Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(8)(2019) pp. 87-109

Some Generalized Contractive Mappings in Generalized Spaces

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Received: 08 November, 2018 / Accepted: 03 March, 2019 / Published online: 01 July, 2019

Abstract. In the following article we have introduced generalized multivalued α-ψ-contractive type mappings and multi-valued (θ, L)-weakly contractive mappings in $\mathcal{G}$-metric spaces and have established common fixed point results for the concerned contractive mappings. We have also proved some coincidence point theorems for multivalued $\mathcal{G}$-α-admissible mappings and relations. Some examples are also included in the support of these results.Our results can not be derived directly from fixed point theorems on metric or qusai-metric spaces as discussed by Jleli *et al.* in [21] and Azam *et al.* in [9]. An interesting application is also added to evaluate the solution of hyperbolic differential inclusion.

AMS (MOS) Subject Classification C odes: 6S40; 47H10; 54H25
Key Words: $\mathcal{G}$-metric spaces, generalized contractive type mappings, fixed point results.

1. Introduction

Fixed point theory is an interdisciplinary topic which can be applied in various disciplines of mathematical sciences like game theory, mathematical economics, optimization theory, approximation theory and variational inequalities. Banach [10] in 1922, presented a contraction principle for the existence of unique fixed p o int. T he fi xed po int th eory (as well as Banach contraction principle) has been studied and generalized in different spaces and various fixed p oint t heorems w ere d e veloped. U sing t he i dea o f f amous B anach contraction theorem, many authors have worked to find t he fi xed po int of se veral contractive mappings and have also introduced several new contractions. In this regard several fixed point theorems, common fixed p o int t h eorems a n d c o incidence r e sults h a ve b e en developed for mappings satisfying certain contractive conditions. Nadler [34] generalized Banach contraction principle to set-valued mappings by using the Hausdorff metric. After that several mathematicians deliberated various fixed point results for multi-valued contraction mappings [16,17,18,19,29,37,38,41].

Alber and Guerre [2] in 1997, presented concept of weak contraction . Berinde in 2004 [11], introduced the concept of (θ, L)-weak contraction and multi-valued (θ, L)-weak contraction [12] and extended the idea of fixed point theory to multi-valued mappings satisfying the related contraction. Samet *et al*. [39], in 2012, presented a new concept of contractive mapping namely α-ψ-contractive type mapping and studied fixed point theorem for concerned contraction. With time concept of α-admissible mappings has been developed in several directions like [4,5,13,14,15,20,24,36,40].

In the perspective of generalization, mathematicians have been focusing towards new metric based structures as well as generalized contractions, in order to obtain more refined fixed point results. So, in 2006, Mustafa and Sims [28] presented a generalized version of metric spaces, namely $\mathcal{G}$-metric spaces and after that various fixed point results have been obtained using various contractive conditions for the concerned metric [1,6,7,22,27-32,35]. This article pursue for fixed point results of multi-valued contractive type mappings and coincidence point results for multi-valued mappings and relation in $\mathcal{G}$-metric spaces. In [25, 26] Khuri and Sayfy applied the fixed point schemes for the numerical solutions of initial and boundry value problems. Inspired by this an interesting application is also added to evaluate the solution of hyperbolic differential inclusion.

2. Preliminaries

The following section is related to some basic concepts and results mostly from [28].

Definition 2.1. *Consider $\Upsilon \neq \emptyset$ and mapping $\mathcal{G} : \Upsilon \times \Upsilon \times \Upsilon \to \mathbb{R}^+$ satisfying subsequent axioms:*
$(g1)$ $\mathcal{G}(\wp, \nu, \varsigma) = 0$ *if* $\wp = \nu = \varsigma$,
$(g2)$ $0 < \mathcal{G}(\wp, \nu, \varsigma)$, $\forall\, \wp, \nu \in \Upsilon$ *with* $\wp \neq \nu$,
$(g3)$ $\mathcal{G}(\wp, \wp, \nu) \leq \mathcal{G}(\wp, \nu, \varsigma)$ $\forall\, \wp, \nu, \varsigma \in \Upsilon$ *with* $\varsigma \neq \nu$,
$(g4)$ $\mathcal{G}$ *preserves the symmetry in each variable,*
$(g5)$ $\mathcal{G}(\wp, \nu, \varsigma) \leq \mathcal{G}(\wp, \wp', \wp') + \mathcal{G}(\wp', \nu, \varsigma)$ $\forall\, \wp, \nu, \varsigma, \wp' \in \Upsilon$.
Then Υ is called $\mathcal{G}$-metric space.

Definition 2.2. *Consider Υbe a $\mathcal{G}$-metric space. Then the sequence $\{\wp_\eta\}$ in Υ is:*
(i) *$\mathcal{G}$-convergent if for any $\gamma > 0$, there is $\wp \in \Upsilon$ and $\aleph \in \mathbb{N}$ so that $\mathcal{G}(\wp, \wp_\eta, \wp_\zeta) < \gamma$, $\forall$ $\eta, \zeta \geq \aleph$,*
(ii) *$\mathcal{G}$-Cauchy if for any $\gamma > 0$, there is $\aleph \in \mathbb{N}$ so that $\mathcal{G}(\wp_\eta, \wp_\zeta, \wp_\tau) < \gamma$, $\forall\, \eta, \zeta, \tau \geq \aleph$.*

Theorem 2.3. *Consider Υ be a $\mathcal{G}$-metric space and $\{\wp_\eta\}$ be a sequence in Υ. Consequently subsequent are equal:*
(i) *$\{\wp_\eta\}$ is $\mathcal{G}$-convergent to $\wp$,*
(ii) *$\mathcal{G}(\wp_\eta, \wp_\eta, \wp) \to 0$, as $\eta \to \infty$,*
(iii) *$\mathcal{G}(\wp_\eta, \wp, \wp) \to 0$, as $\eta \to \infty$,*
(iv) *$\mathcal{G}(\wp_\zeta, \wp_\eta, \wp) \to 0$, as $\zeta, \eta \to \infty$.*

Proposition 2.4. *Every $\mathcal{G}$-metric space Υ, defines a metric space $(\Upsilon, d_\mathcal{G})$ by $d_\mathcal{G}(\wp, \nu) = \mathcal{G}(\wp, \nu, \nu) + \mathcal{G}(\wp, \wp, \nu)$, $\forall\, \wp, \nu \in \Upsilon$.*

Theorem 2.5. *Consider Υ be a $\mathcal{G}$-metric space and $\{\wp_\eta\}$ a sequence in Υ respectively. Consequently subsequent are equal:*

(i) $\{\wp_\eta\}$ *is* $\mathcal{G}$*-Cauchy.*
(ii) *For every* $\varepsilon > 0$, *there exists* $\aleph \in \mathbb{N}$ *such that* $\mathcal{G}(\wp_\eta, \wp_\zeta, \wp_\zeta) < \varepsilon, \forall\, \eta, \zeta \geq \aleph$.
(iii) $\{\wp_\eta\}$ *is a Cauchy sequence in the metric space* $(\Upsilon, d_{\mathcal{G}})$.

Υ will be complete if every $\mathcal{G}$-Cauchy sequence in Υ is a $\mathcal{G}$-convergent sequence in Υ.
Denote the class of all nonempty closed and bounded subsets of Υ as $F(\Upsilon)$.
Consider $H_{\mathcal{G}}(\cdot,\cdot,\cdot)$ be a Hausdorff $\mathcal{G}$-distance on $F(\Upsilon)$ [23], i.e.

$$H_{\mathcal{G}}(\check{C}, \check{D}, \check{E}) = \max\{\sup_{\wp\in\check{C}} \mathcal{G}(\wp, \check{D}, \check{E}), \sup_{\wp\in\check{D}} \mathcal{G}(\wp, \check{E}, \check{C}), \sup_{\wp\in\check{E}} \mathcal{G}(\wp, \check{C}, \check{D})\}$$

where

$$\begin{aligned}
\mathcal{G}(\wp, \check{D}, \check{E}) &= d_{\mathcal{G}}(\wp, \check{D}) + d_{\mathcal{G}}(\check{D}, \check{E}) + d_{\mathcal{G}}(\wp, \check{E}) \\
d_{\mathcal{G}}(\wp, \check{D}) &= \inf\{d_{\mathcal{G}}(\wp, \nu) : \nu \in \check{D}\} \\
d_{\mathcal{G}}(\check{C}, \check{D}) &= \inf\{d_{\mathcal{G}}(a, b) : a \in \check{C}, b \in \check{D}\} \\
\mathcal{G}(\wp, \nu, \check{E}) &= \inf\{\mathcal{G}(\wp, \nu, \varsigma) : \varsigma \in \check{E}\}.
\end{aligned}$$

Consider $\Gamma : \Upsilon \to 2^{\Upsilon}$ and $\wp \in \Upsilon$ then $\wp$ is a fixed point of Γ if $\wp \in \Gamma\wp$..

Remark 2.6. *Consider* Υ *be a* $\mathcal{G}$*-metric space,* $\wp \in \Upsilon$ *and* $\check{D} \subseteq \Upsilon$. *For each* $\nu \in \check{D}$,

$$\mathcal{G}(\wp, \check{D}, \check{D}) \leq 6\,\mathcal{G}(\wp, \nu, \nu).$$

Lemma 2.7. *(*[33]*) Consider* Υ *be a* $\mathcal{G}$*-metric space and* $\check{C}, \check{D} \in F(\Upsilon)$*, then for each* $a \in \check{C}$*:*

$$\mathcal{G}(a, \check{D}, \check{D}) \leq H_{\mathcal{G}}(\check{C}, \check{D}, \check{D}).$$

Lemma 2.8. *(*[33]*) Consider* Υ *be a* $\mathcal{G}$*-metric space. If* $\check{C}, \check{D} \in F(\Upsilon)$ *and* $\wp \in \check{C}$*, then for each* $\epsilon > 0$ *there exists* $\nu \in \check{D}$ *such that:*

$$\mathcal{G}(\wp, \nu, \nu) \leq H_{\mathcal{G}}(\check{C}, \check{D}, \check{D}) + \epsilon.$$

Definition 2.9. *(*[12]*)Consider* $\Upsilon \neq \emptyset$ *and mapping* $S : \Upsilon \to F(\Upsilon)$. S *is said to be a multivalued* (θ, L)*-weak contraction if there exists two constants* $\theta \in (0, 1)$ *and* $L \geq 0$ *such that*

$$H(S\wp, S\nu) \leq \theta d(\wp, \nu) + LD(\wp, S\nu) \quad \textit{for all } \wp, \nu \in \Upsilon.$$

3. Results

Let Ψ be the family of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{\eta=1}^{\infty} \psi^\eta(t) < \infty$ and $\psi(t) < t$, for each $t > 0$.

