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Time Optimization of a Draining Tank and Some Similar Problems on Star Graphs

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Abstract. We consider a problem of calculus of variations motivated by the model of a tank filled with a given volume of liquid and draining through a small orifice according to Torricelli's law. We prove that given any length of time, some tank exists which drains in this time. Our main interest in this optimization (i.e., minimization and maximization) problem is that the usual Euler–Lagrange equation may not be used here, at least directly. We consider optimization for some similar physical models where Torricelli's law has to be modified. We also study optimization problems on the star graphs that are inspired by our physical model.

AMS (MOS) Subject Classification Codes: 49J40; 49J30; 49K30; 49K99; 76A99; 76M30
Key Words: Calculus of variations, Draining tank, Torricelli's law, Optimization on graphs

1. INTRODUCTION

When a finite volume V of liquid inside a tank is allowed to drain through an aperture according to Toricelli's law, the time it takes to drain will be determined by the tank's shape. This is because the only factor influencing the exit speed is the liquid's height. As the volume of liquid decreases, the height decreases accordingly, thereby changing the exit velocity.

For simplicity's sake, we assumed Torricelli's Law rather than conditions which engender it. We have effectively stated that our liquid is Newtonian, incompressible, unaffected by capillary action, non-viscous, and not affected by friction.

We show that a tank short, wide allows for an arbitrarily large draining time and a tall, narrow tank can give an arbitrarily small draining time. We also demonstrate that for any positive number, there exists a tank draining in this many units of time.

Our main interest in this problem is that the usual Euler–Lagrange equation may not be used here, at least directly. Instead, we proceed as in [4], where the mathematical situation is similar though not identical. In short, we *guess* a specific family of tanks that has a wide range of possible draining times. This family is constructed by revolving the graph of a continuous function about the y-axis with its lowest point located at the origin. In the optimization problem, we only consider the form of the tank as the input; the orifice surface is supposed to be fixed.

In Section 2 we derive the representation of draining time based on Torricelli's law and formulate the main result for a single tank. We describe, in Section 3, simple families of tanks which provided intuition when solving our main theorem in Section 4. Optimization problems with similar physical models are explored in Section 5. In the sixth and final section, we study a few optimization problems on star graphs that are inspired by the draining tank model. Everywhere, by *optimization*, we understand that both maximization and minimization are sought.

Finally, our recent search of the literature has revealed several papers dealing with some general variants of our work, namely [1, 2, 3, 5, 13, 14, 15, 16, 17, 18, 19, 20].

2. Problem Statement

Though Torricelli's law has its limitations (see Remark 11) we proceed within its scope. Firstly, we introduce the class of the tanks $\mathcal{T}$ we consider. In Cartesian coordinates (x, y), let the y-axis be the vertical axis of a tank located at $y \geq 0$, so that $y_{\min} = 0$.

Definition 1. *The tank $\mathcal{T}$ is formed by rotating the curve $x = x(y)$ about the y-axis. The function $x(y) \in \mathcal{F} := \{x(y) \Big| x : [0, y_0] \to \mathbf{R}_+, \; x(y) \in C[0, y_0]\}$.*

Here $C[0, y_0]$ is the class of continuous functions on $[0, y_0]$, y_0 the initial height of the liquid, S the (given) area of the orifice, $A(y)$ the cross sectional area at height y, and $v(y)$ the velocity of the liquid exiting from the orifice when the height is y (see Fig. 1). The volume which drains from the orifice in a given time interval dt is given by $dV = Sv(y)dt$, and the liquid removed from the top of the tank during this same interval by $A(y)dy$. Setting the two expressions equal yields $Sv(y)dt = A(y)dy$. We find the following differential equation

$$\frac{dy}{dt} = \frac{Sv(y)}{A(y)}. \tag{2. 1}$$

We now apply Torricelli's law [6]

$$v(y) = -\sqrt{2gy}, \tag{2. 2}$$

where g is the acceleration due to gravity.

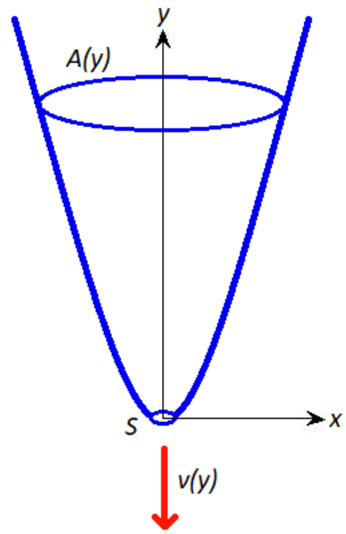

Figure 1: Definition Diagram

Torricelli's Law often contains a flow coefficient [12] that is generally found empirically; we have set this coefficient to 1 in (2. 2) because the specific value of the coefficient does not affect the results. We denote the time taken for the tank to drain completely as t_d. Separating variables in (2. 1) we then integrate, keeping in mind that $y = y_0$ when $t = 0$ and $y = 0$ when $t = t_d$

$$\int_0^{y_0} \frac{A(\hat{y})}{\sqrt{\hat{y}}} d\hat{y} = -\int_{y_0}^{0} \frac{A(\hat{y})}{\sqrt{\hat{y}}} d\hat{y} = \int_0^{t_d} S\sqrt{2g}dt = S\sqrt{2g}t_d. \tag{2. 3}$$

