10^{-4}
0.5	0.25	1.71901×10^{-3}	4.74974×10^{-4}	6.0926×10^{-4}
0.6	0.36	1.84071×10^{-3}	5.07168×10^{-4}	6.7714×10^{-4}
0.7	0.49	1.9497×10^{-3}	5.36019×10^{-4}	8.7750×10^{-4}
0.8	0.64	2.04533×10^{-3}	5.6132×10^{-4}	9.1878×10^{-4}
0.9	0.81	2.12435×10^{-3}	5.82158×10^{-4}	7.9774×10^{-4}
1.0	1.00	$.218021 \times 10^{-3}$	5.80052×10^{-4}	7.9890×10^{-4}

few better than the numerical solutions obtained in the method of [27]. Also, by comparing the absolute error of y_{j_u} for $j_u = 3, 4$ in Table 1, it can be concluded that the error term becomes smaller as j_u increases.



(a) (b)

FIGURE 2. a. Comparison between the approximation and exact solution b. Absolute error

Example 2. Consider the linear fractional Fredholm integro-differential equation

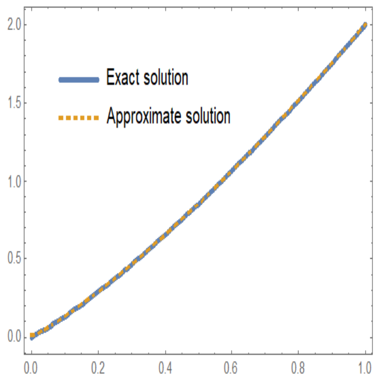
$$ {}_{C_0}D_x^{\frac{1}{2}} y(x) = 2\sqrt{\frac{x}{\pi}} + \frac{3x\sqrt{\pi}}{4} - \frac{9}{10} + \int_0^1 y(t)dt $$

subject to the initial condition $y(0) = 0$.
The exact solution is $y(x) = x + x^{\frac{3}{2}}$. This problem has been solved by the proposed method for $j_u = 4$. Table 2 shows the exact and approximate solutions and absolute error

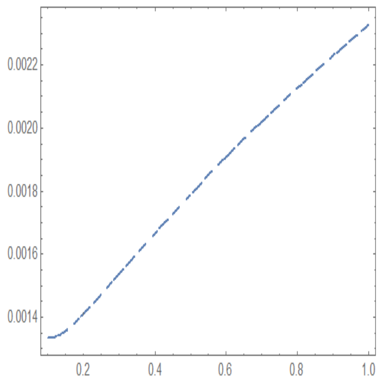
TABLE 2. The results for example 2 by cubic B-spline scaling functions and wavelets($j_u = 4$).

t_i	Exact solution	$y_{j_u=4}$	Approximat solution the method of [31]	Absolute error of this proposed method	Absolute error the method of [31]
0.1	0.131623	0.132957	0.1333	1.3338×10^{-3}	1.7×10^{-3}
0.2	0.289443	0.290853	0.2911	1.40988×10^{-3}	1.7×10^{-3}
0.3	0.464317	0.465853	0.4661	1.53657×10^{-3}	1.8×10^{-3}
0.4	0.652982	0.654648	0.6549	1.6655×10^{-3}	1.9×10^{-3}
0.5	0.853553	0.855343	0.8556	1.78953×10^{-3}	2.0×10^{-3}
0.6	1.06476	1.06667	1.0670	1.90756×10^{-3}	2.2×10^{-3}
0.7	1.28566	1.28768	1.2880	2.01979×10^{-3}	2.3×10^{-3}
0.8	1.51554	1.51767	1.5180	2.12674×10^{-3}	2.5×10^{-3}
0.9	1.75381	1.75604	1.7564	2.22898×10^{-3}	2.6×10^{-3}
1.0	2.00	2.00233	2.0027	2.32833×10^{-3}	2.7×10^{-3}

of Example 2 by this proposed method and the method of [31] in some arbitrary points. The root mean square error with our method is $\| e_4(x) \|_2 = 1.76336 \times 10^{-3}$ and the method of [31] is $\| e(x) \|_2 = 6.9 \times 10^{-3}$. By comparing the two methods, it can be seen that our method has less error, therefore, it is more accurate. This example has been solved by using second kind Chebyshev wavelet($k = 4, M = 4$) in [31]. Also Fig. 3 shows that the numerical solution is in a very good agreement with the exact solution.