Definition 3.1. *(*[21]*) Consider* Υ *be a* $\mathcal{G}$*-metric space,* $\Gamma : \Upsilon \to \Upsilon$ *be a mapping and* $\alpha : \Upsilon \times \Upsilon \times \Upsilon \to [0, \infty)$. *The mapping* Γ *will be* $\mathcal{G}$*-*α*-admissible if for* $\wp, \nu, \varsigma \in \Upsilon$*:*

$$\alpha(\wp, \nu, \varsigma) \geq 1 \Longrightarrow \alpha(\Gamma\wp, \Gamma\nu, \Gamma\varsigma) \geq 1.$$

Definition 3.2. *([3]) Consider Υ be a $\mathcal{G}$-metric space, $\Gamma:\Upsilon\to\Upsilon$ be a mapping and $\alpha:\Upsilon\times\Upsilon\times\Upsilon\to[0,\infty)$. If a function $\psi\in\Psi$, such that*

$$\alpha(\wp,\nu,\varsigma)\mathcal{G}(\Gamma\wp,\Gamma\nu,\Gamma\varsigma)\leq\psi(\mathcal{G}(\wp,\nu,\varsigma))$$

$\forall\ \wp,\nu,\varsigma\in\Upsilon$, *then Γ is called $\mathcal{G}$-α-ψ- contractive type mapping.*

Definition 3.3. *Consider Υ be a $\mathcal{G}$-metric space, (Γ_1,Γ_2) be a pair of self mappings and $\alpha:\Upsilon\times\Upsilon\times\Upsilon\to[0,\infty)$. The pair (Γ_1,Γ_2) will be $\mathcal{G}$-α-admissible if for $\wp,\nu,\varsigma\in\Upsilon$,*

$$\alpha(\wp,\nu,\varsigma)\geq 1\Longrightarrow\alpha(\Gamma_1\wp,\Gamma_1\nu,\Gamma_2\varsigma)\geq 1,\alpha(\Gamma_2\wp,\Gamma_1\nu,\Gamma_1\varsigma)\geq 1.$$

Definition 3.4. *Consider Υ be a $\mathcal{G}$-metric space, $\Gamma:\Upsilon\to F(\Upsilon)$ be a multi-valued mapping, $\alpha:\Upsilon\times\Upsilon\times\Upsilon\to[0,\infty)$. The mapping Γ is called multi-vlaued $\mathcal{G}$-α-admissible mapping if for $\wp_0,\nu_0,\varsigma_0\in\Upsilon$,*

$$\alpha(\wp_0,\nu_0,\varsigma_0)\geq 1\Longrightarrow\alpha(\wp_1,\nu_1,\varsigma_1)\geq 1,\quad \textit{where } \wp_1\in\Gamma\wp_0,\nu_1\in\Gamma\nu_0,\varsigma_1\in\Gamma\varsigma.$$

The pair (Γ_1,Γ_2) of multi-valued mappings is called multi-valued $\mathcal{G}$-α-admissible if for $\wp_0,\nu_0,\varsigma_0\in\Upsilon$, $\alpha(\wp_0,\nu_0,\varsigma_0)\geq 1$ imply
(i) $\alpha(\wp_1,\nu_1,\varsigma_1)\geq 1$, whenever $\wp_1\in\Gamma_1\wp_0,\nu_1\in\Gamma_2\nu_0,\varsigma_1\in\Gamma_2\varsigma_0$,
(ii) $\alpha(\wp_1,\nu_1,\varsigma_1)\geq 1$, whenever $\wp_1\in\Gamma_2\wp_0,\nu_1\in\Gamma_1\nu_0,\varsigma_1\in\Gamma_1\varsigma_0$.

Theorem 3.5. *Consider Υ be a complete $\mathcal{G}$-metric space, $\Gamma_1,\Gamma_2:\Upsilon\to F(\Upsilon)$ be a pair of generalized multi-valued $\mathcal{G}$-α-ψ- contractive type mappings. Suppose $\forall\ \wp,\nu,\varsigma\in\Upsilon, L\geq 0$,*

$$\alpha(\wp,\nu,\varsigma)H_{\mathcal{G}}(\Gamma_1\wp,\Gamma_2\nu,\Gamma_2\varsigma)\leq\psi\left[\frac{1}{6}\mathcal{Y}_{i,j}\right]+L\mathcal{Z}_{i,j}\tag{3. 1}$$

where

$$\mathcal{Y}_{i,j}=\max\begin{pmatrix}6\mathcal{G}(\wp,\nu,\varsigma),\mathcal{G}(\wp,\Gamma_i\wp,\Gamma_i\wp),\mathcal{G}(\nu,\Gamma_j\nu,\Gamma_j\nu),\\ \frac{\mathcal{G}(\nu,\Gamma_i\wp,\Gamma_i\wp)+\mathcal{G}(\wp,\Gamma_j\nu,\Gamma_j\nu)}{2},\\ \frac{\mathcal{G}(\nu,\Gamma_j\nu,\Gamma_j\nu)[1+\mathcal{G}(\wp,\Gamma_i\wp,\Gamma_i\wp)]}{1+6\mathcal{G}(\wp,\nu,\varsigma)},\\ \frac{\mathcal{G}(\nu,\Gamma_i\wp,\Gamma_i\wp)[1+\mathcal{G}(\wp,\Gamma_j\nu,\Gamma_j\nu)]}{1+6\mathcal{G}(\wp,\nu,\varsigma)}\end{pmatrix},$$

$$\mathcal{Z}_{i,j}=\min\left[\begin{array}{c}\mathcal{G}(\wp,\Gamma_i\wp,\Gamma_i\wp),\mathcal{G}(\nu,\Gamma_j\nu,\Gamma_j\nu),\\ \mathcal{G}(\wp,\Gamma_j\nu,\Gamma_j\nu),\mathcal{G}(\nu,\Gamma_i\wp,\Gamma_i\wp)\end{array}\right].$$

Also assume that $\alpha(\wp,\nu,\nu)\geq 1\ \forall\ \wp,\nu\in\Upsilon$. Then there exists a common fixed point of the multi-valued mappings Γ_i and Γ_j.

Proof. Let $\wp_0\in\Upsilon$. As $\Gamma_1\wp_0\neq\varphi$ so there exists $\wp_1\in\Gamma_1\wp_0$. From Lemma 2, for a nonnegative real number $k<1$ there exists $\wp_2\in\Gamma_2\wp_1$ such that

$$\mathcal{G}(\wp_1,\wp_2,\wp_2)\leq kH_{\mathcal{G}}(\Gamma_1\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1)$$

As $\alpha(\wp_0,\wp_1,\wp_1)\geq 1$, so we conclude that

$$\mathcal{G}(\wp_1,\wp_2,\wp_2)\leq\alpha(\wp_0,\wp_1,\wp_1)H_{\mathcal{G}}(\Gamma_1\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1).$$

By condition (3.1) we get,

$$\begin{array}{rcl}\mathcal{G}(\wp_1,\wp_2,\wp_2) & \leq & \psi\left[\frac{1}{6}\max\left(\begin{array}{c}6\mathcal{G}(\wp_0,\wp_1,\wp_1),\mathcal{G}(\wp_0,\Gamma_1\wp_0,\Gamma_1\wp_0),\\ \mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1),\\ \frac{\mathcal{G}(\wp_1,\Gamma_1\wp_0,\Gamma_1\wp_0)+\mathcal{G}(\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1)}{2},\\ \frac{\mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1)[1+\mathcal{G}(\wp_0,\Gamma_1\wp_0,\Gamma_1\wp_0)]}{1+6\mathcal{G}(\wp_0,\wp_1,\wp_1)},\\ \frac{\mathcal{G}(\wp_1,\Gamma_1\wp_0,\Gamma_1\wp_0)[1+\mathcal{G}(\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1)]}{1+6\mathcal{G}(\wp_0,\wp_1,\wp_1)}\end{array}\right)\right] \\ & & +L\min\left[\begin{array}{c}\mathcal{G}(\wp_0,\Gamma_1\wp_0,\Gamma_1\wp_0),\mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1),\\ \mathcal{G}(\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1),\mathcal{G}(\wp_1,\Gamma_1\wp_0,\Gamma_1\wp_0)\end{array}\right] \\ & \leq & \psi\left[\max\left(\mathcal{G}(\wp_0,\wp_1,\wp_1),\mathcal{G}(\wp_1,\wp_2,\wp_2)\right)\right].\end{array}$$

By using properties of ψ we have

$$\mathcal{G}(\wp_1,\wp_2,\wp_2)\leq\psi\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right). \tag{3. 2}$$

Now for $\wp_2\in\Gamma_2\wp_1$ there exists $\wp_3\in\Gamma_1\wp_2$. By Lemma 2 and condition (3.1) we get,

$$\begin{array}{rcl}\mathcal{G}(\wp_2,\wp_3,\wp_3) & \leq & \alpha(\wp_1,\wp_2,\wp_2)H_{\mathcal{G}}(\Gamma_2\wp_1,\Gamma\wp_2,\Gamma\wp_2) \\ & \leq & \psi\left[\frac{1}{6}\max\left(\begin{array}{c}6\mathcal{G}(\wp_1,\wp_2,\wp_2),\mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1),\\ \mathcal{G}(\wp_2,\Gamma_1\wp_2,\Gamma_1\wp_2),\\ \frac{\mathcal{G}(\wp_2,\Gamma_2\wp_1,\Gamma_2\wp_1)+\mathcal{G}(\wp_1,\Gamma_1\wp_2,\Gamma_1\wp_2)}{2},\\ \frac{\mathcal{G}(\wp_2,\Gamma_1\wp_2,\Gamma_1\wp_2)[1+\mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1)]}{1+6\mathcal{G}(\wp_1,\wp_2,\wp_2)},\\ \frac{\mathcal{G}(\wp_2,\Gamma_2\wp_1,\Gamma_2\wp_1)[1+\mathcal{G}(\wp_1,\Gamma_1\wp_2,\Gamma_1\wp_2)]}{1+6\mathcal{G}(\wp_1,\wp_2,\wp_2)}\end{array}\right)\right] \\ & & +L\min\left[\begin{array}{c}\mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1),\mathcal{G}(\wp_2,\Gamma_1\wp_2,\Gamma_1\wp_2),\\ \mathcal{G}(\wp_2,\Gamma_2\wp_1,\Gamma_2\wp_1),\mathcal{G}(\wp_1,\Gamma_1\wp_2,\Gamma_1\wp_2)\end{array}\right] \\ & \leq & \psi\left[\max\left(\mathcal{G}(\wp_1,\wp_2,\wp_2),\mathcal{G}(\wp_2,\wp_3,\wp_3)\right)\right].\end{array}$$