We introduce the notation

$$T_d := S\sqrt{2g}t_d \tag{2. 4}$$

and (to the end of Section 5) call this the "*draining time*." Thus, we have

$$T_d = \int_0^{y_0} \frac{A(y)}{\sqrt{y}} dy. \tag{2. 5}$$

Note that the prescribed initial volume of the liquid in the tank is given by

$$V = \int_0^{y_0} A(y)dy. \tag{2. 6}$$

We consider the following

Problem I. *Given the initial volume $V > 0$, find the range $\mathcal{R}(T_d)$ of the functional T_d.*

Obviously $\mathcal{R}(T_d) \subseteq (0, \infty)$.

Remark 1. We emphasize that the height of the tank is not fixed in our problem. The case when the height is fixed is considered in [10]. It appears that this constraint, $y_0 = H$, results in the existence of the positive lower bound for the draining time (which is not attained though). We cite [10] for this bound

$$t^* = \frac{V}{S\sqrt{2gH}}. \tag{2. 7}$$

In our case, the lower bound appears to be zero. The author of [10] also finds the draining time for several specific shapes of tank

Remark 2. The optimization problem for the functional (2. 5) subject to the constraint (2. 6) does not belong to the traditional Calculus of Variations [8] in the sense that we cannot use the Euler–Lagrange equation, at least directly. Instead, we proceed as in [4], where the situation is similar though not identical. In short, we *guess* a specific family of functions $x = x(y) \in \mathcal{F}$ and find the range $\mathcal{R}(T_d)$ of the functional T_d. Specifically, we prove the following result.

Theorem 1. *We have*

$$\mathcal{R}(T_d) = (0, \infty). \tag{2. 8}$$

3. Draining Time for Parabolic and Cylindrical Tanks

This section shows how we *guess* the family of tanks satisfying (2. 8). Firstly, we consider a two-parametric family of parabolic tanks $\mathcal{T}_{parab.} \in \mathcal{F}$ obtained by revolving a "parabola" of the form $x = \alpha y^\gamma,\ y \in [0, y_0]$, about the y-axis (Fig. 2) (here α, γ, y_0 are positive parameters).

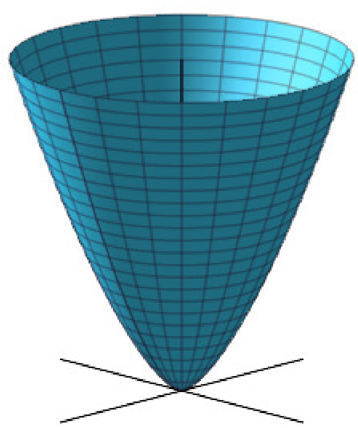

Figure 2: Paraboloid Tank

In this case, we can find an explicit expression for the draining time by substituting the cross sectional area at height y, $A(y) = \pi x^2 = \pi(\alpha y^\gamma)^2$ into the general formulas (2. 5) and (2. 6) for volume and draining time, so that

$$V = \int_0^{y_0} A(y)dy = \int_0^{y_0} \pi(\alpha y^\gamma)^2 dy = \frac{\pi\alpha^2}{2\gamma+1}\, y_0^{2\gamma+1} \tag{3. 9}$$

and

$$T_d = \int_0^{y_0} \frac{A(y)}{\sqrt{y}} dy = \int_0^{y_0} \frac{\pi(\alpha y^\gamma)^2}{\sqrt{y}} dy = \frac{\pi\alpha^2}{2\gamma+\frac{1}{2}}\, y_0^{2\gamma+\frac{1}{2}}. \tag{3. 10}$$

We eliminate α^2 to find the representation for the draining time in terms of γ

$$T_d(\gamma) = V\,\frac{2\gamma+1}{2\gamma+\frac{1}{2}}\, y_0^{-\frac{1}{2}}. \tag{3. 11}$$

Evaluating its derivative we conclude that the function $T(\gamma)$ has no critical points. Hence, its the supremum and infimum occur at the limiting cases, i.e.