(a)



(b)

FIGURE 3. a. Comparison between the approximation and exact solution b. Absolute error

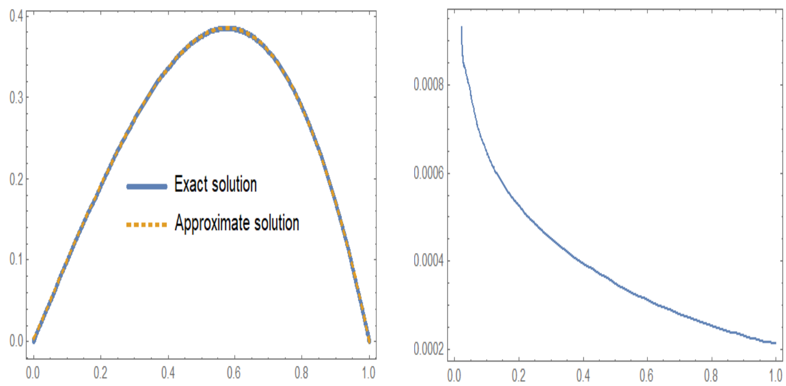
Example 3. Consider the nonlinear fractional Fredholm integro-differential equation

$$ {}_{C_0}D_x^{\frac{5}{6}} y(x) = \frac{3}{\Gamma(\frac{1}{6})}\left(2\sqrt[6]{x} - \frac{432}{91}\sqrt[6]{x^{13}}\right) + x\left(248e - 674\right) + \int_0^1 x e^t (y(t))^2 dt, $$

TABLE 3. The results for example 3 by cubic B-spline scaling functions and wavelets($j_u = 4$).

t_i	Exact solution	$y_{j_u=4}$	Absolute error proposed method ($j_u = 4$)	Absolute error the method of [28]
0.1	0.099	0.0996469	6.46882×10^{-4}	7.3210×10^{-4}
0.2	0.192	0.192525	5.24887×10^{-4}	7.3841×10^{-4}
0.3	0.273	0.273451	4.50886×10^{-4}	6.7841×10^{-4}
0.4	0.336	0.336395	3.95241×10^{-4}	5.8431×10^{-4}
0.5	0.375	0.37535	3.50174×10^{-4}	5.5893×10^{-4}
0.6	0.384	0.384312	3.12495×10^{-4}	5.7452×10^{-4}
0.7	0.375	0.357281	2.8058×10^{-4}	5.3282×10^{-4}
0.8	0.288	0.288253	2.53487×10^{-4}	5.8422×10^{-4}
0.9	0.171	0.171231	2.3062×10^{-4}	6.4470×10^{-4}
1.0	0.000	0.000211818	2.11818×10^{-4}	6.7731×10^{-4}

The exact solution and initial condition are $y(x) = x - x^3$ and $y(0) = 0$, respectively. Fig. 4 shows the behavior of the approximate and exact solution and the graphs of the absolute error of the proposed method. Also, it can be seen the exact and approximate solutions, and absolute error for some arbitrary points. The root mean square error with our method is $\| e_4(x) \|_2 = 3.65707 \times 10^{-4}$ and the method of [28] is $\| e(x) \|_2 = 6.3440 \times 10^{-4}$. This example has been solved by using the CAS wavelet method in [28].



(a) (b)

FIGURE 4. a. Comparison between the approximation and exact solution b. Absolute error

8. Conclusion

In this present work, the Haar scaling functions and the collocation method has been successfully applied to obtain the operational matrix of fractional integration for cubic B-spline scaling functions and wavelets. Then, the nonlinear fractional integro-differential equation reduced into a nonlinear system that can be solved with Newton method. The operation matrix can be simply obtained for any basis of the approximation space and it is always available, therefore it can be applied for different varieties of problems with fractional derivatives. Illustrative examples demonstrate that approximation solution fairly matches with the exact solution. The upper bound of error exponentially decreases by growing of approximation space.

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