Again by using properties of ψ, we get

$$\mathcal{G}(\wp_2,\wp_3,\wp_3)\leq\psi^2\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right).$$

Continuing in the same manner, we get a sequence $\{\wp_\eta\}$ in Υ for $\alpha(\wp_\eta,\wp_{\eta+1},\wp_{\eta+1})\geq 1$ such that

$$\mathcal{G}(\wp_\eta,\wp_{\eta+1},\wp_{\eta+1})\leq\psi^\eta\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right).$$

So by $(g5)$ and by a property of ψ, we have

$$\mathcal{G}(\wp_\eta,\wp_\zeta,\wp_\zeta)\leq\sum_{k=\eta}^{\zeta-1}\psi^k\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right)\leq\sum_{k=0}^{\infty}\psi^k\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right)<\infty.$$

This proves that the sequence $\{\wp_\eta\}$ is $\mathcal{G}$-Cauchy. The completeness of Υ ensures an element $\wp^*\in\Upsilon$ such that the sequence $\{\wp_\eta\}$ is $\mathcal{G}$-convergent to $\wp^*$ as $\eta\to\infty$.
Using the fact that $\wp_{2\eta+1}\in\Gamma_1\wp_{2\eta}$ and $\wp_{2\eta+2}\in\Gamma_2\wp_{2\eta+1}$, now it is shown that $\wp^*\in\Gamma_1\wp^*$

and $\wp^* \in \Gamma_2\wp^*$.

$$\begin{aligned}
\mathcal{G}(\wp_{2\eta+2},\Gamma_1\wp^*,\Gamma_1\wp^*) &\leq H_{\mathcal{G}}(\Gamma_2\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*) \\
&\leq \alpha(\wp_{2\eta+1},\wp^*,\wp^*)H_{\mathcal{G}}(\Gamma_2\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*) \\
&\leq \psi\left[\frac{1}{6}\max\left(\begin{array}{c} 6\mathcal{G}(\wp_{2\eta+1},\wp^*,\wp^*), \\ 6\mathcal{G}(\wp_{2\eta+1},\wp_{2\eta+2},\wp_{2\eta+2}), \\ \mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*), \\ \frac{6\mathcal{G}(\wp^*,\wp_{2\eta+2},\wp_{2\eta+2})+\mathcal{G}(\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*)}{2}, \\ \frac{\mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*)[1+6\mathcal{G}(\wp_{2\eta+1},\wp_{2\eta+2},\wp_{2\eta+2})]}{1+6\mathcal{G}(\wp_{2\eta+1},\wp^*,\wp^*)}, \\ \frac{6\mathcal{G}(\wp^*,\wp_{2\eta+2},\wp_{2\eta+2})[1+\mathcal{G}(\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*)]}{1+6\mathcal{G}(\wp_{2\eta+1},\wp^*,\wp^*)} \end{array}\right)\right] \\
&\quad +L\min\left[\begin{array}{c} 6\mathcal{G}(\wp_{2\eta+1},\wp_{2\eta+2},\wp_{2\eta+2}), \\ \mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*), \\ \mathcal{G}(\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*), \\ 6\mathcal{G}(\wp^*,\wp_{2\eta+2},\wp_{2\eta+2}) \end{array}\right].
\end{aligned}$$

Applying $\lim \eta \to \infty$, we get,

$$\begin{aligned}
\lim_{\eta\to\infty}\mathcal{G}(\wp_{2\eta+2},\Gamma_1\wp^*,\Gamma_1\wp^*) &\leq \lim_{\eta\to\infty}\psi\left[\frac{1}{6}\max\left(\begin{array}{c} 6\mathcal{G}(\wp_{2\eta+1},\wp^*,\wp^*), \\ 6\mathcal{G}(\wp_{2\eta+1},\wp_{2\eta+2},\wp_{2\eta+2}), \\ \mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*), \\ \frac{6\mathcal{G}(\wp^*,\wp_{2\eta+2},\wp_{2\eta+2})+ \mathcal{G}(\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*)}{2}, \\ \frac{\mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*) [1+ 6\mathcal{G}(\wp_{2\eta+1},\wp_{2\eta+2},\wp_{2\eta+2})]}{1+6\mathcal{G}(\wp_{2\eta+1},\wp^*,\wp^*)}, \\ \frac{6\mathcal{G}(\wp^*,\wp_{2\eta+2},\wp_{2\eta+2}) [1+ \mathcal{G}(\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*)]}{1+6\mathcal{G}(\wp_{2\eta+1},\wp^*,\wp^*)} \end{array}\right)\right] \\
&\quad +\lim_{\eta\to\infty}\left\{L\min\left[\begin{array}{c} 6\mathcal{G}(\wp_{2\eta+1},\wp_{2\eta+2},\wp_{2\eta+2}), \\ \mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*), \\ \mathcal{G}(\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*), \\ 6\mathcal{G}(\wp^*,\wp_{2\eta+2},\wp_{2\eta+2}) \end{array}\right]\right\}.
\end{aligned}$$

Assuming $\mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*) \neq 0$, and using property of ψ we get

$$\mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*) \leq \psi\left[\frac{\mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*)}{6}\right] < \frac{\mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*)}{6},$$

a contradiction. Hence

$$\mathcal{G}(\wp^*,\Gamma_1\wp^*,\Gamma_1\wp^*) = 0,$$

that is $\wp^* \in \Gamma_1\wp^*$.
Similarly we can prove that $\wp^* \in \Gamma_2\wp^*$ and $\wp^*$ is the common fixed point of the pair (Γ_1, Γ_2).

Example 3.6. *Let* $\Upsilon = [0,\infty)$. *Define* $\Gamma_1, \Gamma_2 : \Upsilon \to F(\Upsilon)$ *by* $\Gamma_1\wp = \left[0, \frac{\wp}{5}\right]$, $\Gamma_2\wp = \left[0, \frac{\wp}{3}\right]$ $\forall\, \wp \in \Upsilon$. *Define a* $\mathcal{G}$*-metric space on* Υ *by* $\mathcal{G}(\wp,\nu,\varsigma) = |\wp-\nu| + |\nu-\varsigma| + |\varsigma-\wp|$ *and*

$$\alpha(\wp,\nu,\varsigma) = \begin{cases} 1 & \text{if } \wp,\nu,\varsigma \in [0,1] \\ 0 & \text{otherwise} \end{cases},$$

then for $\psi(t) = \frac{t}{2}$, *conditions of above theorem are satisfied and* 0 *is the common fixed point of mappings* Γ_1 *and* Γ_2.

Corollary 3.7. *Consider* Υ *be a complete* $\mathcal{G}$*-metric space,* $\Gamma : \Upsilon \to F(\Upsilon)$ *be a generalized multi-valued* $\mathcal{G}$-α-ψ*- contractive type mapping. Suppose* $\forall\, \wp,\nu,\varsigma \in \Upsilon$,

$$\alpha(\wp,\nu,\varsigma)H_{\mathcal{G}}(\Gamma\wp,\Gamma\nu,\Gamma\varsigma) \le \psi\left(\frac{1}{6}\mathcal{Y}\right) + L\mathcal{Z} \tag{3. 3}$$

where

$$\mathcal{Y} = \max\begin{pmatrix} 6\mathcal{G}(\wp,\nu,\varsigma), \mathcal{G}(\wp,\Gamma\wp,\Gamma\wp), \mathcal{G}(\nu,\Gamma\nu,\Gamma\nu), \\ \frac{\mathcal{G}(\nu,\Gamma\wp,\Gamma\wp)+\mathcal{G}(\wp,\Gamma\nu,\Gamma\nu)}{2}, \\ \frac{\mathcal{G}(\nu,\Gamma\nu,\Gamma\nu)[1+\mathcal{G}(\wp,\Gamma\wp,\Gamma\wp)]}{1+6\mathcal{G}(\wp,\nu,\varsigma)}, \\ \frac{\mathcal{G}(\nu,\Gamma\wp,\Gamma\wp)[1+\mathcal{G}(\wp,\Gamma\nu,\Gamma\nu)]}{1+6\mathcal{G}(\wp,\nu,\varsigma)} \end{pmatrix},$$

$$\mathcal{Z} = \min\left[\begin{array}{c} \mathcal{G}(\wp,\Gamma\wp,\Gamma\wp), \mathcal{G}(\nu,\Gamma\nu,\Gamma\nu), \\ \mathcal{G}(\wp,\Gamma\nu,\Gamma\nu), \mathcal{G}(\nu,\Gamma\wp,\Gamma\wp) \end{array}\right].$$

Also assume that $\alpha(\wp,\nu,\nu) \ge 1$ $\forall\, \wp,\nu \in \Upsilon$. *Then there exists a fixed point of the multi-valued mapping* Γ.

□

Definition 3.8. *Consider* Υ *be a* $\mathcal{G}$*-metric space and* $\Gamma : \Upsilon \to F(\Upsilon)$. *Then* Γ *is called a multi-valued* (θ, L)*-weak contraction in* $\mathcal{G}$*-metric space if there exist two constants* $\theta \in (0,1)$ *and* $L \ge 0$ *such that*

$$H_{\mathcal{G}}(\Gamma\wp,\Gamma\nu,\Gamma\varsigma) \le \theta\mathcal{G}(\wp,\nu,\varsigma) + Ld_{\mathcal{G}}(\nu,\Gamma\wp) \quad \forall\, \wp,\nu,\varsigma \in \Upsilon.$$

Theorem 3.9. *Consider* Υ *be a complete* $\mathcal{G}$*-metric space and* $\Gamma : \Upsilon \to F(\Upsilon)$ *a multi-valued* (θ, L)*-weakly contractive mapping. Then there exists* $\wp^* \in \Upsilon$ *such that* $\wp^*$ *is a fixed point of* Γ.