$$\sup_{\mathcal{T}_{parab.}} T_d = \lim_{\gamma\to 0} \frac{V(2\gamma+1)}{2\gamma+\frac{1}{2}} y_0^{-1/2} = 2V y_0^{-1/2} \tag{3. 12}$$

and

$$\inf_{\mathcal{T}_{parab.}} T_d = \lim_{\gamma\to\infty} \frac{V(2\gamma+1)}{2\gamma+\frac{1}{2}} y_0^{-1/2} = V y_0^{-1/2}. \tag{3. 13}$$

In view of monotonicity of $T_d(\gamma)$, we conclude that for any $T_* \in (\inf T_d, \sup T_d)$ there exists a unique value of $\gamma_* \in (0,\infty)$ such that $T_d(\gamma_*) = T_*$. Hence, we find the range of time for the family $\mathcal{F}_{parab.}$ to be $\mathcal{R}(T)_{parab.} = (V y_0^{-1/2}, 2V y_0^{-1/2})$, so that for the given volume of liquid and height of the tank, the draining time for this family is bounded both above and below.

We further briefly consider the class of tanks $\mathcal{T}_{cyl.}$ that have a cylindrical shape. Let the tank have a base area of A_0 and a height a for some $a \in (0, y_0]$. We intentionally chose the height of the "parabola" y_0 from Section 3 to be the upper limit for the parameter a to compare the corresponding draining times. The volume and the draining time for the class $\mathcal{T}_{cyl.}$ are given as follows

$$V = \int_0^a A_0 dy = A_0 a, \quad \text{so that} \quad A_0 = \frac{V}{a}; \quad T_d = \int_0^a \frac{A_0}{\sqrt{y}} dy = 2V a^{-1/2}. \tag{3. 14}$$

Obviously, $T_d(a)$ is a monotonically decreasing function of a. Thus,

$$\sup_{\mathcal{T}_{cyl.}} T_d = \lim_{a\to 0^+} T_d(a) = \lim_{a\to 0^+} \frac{2V}{\sqrt{a}} = \infty; \quad \min_{\mathcal{T}_{cyl.}} T_d = T_d(y_0) = 2V y_0^{-1/2}. \tag{3. 15}$$

We observe that

$$\mathcal{R}(T)_{cyl.} = [2V\, y_0^{-1/2},\, \infty) \quad \text{and} \quad \sup_{\mathcal{T}_{parab.}} T_d = \min_{\mathcal{T}_{cyl.}} T_d, \tag{3. 16}$$

i.e., the minimum of the draining time for the cylindrical tank coincides with the supremum of the drain time for the parabolic tank. We may guess at this moment that the "optimal" tank is a combination of a cylindrical and parabolic tanks. The goal of the next section is to construct this combination.

4. A Family of Tanks with an Arbitrary Draining Time

4.1. **Heuristic Approach.** Here, we are making the first step toward the proof of Theorem 1; the complete proof to be shown in Subsection 4.2. Specifically, we construct a family of the curves $x(y) \in \mathcal{F}$ that is "rich" enough to produce a broader range of draining times than in Section 3. The corresponding tanks imitate a combination of a cylinder and paraboloid. We proceed in terms of the cross section $A(y)$. We let this function $A(y)$ be a multiple of some positive power of y, y^ξ, to imitate a paraboloid, and to cause a rapid decrease in area after a certain point, we let this function be a multiple of another function $\omega_\epsilon(y) := e^{-y^2/\epsilon}$ with $\epsilon > 0$ to be a parameter. Hence, we define the two-parametric family of functions $A(y)$ by

$$A(y) := C_\epsilon y^\xi e^{-y^2/\epsilon}, \quad \text{or} \quad x(y) := \Big(C_\epsilon y^\xi e^{-y^2/\epsilon}\Big)^{1/2} / \sqrt{\pi}, \quad y \in [0, y_0] \tag{4. 17}$$

with a positive constant factor C_ϵ and the corresponding set of tanks as $\mathcal{T}_\epsilon \in \mathcal{F}$. We find for the corresponding volume that is supposed to be given

$$V = \int\limits_0^{y_0} A(y)dy = \int\limits_0^{y_0} y^\xi C_\epsilon e^{-y^2/\epsilon} dy, \quad \text{so that} \quad C_\epsilon = V / \int\limits_0^{y_0} y^\xi e^{-y^2/\epsilon} dy\,. \tag{4. 18}$$

Furthermore, we have

$$\int\limits_0^{y_0} y^\xi e^{-y^2/\epsilon} dy = \epsilon^{(\xi+1)/2} \int\limits_0^{y_0/\sqrt{\epsilon}} s^\xi e^{-s^2} ds.$$