Proof. Consider $\wp_0 \in \Upsilon$, then there exists $\wp_1 \in \Upsilon$ such that $\wp_1 \in \Gamma\wp_0$. Similarly there exists $\wp_2 \in \Upsilon$ such that $\wp_2 \in \Gamma\wp_1$.
Then for some $k_1 > 0$,

$$\begin{aligned} \mathcal{G}(\wp_1,\wp_2,\wp_2) &\le k_1 H_{\mathcal{G}}(\Gamma\wp_0,\Gamma\wp_1,\Gamma\wp_1) \\ &\le k_1\left[\theta\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right) + Ld_{\mathcal{G}}(\wp_1,\Gamma\wp_0)\right] \\ &\le k_1\left[\theta\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right) + L\inf\{d_{\mathcal{G}}(\wp_1,\nu) : \nu \in \Gamma\wp_0\}\right] \\ &\le k_1\theta\left(\mathcal{G}(\wp_0,\wp_1,\wp_1)\right). \end{aligned}$$

Similarly, for some $k_2 > 0$,

$$\begin{aligned} \mathcal{G}(\wp_2,\wp_3,\wp_3) &\le k_2 H_{\mathcal{G}}(\Gamma\wp_1,\Gamma\wp_2,\Gamma\wp_2) \\ &\le k_1k_2\theta^2\mathcal{G}(\wp_0,\wp_1,\wp_1). \end{aligned}$$

In general, $\wp_{\eta+1} \in \Gamma\wp_\eta, \eta \in \mathbb{N}$. Continuing in this way, we get $k_i > 0, i = 3, 4, ..., \eta$ such that

$$\mathcal{G}(\wp_\eta, \wp_{\eta+1}, \wp_{\eta+1}) \leq k_1 k_2 ... k_\eta \theta^\eta \mathcal{G}(\wp_0, \wp_1, \wp_1) = l_1 \theta^\eta \mathcal{G}(\wp_0, \wp_1, \wp_1),$$

where $l_1 = k_1 k_2 ... k_\eta$.
Consider for $\zeta > \eta$, then by $(g5)$:

$$\mathcal{G}(\wp_\eta, \wp_\zeta, \wp_\zeta) \leq \theta^\eta (l_1 + l_2\theta + ... + l_{\zeta-\eta}\theta^{\zeta-\eta-1})\mathcal{G}(\wp_0, \wp_1, \wp_1).$$

Applying limit $\zeta, \eta \to \infty, \mathcal{G}(\wp_\eta, \wp_\zeta, \wp_\zeta) \to 0$. Hence the sequence $\{\wp_\eta\}$ is $\mathcal{G}$-Cauchy. Completeness of Υ ensures that, $\wp^* \in \Upsilon$ such that $\wp_\eta \to \wp^*$ as $\eta \to \infty$.
Now, for some $k > 0$

$$\begin{aligned} \mathcal{G}(\wp_\eta, \Gamma\wp^*, \Gamma\wp^*) &\leq kH_{\mathcal{G}}(\Gamma\wp_{\eta-1}, \Gamma\wp^*, \Gamma\wp^*) \\ &\leq k\left[\theta\mathcal{G}(\wp_{\eta-1}, \wp^*, \wp^*) + Ld_{\mathcal{G}}(\wp^*, \Gamma\wp_{\eta-1})\right] \\ &\leq k\left[\theta\mathcal{G}(\wp_{\eta-1}, \wp^*, \wp^*) + L\inf\{d_{\mathcal{G}}(\wp^*, \nu) : \nu \in \Gamma\wp_{\eta-1}\}\right]. \end{aligned}$$

Applying $\lim \eta \to \infty$, we get

$$\lim_{\eta\to\infty} \mathcal{G}(\wp_\eta, \Gamma\wp^*, \Gamma\wp^*) \leq k \lim_{\eta\to\infty} \left[\theta\mathcal{G}(\wp_{\eta-1}, \wp^*, \wp^*) + L\inf\{d_{\mathcal{G}}(\wp^*, \nu) : \nu \in \Gamma\wp_{\eta-1}\}\right],$$

which gives,

$$\mathcal{G}(\wp^*, \Gamma\wp^*, \Gamma\wp^*) = 0, \text{as } \wp_\eta \in \Gamma\wp_{\eta-1}.$$

This implies $\wp^* \in \Gamma\wp^*$. Hence $\wp^*$ is fixed point of the mapping Γ. □

Example 3.10. *Let $\Upsilon = [0, 1]$ and $\Gamma : \Upsilon \to F(\Upsilon)$ be a multi-valued (θ, L)-weak contractive mapping defined as,*

$$\Gamma\wp = \left[0, \frac{\wp}{7}\right] \qquad \forall\, \wp \in \Upsilon,$$

then for $\theta = \frac{1}{2}$ and $L \geq 0$, conditions of above Theorem are satisfied and 0 is the fixed point of mapping Γ.

Corollary 3.11. *Consider Υ a complete $\mathcal{G}$-metric space. Let $\Gamma : \Upsilon \to \Upsilon$ be a (θ, L)-weakly contractive mapping, that is, there exists $\theta \in (0, 1)$, $L \geq 0$ and $\forall\, \wp, \nu, \varsigma \in \Upsilon, L \geq 0$, the following hold*

$$\mathcal{G}(\Gamma\wp, \Gamma\nu, \Gamma\varsigma) \leq \theta\mathcal{G}(\wp, \nu, \varsigma) + Ld_{\mathcal{G}}(\nu, \Gamma\wp).$$

Then Γ has a fixed point in Υ.

Theorem 3.12. *Consider Υ a complete $\mathcal{G}$-metric space. Let $\Gamma_1, \Gamma_2 : \Upsilon \to F(\Upsilon)$ be a pair of multi valued (θ, L)-weakly contractive type mappings satisfying*

$$H_{\mathcal{G}}(\Gamma_i\wp, \Gamma_j\nu, \Gamma_j\varsigma) \leq \frac{\theta}{6}\mathcal{N}_i + L_1 d_{\mathcal{G}}(\nu, \Gamma_1\wp) + L_2\mathcal{M}_{i,j} \tag{3. 4}$$

where

$$\mathcal{N}_i = \max\left(6\mathcal{G}(\wp, \nu, \varsigma), \mathcal{G}(\wp, \Gamma_i\wp, \Gamma_i\wp), \frac{\mathcal{G}(\nu, \Gamma_i\wp, \Gamma_i\wp) + \mathcal{G}(\varsigma, \Gamma_i\wp, \Gamma_i\wp)}{2}\right)$$

$$\mathcal{M}_{i,j} = \min\left[\mathcal{G}(\nu, \Gamma_j\nu, \Gamma_j\nu), \mathcal{G}(\varsigma, \Gamma_i\wp, \Gamma_i\wp), \mathcal{G}(\wp, \Gamma_j\varsigma, \Gamma_j\varsigma)\right]$$

$\forall\ \wp,\nu,\varsigma\in\Upsilon$, $\theta\in(0,1)$ *and* $L_1,L_2\geq 0$, *where* $i\neq j$ $i,j=1,2$.
Then there exists $\wp^*\in\Upsilon$ *such that* $\wp^*\in\Gamma_i\wp^*$ *and* $\wp^*\in\Gamma_j\wp^*$ *i.e* $\wp^*$ *is a common fixed point of* Γ_i *and* Γ_j.

Proof. Consider $\wp_0\in\Upsilon$. As $\Gamma_1\wp_0\neq\varphi$, define $\wp_1\in\Gamma_1\wp_0$ and similarly $\wp_2\in\Gamma_2\wp_1$. By condition (3.3) and Lemma 2, there exists $k_1>0$ such that

$$\begin{aligned}\mathcal{G}(\wp_1,\wp_2,\wp_2) &\leq k_1H_{\mathcal{G}}(\Gamma_1\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1)\\ &\leq k_1\left[\begin{array}{c}\frac{\theta}{6}\max\left(\begin{array}{c}6\mathcal{G}(\wp_0,\wp_1,\wp_1),\mathcal{G}(\wp_0,\Gamma_1\wp_0,\Gamma_1\wp_0),\\ \frac{\mathcal{G}(\wp_1,\Gamma_1\wp_0,\Gamma_1\wp_0)+\mathcal{G}(\wp_1,\Gamma_1\wp_0,\Gamma_1\wp_0)}{2}\end{array}\right)\\ +L_1d_{\mathcal{G}}(\wp_1,\Gamma_1\wp_0)+L_2\min\left[\begin{array}{c}\mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1),\\ \mathcal{G}(\wp_1,\Gamma_1\wp_0,\Gamma_1\wp_0),\\ \mathcal{G}(\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1)\end{array}\right]\end{array}\right]\\ &\leq k_1\theta\mathcal{G}(\wp_0,\wp_1,\wp_1).\end{aligned}$$

Now for $\wp_2\in\Gamma_2\wp_1$ and $\wp_3\in\Gamma_1\wp_2$, there exists $k_2>0$ such that

$$\begin{aligned}\mathcal{G}(\wp_2,\wp_3,\wp_3) &\leq k_2H_{\mathcal{G}}(\Gamma_2\wp_1,\Gamma_1\wp_2,\Gamma_1\wp_2)\\ &\leq k_2\left[\begin{array}{c}\frac{\theta}{6}\max\left(\begin{array}{c}6\mathcal{G}(\wp_1,\wp_2,\wp_2),\mathcal{G}(\wp_1,\Gamma_2\wp_1,\Gamma_2\wp_1),\\ \frac{\mathcal{G}(\wp_2,\Gamma_2\wp_1,\Gamma_2\wp_1)+\mathcal{G}(\wp_2,\Gamma_2\wp_1,\Gamma_2\wp_1)}{2}\end{array}\right)\\ +L_3d_{\mathcal{G}}(\wp_2,\Gamma_2\wp_1)\\ +L_4\min\left\{\begin{array}{c}\mathcal{G}(\wp_2,\Gamma_1\wp_2,\Gamma_1\wp_2),\mathcal{G}(\wp_2,\Gamma_2\wp_1,\Gamma_2\wp_1),\\ \mathcal{G}(\wp_1,\Gamma_1\wp_2,\Gamma_1\wp_2)\end{array}\right\}\end{array}\right]\\ &\leq k_2\theta\mathcal{G}(\wp_1,\wp_2,\wp_2)\\ &\leq k_1k_2\theta^2\mathcal{G}(\wp_0,\wp_1,\wp_1)\end{aligned}$$

Continuing in this way, we get

$$\mathcal{G}(\wp_\eta,\wp_{\eta+1},\wp_{\eta+1})\leq k_1k_2...k_\eta\theta^\eta\mathcal{G}(\wp_0,\wp_1,\wp_1)=l_1\theta^\eta\mathcal{G}(\wp_0,\wp_1,\wp_1)$$

$\forall\ k_i>0,i=3,4,...,\eta$, where $l_1=k_1k_2...k_\eta$.
Consider for $\zeta>\eta$, then by $(g5)$:

$$\mathcal{G}(\wp_\eta,\wp_\zeta,\wp_\zeta)\leq\theta^\eta(l_1+l_2\theta+...+l_{\zeta-\eta}\theta^{\zeta-\eta-1})\mathcal{G}(\wp_0,\wp_1,\wp_1).$$

Applying limit $\zeta,\eta\to\infty,\mathcal{G}(\wp_\eta,\wp_\zeta,\wp_\zeta)\to 0$. Hence the sequence $\{\wp_\eta\}$ is $\mathcal{G}$-Cauchy. The completeness of Υ ensures that, $\wp^*\in\Upsilon$ so that $\wp_\eta\to\wp^*$ as $\eta\to\infty$.
Since $\wp_{2\eta+1}\in\Gamma_1\wp_{2\eta}$ and $\wp_{2\eta+2}\in\Gamma_2\wp_{2\eta+1}$, now it is shown that $\wp^*\in\Gamma_1\wp^*$ and

$\wp^* \in \Gamma_2\wp^*$.

$$
\begin{aligned}
\mathcal{G}(\wp_{2\eta+1}, \Gamma_2\wp^*, \Gamma_2\wp^*) &\leq kH_{\mathcal{G}}(\Gamma_1\wp_{2\eta}, \Gamma_2\wp^*, \Gamma_2\wp^*) \\
&\leq k\left[\begin{array}{c}
\frac{\theta}{6}\max\left(\begin{array}{c}
6\mathcal{G}(\wp_{2\eta}, \wp^*, \wp^*), \\
\mathcal{G}(\wp_{2\eta}, \Gamma_1\wp_{2\eta}, \Gamma_1\wp_{2\eta}), \\
\frac{\mathcal{G}(\wp^*, \Gamma_1\wp_{2\eta}, \Gamma_1\wp_{2\eta}) + \mathcal{G}(\wp^*, \Gamma_1\wp_{2\eta}, \Gamma_1\wp_{2\eta})}{2}
\end{array}\right) \\
+L_1 d_{\mathcal{G}}(\wp^*, \Gamma_1\wp_{2\eta}) + \\
L_2\min\left[\begin{array}{c}
\mathcal{G}(\wp^*, \Gamma_2\wp^*, \Gamma_2\wp^*), \\
\mathcal{G}(\wp^*, \Gamma_1\wp_{2\eta}, \Gamma_1\wp_{2\eta}), \\
\mathcal{G}(\wp_{2\eta}, \Gamma_2\wp^*, \Gamma_2\wp^*)
\end{array}\right]
\end{array}\right] \\
&\leq k\left[\begin{array}{c}
\frac{\theta}{6}\max\left(\begin{array}{c}
6\mathcal{G}(\wp_{2\eta}, \wp^*, \wp^*), \\
6\mathcal{G}(\wp_{2\eta}, \wp_{2\eta+1}, \wp_{2\eta+1}), \\
\frac{6\mathcal{G}(\wp^*, \wp_{2\eta+1}, \wp_{2\eta+1}) + \mathcal{G}(\wp^*, \wp_{2\eta+1}, \wp_{2\eta+1})}{2}
\end{array}\right) \\
+L_1\inf\{d_{\mathcal{G}}(\wp^*, \nu) : \nu \in \Gamma_1\wp_{2\eta}\} \\
+L_2\min\left[\begin{array}{c}
\mathcal{G}(\wp^*, \Gamma_2\wp^*, \Gamma_2\wp^*), \\
6\mathcal{G}(\wp^*, \wp_{2\eta+1}, \wp_{2\eta+1}), \\
\mathcal{G}(\wp_{2\eta}, \Gamma_2\wp^*, \Gamma_2\wp^*)
\end{array}\right]
\end{array}\right].
\end{aligned}
$$

Applying $\lim \eta \to \theta$, we get $\mathcal{G}(\wp^*, \Gamma_2\wp^*, \Gamma_2\wp^*) = 0$, that is $\wp^* \in \Gamma_2\wp^*$.
Now,

$$
\begin{aligned}
\mathcal{G}(\wp_{2\eta+2}, \Gamma_1\wp^*, \Gamma_1\wp^*) &\leq kH_{\mathcal{G}}(\Gamma_2\wp_{2\eta+1}, \Gamma_1\wp^*, \Gamma_1\wp^*) \\
&\leq k\left[\begin{array}{c}
\frac{\theta}{6}\left\{\max\left(\begin{array}{c}
6\mathcal{G}(\wp_{2\eta+1}, \wp^*, \wp^*), \\
\mathcal{G}(\wp_{2\eta+1}, \Gamma_2\wp_{2\eta+1}, \Gamma_2\wp_{2\eta+1}), \\
\frac{\mathcal{G}(\wp^*, \Gamma_2\wp_{2\eta+1}, \Gamma_2\wp_{2\eta+1}) + \mathcal{G}(\wp^*, \Gamma_2\wp_{2\eta+1}, \Gamma_2\wp_{2\eta+1})}{2}
\end{array}\right)\right\} \\
+L_3 d_{\mathcal{G}}(\wp^*, \Gamma_2\wp_{2\eta+1}) + \\
L_4\min\left[\begin{array}{c}
\mathcal{G}(\wp^*, \Gamma_1\wp^*, \Gamma_1\wp^*), \\
\mathcal{G}(\wp^*, \Gamma_2\wp_{2\eta+1}, \Gamma_2\wp_{2\eta+1}), \\
\mathcal{G}(\wp_{2\eta+1}, \Gamma_1\wp^*, \Gamma_1\wp^*)
\end{array}\right]
\end{array}\right] \\
&\leq k\left[\begin{array}{c}
\frac{\theta}{6}\max\left(\begin{array}{c}
6\mathcal{G}(\wp_{2\eta+1}, \wp^*, \wp^*), \\
6\mathcal{G}(\wp_{2\eta+1}, \wp_{2\eta+2}, \wp_{2\eta+2}), \\
\frac{6\mathcal{G}(\wp^*, \wp_{2\eta+2}, \wp_{2\eta+2}) + \mathcal{G}(\wp^*, \wp_{2\eta+2}, \wp_{2\eta+2})}{2}
\end{array}\right) \\
+L_3\inf\{d_{\mathcal{G}}(\wp^*, \nu) : \nu \in \Gamma_2\wp_{2\eta+1}\} \\
+L_4\min\left[\begin{array}{c}
\mathcal{G}(\wp^*, \Gamma_1\wp^*, \Gamma_1\wp^*), \\
6\mathcal{G}(\wp^*, \wp_{2\eta+2}, \wp_{2\eta+2}), \\
\mathcal{G}(\wp_{2\eta+1}, \Gamma_1\wp^*, \Gamma_1\wp^*)
\end{array}\right]
\end{array}\right].
\end{aligned}
$$

Applying $\lim \eta \to \infty$, we get $\mathcal{G}(\wp^*, \Gamma_1\wp^*, \Gamma_1\wp^*) = 0$ which implies $\wp^* \in \Gamma_1\wp^*$.
So $\wp^*$ is the common fixed point of mappings Γ_1 and Γ_2 in a complete $\mathcal{G}$-metric space. □

Example 3.13. *Consider* $\Upsilon = [0,1]$. *Suppose* $\Gamma_1, \Gamma_2 : [0,1] \to F(\Upsilon)$ *be a pair of multi-valued* (θ, L)*-weakly contractive type mapping defined as,*

$$\Gamma_1\wp = \left[0, \frac{\wp}{5}\right] \qquad \Gamma_2\wp = \left[0, \frac{\wp}{7}\right] \quad \forall\, \wp \in \Upsilon,$$

Define a $\mathcal{G}$*-metric space on* Υ *as in Example 1, then for* $\theta = \frac{1}{2}$ *and* $L_1, L_2, L_3, L_4 \geq 0$, *conditions of above theorem are satisfied and* 0 *is the common fixed point of given mappings.*

By setting $\Gamma_1 = \Gamma_2$ in the last theorem, the following result is induced.

Corollary 3.14. *Consider* Υ *a complete* $\mathcal{G}$*-metric space. Let* $\Gamma : \Upsilon \to F(\Upsilon)$ *be a multi-valued* (θ, L)*-weakly contractive type mapping such that* $\forall\ \wp, \nu, \varsigma \in \Upsilon$ *and* $\theta \in (0,1), L_1, L_2 \geq 0$, *the following holds*

$$\begin{aligned} H_{\mathcal{G}}(\Gamma\wp, \Gamma\nu, \Gamma\varsigma) \quad &\leq \quad \frac{\theta}{6}\max\begin{pmatrix} 6\mathcal{G}(\wp,\nu,\varsigma), \mathcal{G}(\wp,\Gamma\wp,\Gamma\wp), \\ \frac{\mathcal{G}(\nu,\Gamma\wp,\Gamma\wp)+\mathcal{G}(\varsigma,\Gamma\wp,\Gamma\wp)}{2} \end{pmatrix} \\ &\quad + L_1 d_{\mathcal{G}}(\nu, \Gamma\wp) + L_2 \min\begin{bmatrix} \mathcal{G}(\nu,\Gamma\nu,\Gamma\nu), \\ \mathcal{G}(\varsigma,\Gamma\wp,\Gamma\wp), \\ \mathcal{G}(\wp,\Gamma\varsigma,\Gamma\varsigma) \end{bmatrix} \end{aligned} \tag{3. 5}$$

Then there exists $\wp^* \in \Upsilon$ *such that* $\wp^* \in \Gamma\wp^*$ *i.e* $\wp^*$ *is a fixed point of* Γ.

Coincidence Points of Mappings and Relation

Consider sets $\check{C}$, $\check{D} \neq \emptyset$. A relation Ω from $\check{C}$ to $\check{D}$ is a subset of $\check{C} \times \check{D}$ represented by $\Omega : \check{C} \rightsquigarrow \check{D}$. The statement $(\wp, \nu) \in \Omega$ is represented by $\wp\Omega\nu$.
A relation $\Omega : \check{C} \rightsquigarrow \check{D}$ is known as left-total if for every $\wp \in \check{C}$ there exists a $\nu \in \check{D}$ so that $\wp\Omega\nu$.
For $\Omega : \check{C} \rightsquigarrow \check{D}, W \subset \check{C}$ define[8]

$$\Omega(W) = \{\nu \in \check{D} : \wp\Omega\nu \text{ for some } \wp \in W\},$$

$$dom(\Omega) = \{\wp \in \check{C} : \Omega(\{\wp\}) \neq \phi\},$$

$$Range(\Omega) = \{\nu \in \check{D} : \nu \in \Omega(\{\wp\}) \text{ for some } \wp \in dom(\Omega)\}.$$

For ease of notation, present $\Omega(\{\wp\})$ by $\Omega_\wp$ and $Range$ by $\varrho(\Omega)$.$(\check{C}, \check{D})$ denotes the class of all relations from $\check{C}$ to $\check{D}$.
An element $\wp \in \check{C}$ is known as coincidence point of mapping $\Gamma : \check{C} \to \check{D}$ and relation $\Omega : \check{C} \rightsquigarrow \check{D}$ if $\Gamma\wp \in \Omega\{\wp\}$.
For $\Omega : \Upsilon \rightsquigarrow \Im$ and $u, v, w \in dom(\Omega)$,we define:

$$D_{\mathcal{G}}\left(\Omega_u, \Omega_v, \Omega_w\right) = \inf_{u\Omega\wp, v\Omega\nu, w\Omega\varsigma} \mathcal{G}(\wp, \nu, \varsigma)$$

Theorem 3.15. *Consider* Υ *as* $\mathcal{G}$*-metric space. Consider* $\Gamma_1, \Gamma_2 : \Upsilon \to F(\Upsilon)$ *be a pair of multi-valued* $\mathcal{G}$*-*α*-admissible mappings and* $\Omega : \Upsilon \rightsquigarrow \Upsilon$ *be any relation in such a way that*
(i) Ω *is left total;*
(ii) $\varrho(\Omega)$ *contains all the closed bounded subsets of* Υ*;*
(iii) $\varrho(\Omega)$ *is complete;*

$(iv)\ \alpha\left(\wp,\nu,\nu\right)\geq 1\ \forall\ \wp,\nu\in\Upsilon$.
Suppose that for $\psi\in\Psi$ *and* $\forall\ \wp,\nu,\varsigma\in\Upsilon$,

$$\alpha(\wp,\nu,\varsigma)H_{\mathcal{G}}(\Gamma_1\wp,\Gamma_2\nu,\Gamma_2\varsigma)\leq\psi\left[D_{\mathcal{G}}\left(\Omega_{\wp},\Omega_{\nu},\Omega_{\varsigma}\right)\right], \tag{3. 6}$$

and

$$\alpha(\wp,\nu,\varsigma)H_{\mathcal{G}}(\Gamma_2\wp,\Gamma_1\nu,\Gamma_1\varsigma)\leq\psi\left[D_{\mathcal{G}}\left(\Omega_{\wp},\Omega_{\nu},\Omega_{\varsigma}\right)\right]. \tag{3. 7}$$

Then there exists $\wp^*\in\Upsilon$ *such that* $\Gamma_1\wp^*\cap\Gamma_2\wp^*\cap\Omega\{\wp^*\}\neq\varphi$.

Proof. Consider $\wp_0\in\Upsilon$. Since Ω is left total, so for $\wp_0\in\Upsilon$ there exists $r_0\in\varrho(\Omega)$ such that $\wp_0\Omega r_{0.}$ Consider the sequences $\{\wp_\eta\}\subset\Upsilon,\{r_\eta\}\subset\varrho(\Omega)$. Let $r_1\in\Gamma_1\wp_0$. As $\Gamma_1\wp_0\subset Ra\eta ge(\Omega)$ select $\wp_1\in\Upsilon$ such that $\wp_1\Omega r_1$. Let $r_2\in\Gamma_2\wp_1$ then there exists $\wp_2\in\Upsilon$ such that $\wp_2\Omega r_2$.
Now from Lemma 2 and (3.5) we have

$$\begin{aligned}\mathcal{G}(r_1,r_2,r_2) &\leq \alpha(\wp_0,\wp_1,\wp_1)H_{\mathcal{G}}(\Gamma_1\wp_0,\Gamma_2\wp_1,\Gamma_2\wp_1)\\ &\leq \psi\left[D_{\mathcal{G}}\left(\Omega_{\wp_0},\Omega_{\wp_1},\Omega_{\wp_1}\right)\right]\\ &\leq \psi\left(\mathcal{G}(r_0,r_1,r_1)\right).\end{aligned}$$

Again using condition (3.6) we have:

$$\begin{aligned}\mathcal{G}(r_2,r_3,r_3) &\leq \alpha(\wp_1,\wp_2,\wp_2)\mathcal{G}(\Gamma_2\wp_1,\Gamma_1\wp_2,\Gamma_1\wp_2)\\ &\leq \psi\left[D_{\mathcal{G}}\left(\Omega_{\wp_1},\Omega_{\wp_2},\Omega_{\wp_2}\right)\right]\\ &\leq \psi\left(\mathcal{G}(r_1,r_2,r_2)\right).\end{aligned}$$

By induction, we can have sequences $\{\wp_\eta\}\subset\Upsilon$ and $\{r_\eta\}\subset\varrho(\Omega)$ such that $r_{2\eta+1}\in\Gamma_1\wp_{2\eta},r_{2\eta+2}\in\Gamma_2\wp_{2\eta+1}$ and $\wp_\eta\Omega r_\eta$ for $\eta=0,1,2...$

$$\begin{aligned}\mathcal{G}(r_\eta,r_{\eta+1},r_{\eta+1}) &\leq \psi\left[D_{\mathcal{G}}\left(\Omega_{\wp_{\eta-1}},\Omega_{\wp_\eta},\Omega_{\wp_\eta}\right)\right]\\ &\leq \psi\left(\mathcal{G}(r_{\eta-1},r_\eta,r_\eta)\right).\end{aligned}$$

By the monotone property of ψ, we have:

$$\mathcal{G}(r_\eta,r_{\eta+1},r_{\eta+1})\leq\psi^\eta\left(\mathcal{G}(r_0,r_1,r_1)\right).$$

So for $\eta>\zeta$, by $(g5)$ we have,

$$\begin{aligned}\mathcal{G}(r_\eta,r_\zeta,r_\zeta) &\leq \psi^\eta\left(\mathcal{G}(r_0,r_1,r_1)\right)+\psi^{\eta+1}\left(\mathcal{G}(r_0,r_1,r_1)\right)+...\\ &\quad +\psi^{\zeta-1}\left(\mathcal{G}(r_0,r_1,r_1)\right)\end{aligned}$$

By a property of ψ, we have

$$\mathcal{G}(r_\eta,r_\zeta,r_\zeta)\leq\sum_{k=\eta}^{\zeta-1}\psi^k\left(\mathcal{G}(r_0,r_1,r_1)\right)\leq\sum_{k=0}^{\infty}\psi^k\left(\mathcal{G}(r_0,r_1,r_1)\right)<\infty.$$

This shows that sequence $\{r_\eta\}$ is Cauchy. The completeness of $\varrho(\Omega)$ ensures an element $w\in\varrho(\Omega)$ such that $r_\eta\to w$ as $\eta\to\infty$. Also $\wp^*\Omega w$ for some $\wp^*\in\Upsilon$.

Now

$$\begin{aligned}\mathcal{G}(r_{2\eta+2},\Gamma_1\wp^*,\Gamma_1\wp^*) &\leq H_{\mathcal{G}}(\Gamma_2\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*)\\ &\leq \alpha(\wp_{2\eta+1},\wp^*,\wp^*)H_{\mathcal{G}}(\Gamma_2\wp_{2\eta+1},\Gamma_1\wp^*,\Gamma_1\wp^*)\\ &\leq \psi\left[D_{\mathcal{G}}\left(\Omega_{\wp_{2\eta+1}},\Omega_{\wp^*},\Omega_{\wp^*}\right)\right]\\ &\leq \psi\left(\mathcal{G}(r_{2\eta+1},w,w)\right)\\ &< \mathcal{G}(r_{2\eta+1},w,w).\end{aligned}$$

Letting $\eta\to\infty$ we have $\mathcal{G}(w,\Gamma_1\wp^*,\Gamma_1\wp^*)=0$. It follows that $w\in\Gamma_1\wp^*$.
Also,

$$\begin{aligned}\mathcal{G}(r_{2\eta+1},\Gamma_2\wp^*,\Gamma_2\wp^*) &\leq H_{\mathcal{G}}(\Gamma_1\wp_{2\eta+2},\Gamma_2\wp^*,\Gamma_2\wp^*)\\ &< \mathcal{G}(\wp_{2\eta+2},w,w).\end{aligned}$$

Letting $\eta\to\infty$ we obtain $\mathcal{G}(w,\Gamma_2\wp^*,\Gamma_2\wp^*)=0$. It follows that $w\in\Gamma_2\wp^*$. Consequently,

$$\Gamma_1\wp^*\cap\Gamma_2\wp^*\cap\Omega_{\wp^*}\neq\emptyset$$

□

Corollary 3.16. *Consider Υ be a $\mathcal{G}$-metric space. Consider $\Gamma:\Upsilon\to F(\Upsilon)$ be a multi-valued $\mathcal{G}$-α-admissible mapping and $\Omega:\Upsilon\rightsquigarrow\Upsilon$ be any relation such that*
(i) Ω is left total;
(ii) $\varrho(\Omega)$ contains all the closed bounded subsets of Υ;
(iii) $\varrho(\Omega)$ is complete;
(iv) $\alpha(\wp,\nu,\nu)\geq 1\ \forall\ \wp,\nu\in\Upsilon$.
Suppose that for $\psi\in\Psi$ and $\forall\ \wp,\nu,\varsigma\in\Upsilon$,

$$\alpha(\wp,\nu,\varsigma)H_{\mathcal{G}}(\Gamma\wp,\Gamma\nu,\Gamma\varsigma)\leq\psi\left[D_{\mathcal{G}}\left(\Omega_{\wp},\Omega_{\nu},\Omega_{\varsigma}\right)\right],$$

Then there exists $\wp^*\in\Upsilon$ such that $\Gamma\wp^*\cap\Omega\{\wp^*\}\neq\varphi$.

Corollary 3.17. *Consider Υ be a $\mathcal{G}$-metric space. Consider $\Gamma:\Upsilon\to\Upsilon$ be a $\mathcal{G}$-α-admissible mapping and $\Omega:\Upsilon\rightsquigarrow\Upsilon$ be any relation such that*
(i) Ω is left total;
(ii) $\varrho(\Omega)$ is complete;
(iii) $\alpha(\wp,\nu,\nu)\geq 1\ \forall\ \wp,\nu\in\Upsilon$.
Suppose that for $\psi\in\Psi$ and $\forall\ \wp,\nu,\varsigma\in\Upsilon$,

$$\alpha(\wp,\nu,\varsigma)\mathcal{G}(\Gamma\wp,\Gamma\nu,\Gamma\varsigma)\leq\psi\left[D_{\mathcal{G}}\left(\Omega_{\wp},\Omega_{\nu},\Omega_{\varsigma}\right)\right],$$

Then there exists $\wp^*\in\Upsilon$ such that $\Gamma\wp^*\in\Omega\{\wp^*\}$.

4. Application

In the following section we provide the application of our main result in terms of an existence result for the following differential inclusion:

$$\begin{cases}\frac{\partial^2 u(x,t)}{\partial x\partial t}\in F\left(x,t,u\left(x,t\right)\right)\ for\ \left(x,t\right)\in\Pi_{\mathbb{R}\times T}\\ u\left(x,0\right)=0,\ x\in\mathbb{R},t\in T=[0,a]\end{cases},\tag{4. 8}$$

where $\Pi_{\mathbb{R}\times T} = \mathbb{R}\times T$. Suppose $F : \Pi_{\mathbb{R}\times T}\times\mathbb{R}\to 2^{\mathbb{R}}$ is a multivalued mapping. For $\Lambda = C(\Pi_{\mathbb{R}\times T},\mathbb{R})$ define a G-metric on Λ as follows:

$$\begin{aligned}(G(v_1,v_2,v_3))(x,t) \quad = \quad & \sup_{(x,t)\in\Pi_{\mathbb{R}\times T}} \|v_1(x,t)-v_2(x,t)\| + \\ & \sup_{(x,t)\in\Pi_{\mathbb{R}\times T}} \left\|\frac{\partial v_1(x,t)}{\partial x}-\frac{\partial v_2(x,t)}{\partial x}\right\| + \\ & \sup_{(x,t)\in\Pi_{\mathbb{R}\times T}} \|v_2(x,t)-v_3(x,t)\| + \\ & \sup_{(x,t)\in\Pi_{\mathbb{R}\times T}} \left\|\frac{\partial v_2(x,t)}{\partial x}-\frac{\partial v_3(x,t)}{\partial x}\right\| + \\ & \sup_{(x,t)\in\Pi_{\mathbb{R}\times T}} \|v_3(x,t)-v_1(x,t)\| + \\ & \sup_{(x,t)\in\Pi_{\mathbb{R}\times T}} \left\|\frac{\partial v_3(x,t)}{\partial x}-\frac{\partial v_1(x,t)}{\partial x}\right\|,\end{aligned}$$

for $v_1,v_2,v_3\in\Lambda$.Then Λ is a complete G-metric space.
Define a partial order $\preceq$ on Λ as follows:

$$\begin{aligned}\text{for } r,s \quad &\in \quad \Lambda, r\preceq s \text{ if and only if} \\ \|r(x,t)\| \quad &\le \quad \|s(x,t)\| \text{ and } \left\|\frac{\partial r(x,t)}{\partial x}\right\| \le \left\|\frac{\partial s(x,t)}{\partial x}\right\|.\end{aligned}$$

Let $\mathcal{L} = L^1(\Pi_{\mathbb{R}\times T},\mathbb{R})$ be the Banach space consisting of all measurable functions $\xi : \Pi_{\mathbb{R}\times T}\to\mathbb{R}$ which are Lebesgue integrable with norm given by

$$\|\xi\|_{\mathcal{L}} = \left|\int_{-\infty}^{\infty}\int_{0}^{a}\xi(\chi,\tau)\,d\tau d\chi\right|, \quad for\ \xi\in\mathcal{L}.$$

For each γ in the space Λ the set of all selections of F is defined as

$$\Upsilon_{F,\gamma} = \{\xi\in\mathcal{L} : \xi\in F(x,t,\gamma(x,t))\ \ a.e\ \ (x,t)\in\Pi_{\mathbb{R}\times T}\}.$$

Now we allocate F a multivalued operator S from Λ to $2^{\mathcal{L}}$ by assuming

$$S(\gamma) = \{\omega\in\mathcal{L} : \xi\in F(x,t,\gamma(x,t))\ \ for\ \ (x,t)\in\Pi_{\mathbb{R}\times T}\}$$

where S is the Niemytsky operator [7].
In order to establish our result we require a continuous mapping $\pi : \mathcal{L}\to *$ defined as

$$\pi(\gamma) = \int_{0}^{t}\int_{-\infty}^{\infty}\gamma(\chi,\tau)\,d\chi d\tau.$$

In the following we use Theorem to obtain the solution of differential inclusion.

Theorem 4.1. *Suppose the multivalued mapping* $F : \Pi_{\mathbb{R}\times T}\times\mathbb{R}\to 2^{\mathbb{R}}$ *fulfills the following conditions:*
A1: $F(x,t,\gamma)$ *is a closed and bounded subset for each* $(x,t,\gamma)\in\Pi_{\mathbb{R}\times T}\times\Lambda$. *Moreover,* $\Upsilon_{F,\gamma}$ *is nonempty for each* γ *in the space* Λ.

A2: For any $\gamma_1, \gamma_2 \in \Lambda$ *if* $\gamma_1 \preceq \gamma_2$ *then for each* $\nu_1 \in F(x,t,\gamma_1)$ *there exists* $\nu_2 \in F(x,t,\gamma_2)$ *such that*

$$|\nu_1(x,t) - \nu_2(x,t)| \leq g_1(x,t)\left(\frac{1}{2}\max\left\{\begin{array}{c} 6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}, \\ |\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|, \\ |\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|, \\ \frac{|\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}|+ |\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|}{2}, \\ \frac{\{|\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|\}\times\{1+|\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}, \\ \frac{\{|\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}|\}\times\{1+|\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}} \end{array}\right\}\right)$$
$$+g_2(x,t)\left[\frac{3L}{k}\min\left\{\begin{array}{c} |\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|, \\ |\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|, \\ |\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|, \\ |\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}| \end{array}\right\}\right], \quad (4.\ 9)$$

where $L \geq 0$ *and* $0 < k < 1$. *Then there exists a solution of the differential inclusion* (4.1).

Proof. The given problem is equivalent to the following integral inclusion

$$\gamma(x,t) \in \left\{\begin{array}{c} \zeta \in \Lambda : \zeta(x,t) = \alpha(t) + \beta(t) - \alpha(0) + \\ \int_0^t \int_{-\infty}^{\infty} \delta(\chi,\tau)\, d\chi d\tau, for\ \delta \in \Upsilon_{F,\gamma} \end{array}\right\}.$$

Define a mapping $S : \Lambda \to 2^{\Lambda}$ by

$$(S\gamma)(x,t) \in \{\zeta \in \Lambda : \zeta(x,t) = \alpha(t) + \beta(t) - \alpha(0) + \pi(\delta(\chi,\tau)), for\ \delta \in \Upsilon_{F,\gamma}\}.$$

Now we prove that $(S\gamma)(x,t)$ is compact for each $\gamma \in \Lambda$ and $\delta \in \Upsilon_{F,\gamma}$. For this purpose it is enough to prove that $\pi \circ \Upsilon_{F,\gamma}$ is compact. Let us assume that $\{\kappa_n\}$ be a sequence in $\Upsilon_{F,\gamma}$. Then by definition of $\Upsilon_{F,\gamma}$

$$\kappa_n \in \mathcal{L} \text{ and } \kappa_n \in F(x,t,\gamma(x,t)) \ \ a.e\ \ (x,t) \in \Pi_{\mathbb{R}\times T}.$$

Since $F(x,t,\gamma(x,t))$ is compact so there exists $\kappa(x,t) \in F(x,t,\gamma(x,t))$ such that $\kappa_n \to \kappa$. By the continuity of π, $\pi \circ \kappa_n \to \pi \circ \kappa$. Since $\kappa \in \mathcal{L}$ and $\kappa(x,t) \in F(x,t,\gamma(x,t))$ therefore $\pi \circ \kappa \in \pi \circ \Upsilon_{F,\gamma}$ which implies the compactness of $\pi \circ \Upsilon_{F,\gamma}$.
Now suppose $\gamma_1, \gamma_2 \in \Lambda$ with $\gamma_1 \preceq \gamma_2$ and $\omega_1 \in \Gamma\gamma_1$, then there exists $\xi_1 \in \Upsilon_{F,\gamma_1}$ such that

$$\begin{aligned} \omega_1(x,t) &= \alpha(t) + \beta(t) - \alpha(0) + \pi(\xi_1(x,t)) \\ &= \alpha(t) + \beta(t) - \alpha(0) + \int_0^t \int_{-\infty}^{\infty} \xi_1(\vartheta,\varrho)\, d\vartheta d\varrho, \\ \text{for } (x,t) &\in \Pi_{\mathbb{R}\times T}. \end{aligned}$$

Now by the given assumption there exists a ς in $F(x,t,\gamma_2)$ such that

$$\begin{aligned}
|\xi_1(x,t)-\varsigma| \leq\ & g_1(x,t)\times \\
& \frac{1}{2}\max\left\{\begin{array}{c}
6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}, \\
|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|, \\
|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|, \\
\frac{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|+ |\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|}{2}, \\
\frac{\{|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|\}\times\{1+|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}, \\
\frac{\{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|\}\times\{1+|\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}
\end{array}\right\} \\
& +g_2(x,t)\times \\
& \left[\frac{3L}{k}\min\left\{\begin{array}{c}
|\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|, \\
|\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|, \\
|\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|, \\
|\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}|
\end{array}\right\}\right].
\end{aligned}$$

Define a multivalued mapping $\mho:\Pi_{\mathbb{R}\times T}\to\mathbb{R}$ by

$$\mho(x.t)=\left\{\begin{array}{c}
\varsigma\in\mathbb{R}:|\xi_1(x,t)-\varsigma|\leq g_1(x,t)\times \\
\left(\frac{1}{2}\max\left\{\begin{array}{c}
6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}, \\
|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|, \\
|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|, \\
\frac{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|+ |\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|}{2}, \\
\frac{\{|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|\}\times\{1+|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}, \\
\frac{\{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|\}\times\{1+|\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}
\end{array}\right\}\right) \\
+g_2(x,t)\times \\
\left[\frac{3L}{k}\min\left\{\begin{array}{c}
|\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|, \\
|\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|, \\
|\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|, \\
|\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}|
\end{array}\right\}\right]
\end{array}\right\};$$

then the mapping $\Phi:\Pi_{\mathbb{R}\times T}\to\mathbb{R}$ defined by

$$\Phi(\varkappa,\gamma)=\mho(x,t)\cap\Upsilon_{F,\gamma_1}$$

is a measurable selection which has non-empty values [7]. So there exists ξ_2 in Φ such that $\xi_2\in F(x,t,\gamma_2)$ and

$$|\xi_1(x,t)-\xi_2(x,t)|\leq g_1(x,t)\times$$

$$\left(\frac{1}{2}\max\left\{\begin{array}{c} 6\left\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\right\}, \\ |\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|, \\ |\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|, \\ \frac{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|+ |\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|}{2}, \\ \frac{\{|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|\}\times \{1+|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}, \\ \frac{\{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|\}\times \{1+|\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}} \end{array}\right\}\right)$$

$$+g_2(x,t)\times \left[\frac{3L}{k}\min\left\{\begin{array}{c} |\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|, \\ |\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|, \\ |\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|, \\ |\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}| \end{array}\right\}\right],$$

for all $(x,t) \in \Pi_{\mathbb{R}\times T}$.

For each $(x,t)\in\Pi_{\mathbb{R}\times T}$, set

$$\begin{aligned} \omega_2(x,t) &= \alpha(t)+\beta(t)-\alpha(0)+\pi(\xi_2(x,t)) \\ &= \alpha(t)+\beta(t)-\alpha(0)+\int_0^t\int_{-\infty}^{\infty}\xi_2(\vartheta,\varrho)\,d\vartheta d\varrho. \end{aligned}$$

Then

$$|\omega_1(x,t)-\omega_2(x,t)| \le \int_0^{\gamma}\int_{-\infty}^{\infty}|\xi_1(\varrho,\tau)-\xi_2(\varrho,\tau)|\,d\varrho d\tau$$

$$\le \frac{1}{2}\max\left\{\begin{array}{c} 6\left\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\right\}, \\ |\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|, \\ |\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|, \\ \frac{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|+ |\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|}{2}, \\ \frac{\{|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|\}\times \{1+|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}, \\ \frac{\{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|\}\times \{1+|\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}} \end{array}\right\}$$

$$\times\int_0^t\int_{-\infty}^{\infty}g_1(\varrho,\tau)\,d\varrho d\tau$$

$$+\frac{3L}{k}\min\left\{\begin{array}{c}|\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|,\\|\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|,\\|\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|,\\|\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}|\end{array}\right\}$$

$$\times\int_0^t\int_{-\infty}^{\infty}g_2(\varrho,\tau)\,d\varrho d\tau$$

$$\leq\frac{\|g_1\|_{\mathcal{L}}}{2}\max\left\{\begin{array}{c}6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\},\\|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|,\\|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|,\\\frac{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|+|\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|}{2},\\\frac{\{|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|\}\times\{1+|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}},\\\frac{\{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|\}\times\{1+|\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}\end{array}\right\}$$

$$+\|g_2\|_{\mathcal{L}}\frac{3L}{k}\min\left\{\begin{array}{c}|\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|,\\|\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|,\\|\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|,\\|\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}|\end{array}\right\}.$$

Taking sup on both sides

$$\sup_{(x,t)\in\Pi_{\mathbb{R}\times T}}|\omega_1(x,t)-\omega_2(x,t)|\leq\frac{\|g_1\|_{\mathcal{L}}}{2}\max\left(\begin{array}{c}6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2),\\\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1),\\\mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2),\\\frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)}{2},\\\frac{\mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2)[1+\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)},\\\frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)[1+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)}\end{array}\right)$$

$$+\|g_2\|_{\mathcal{L}}\frac{3L}{k}\min\left\{\begin{array}{c}\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1),\\\mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2),\\\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2),\\\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)\end{array}\right\}$$

$$\leq \frac{k}{12}\max\left(\begin{array}{c} 6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2), \\ \mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1), \\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2), \\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)+ \\ \mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)}{2}, \\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2) \\ [1+\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)}, \\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1) \\ [1+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)} \end{array}\right)$$

$$+\frac{L}{2}\min\left\{\begin{array}{c} \mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1), \\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2), \\ \mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2), \\ \mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1) \end{array}\right\},$$

where $\frac{k}{6} = \|g_i\|_{\mathcal{L}} \leq 1$ for $i = 1, 2$.
Now

$$|\omega_{1x}(x,t) - \omega_{2x}(x,t)| \leq \int_0^{\gamma}\int_{-\infty}^{\infty} |\xi_{1x}(\varrho,\tau) - \xi_{2x}(\varrho,\tau)|\, d\varrho d\tau$$

$$\leq \frac{1}{2}\max\left\{\begin{array}{c} 6\{|\gamma_1-\gamma_2| + |\gamma_{1x}-\gamma_{2x}|\}, \\ |\xi_1-\gamma_1| + |\xi_{1x}-\gamma_{1x}|, \\ |\xi_2-\gamma_2| + |\xi_{2x}-\gamma_{2x}|, \\ \frac{|\xi_1-\gamma_2| + |\xi_{1x}-\gamma_{2x}| + \\ |\xi_2-\gamma_1| + |\xi_{2x}-\gamma_{1x}|}{2}, \\ \frac{\{|\xi_1-\gamma_1| + |\xi_{1x}-\gamma_{1x}|\}\times \\ \{1+|\xi_2-\gamma_2| + |\xi_{2x}-\gamma_{2x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}, \\ \frac{\{|\xi_1-\gamma_2| + |\xi_{1x}-\gamma_{2x}|\}\times \\ \{1+|\xi_2-\gamma_1| + |\xi_{2x}-\gamma_{1x}|\}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}} \end{array}\right\}$$

$$\times \int_0^{t}\int_{-\infty}^{\infty} g_{1x}(\varrho,\tau)\, d\varrho d\tau$$

$$+\frac{3L}{k}\min\left\{\begin{array}{c} |\nu_1-\gamma_1| + |\nu_{1x}-\gamma_{1x}|, \\ |\nu_2-\gamma_2| + |\nu_{2x}-\gamma_{2x}|, \\ |\nu_2-\gamma_1| + |\nu_{2x}-\gamma_{1x}|, \\ |\nu_1-\gamma_2| + |\nu_{1x}-\gamma_{2x}| \end{array}\right\}$$

$$\times \int_0^{t}\int_{-\infty}^{\infty} g_{2x}(\varrho,\tau)\, d\varrho d\tau$$

$$\leq \frac{\|g_{1x}\|_{\mathcal{L}}}{2}\max\left\{\begin{array}{c} 6\left\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\right\}, \\ |\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|, \\ |\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|, \\ \frac{\begin{array}{c}|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|+ \\ |\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|\end{array}}{2}, \\ \frac{\begin{array}{c}\{|\xi_1-\gamma_1|+|\xi_{1x}-\gamma_{1x}|\}\times \\ \{1+|\xi_2-\gamma_2|+|\xi_{2x}-\gamma_{2x}|\}\end{array}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}, \\ \frac{\begin{array}{c}\{|\xi_1-\gamma_2|+|\xi_{1x}-\gamma_{2x}|\}\times \\ \{1+|\xi_2-\gamma_1|+|\xi_{2x}-\gamma_{1x}|\}\end{array}}{1+6\{|\gamma_1-\gamma_2|+|\gamma_{1x}-\gamma_{2x}|\}}\end{array}\right\}$$

$$+\|g_{2x}\|_{\mathcal{L}}\frac{3L}{k}\min\left\{\begin{array}{c}|\nu_1-\gamma_1|+|\nu_{1x}-\gamma_{1x}|, \\ |\nu_2-\gamma_2|+|\nu_{2x}-\gamma_{2x}|, \\ |\nu_2-\gamma_1|+|\nu_{2x}-\gamma_{1x}|, \\ |\nu_1-\gamma_2|+|\nu_{1x}-\gamma_{2x}|\end{array}\right\}.$$

Taking sup on both sides

$$\sup_{(x,t)\in\Pi_{\mathbb{R}\times T}}|\omega_{1x}(x,t)-\omega_{2x}(x,t)|\leq\frac{\|g_{1x}\|_{\mathcal{L}}}{2}\times$$

$$\max\left(\begin{array}{c}6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2), \\ \mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1), \\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2), \\ \frac{\begin{array}{c}\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)+ \\ \mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)\end{array}}{2}, \\ \frac{\begin{array}{c}\mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2) \\ [1+\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1)]\end{array}}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)}, \\ \frac{\begin{array}{c}\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1) \\ [1+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)]\end{array}}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)}\end{array}\right)$$

$$+\|g_{2x}\|_{\mathcal{L}}\frac{3L}{k}\min\left\{\begin{array}{c}\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1), \\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2), \\ \mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2), \\ \mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)\end{array}\right\}$$

$$\leq\frac{k}{12}\max\left(\begin{array}{c}6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2), \\ \mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1), \\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2), \\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)}{2}, \\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2)[1+\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)}, \\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)[1+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)}\end{array}\right)$$

$$+\frac{L}{2}\min\left\{\begin{array}{c}\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1),\\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2),\\ \mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2),\\ \mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)\end{array}\right\}.$$

Thus we have

$$H_{\mathcal{G}}(\Gamma\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)\leq\frac{k}{6}\max\left(\begin{array}{c}6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2),\\ \mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1),\\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2),\\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)}{2},\\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2)[1+\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)},\\ \frac{\mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)[1+\mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2)]}{1+6\mathcal{G}(\gamma_1,\gamma_2,\gamma_2)}\end{array}\right)$$

$$+L\min\left\{\begin{array}{c}\mathcal{G}(\gamma_1,\Gamma\gamma_1,\Gamma\gamma_1),\\ \mathcal{G}(\gamma_2,\Gamma\gamma_2,\Gamma\gamma_2),\\ \mathcal{G}(\gamma_1,\Gamma\gamma_2,\Gamma\gamma_2),\\ \mathcal{G}(\gamma_2,\Gamma\gamma_1,\Gamma\gamma_1)\end{array}\right\}.$$

Assuming $\psi(t)=kt$ for all $t\in[0,\infty)$ and $\alpha(\wp,\nu,\nu)=1$ for all $\wp,\nu\in\Lambda$ we get the hypothesis of the Corollary 1, which gives $\gamma\in\Lambda$ such that $\gamma\in\Gamma\gamma$, which is the solution of the Problem (4.1). □

5. Acknowledgments

We are grateful to the referees for their careful reading of our paper and many valuable suggestions.

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