The integrand here has a very small "tail". Hence, its asymptotic approximation (as $\epsilon \to 0$) is given by the integral over the semi-axis $[0, \infty)$, i.e.,

$$\epsilon^{(\xi+1)/2} \int\limits_0^{y_0/\sqrt{\epsilon}} s^\xi e^{-s^2} ds \sim \epsilon^{(\xi+1)/2} \int\limits_0^{\infty} s^\xi e^{-s^2} ds = \frac{1}{2}\, \epsilon^{(\xi+1)/2}\, \Gamma\Big(\frac{\xi+1}{2}\Big). \tag{4. 19}$$

Here and below, we use the Bachmann–Landau symbol for the asymptotic relation, $f \sim g$. Formulas (4. 18) and (4. 19) imply the asymptotic expression for C_ϵ is

$$C_\epsilon \sim \frac{V}{\epsilon^{(\xi+1)/2} \int\limits_0^\infty s^\xi e^{-s^2} ds} = \frac{2V}{\Gamma\Big(\frac{\xi+1}{2}\Big)}\, \epsilon^{-(\xi+1)/2}. \tag{4. 20}$$

Similarly, we find an asymptotic representation for the draining time (2. 5)

$$\begin{aligned} T_d(\epsilon) &= \int\limits_0^{y_0} \frac{A(y)}{\sqrt{y}} dy = \int\limits_0^{y_0} \frac{y^\xi C_\epsilon e^{-y^2/\epsilon}}{\sqrt{y}} dy \\ &\sim C_\epsilon \int\limits_0^\infty (s\sqrt{\epsilon})^{\xi - 1/2} e^{-s^2} \sqrt{\epsilon} ds \\ &= \frac{1}{2} C_\epsilon \epsilon^{(\xi+1/2)/2} \Gamma\Big(\frac{\xi}{2} + \frac{1}{4}\Big). \end{aligned} \tag{4. 21}$$

Excluding C_ϵ from (4. 20) and (4. 21), we obtain

$$\begin{aligned} T_d(\epsilon) &\sim \frac{2V}{\Gamma\left(\frac{\xi+1}{2}\right)} \epsilon^{-(\xi+1)/2} \epsilon^{(\xi+\frac{1}{2})/2} \frac{1}{2} \Gamma\left(\frac{\xi}{2}+\frac{1}{4}\right) \\ &= V\,\frac{\Gamma(\frac{\xi}{2}+\frac{1}{4})}{\Gamma(\frac{\xi+1}{2})} \epsilon^{-1/4} \\ &= v\,\epsilon^{-1/4}, \end{aligned} \tag{4. 22}$$

where

$$v := V\,\frac{\Gamma(\frac{\xi}{2}+\frac{1}{4})}{\Gamma(\frac{\xi+1}{2})}.$$

Hence, we come to the asymptotic expression for draining time

$$T_d(\epsilon) \sim v\,\epsilon^{-1/4} \tag{4. 23}$$

with a positive constant v. Since the asymptotic formula for draining time is a monotonic function of ϵ, we may hope that

$$\sup_{\mathcal{T}_\epsilon} T_d(\epsilon) = \lim_{\epsilon \to 0} v\,\epsilon^{-1/4} = \infty. \tag{4. 24}$$

Remark 3. It is imperative to note that the asymptotic formulas above were constructed only for small ϵ and we may not claim these formulas for all $\epsilon > 0$.

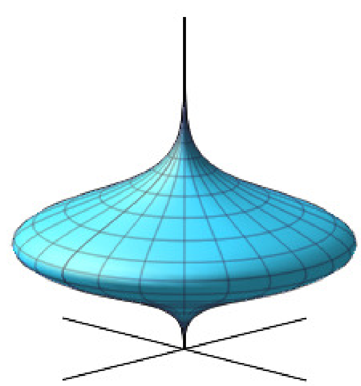

Figure 3: "Ideal Tank"

4.2. **A Family of Tanks with an Arbitrarily Large or Small Draining Time—Proof of Theorem 1.** We construct a family of tanks $\mathcal{T}_{\epsilon,*}$ similar to one in Subsection 4.1, for which Theorem 1 holds. The family is defined by the cross-section area

$$A(y) := C_{\epsilon,*} y^{\xi} e^{-y^2/\epsilon}, \quad y \in [0, y_*], \tag{4. 25}$$

where

$$y_* := \sqrt{\frac{\xi\epsilon}{2}}, \quad C_\epsilon := \frac{V}{\epsilon^{(\xi+1)/2} \int\limits_0^{\sqrt{\xi/2}} s^\xi e^{-s^2} ds}. \tag{4. 26}$$

We will show that the corresponding draining time is

$$T_d(\epsilon) = \frac{V \int\limits_0^{\sqrt{\xi/2}} s^{\xi-\frac{1}{2}} e^{-s^2} ds}{\int\limits_0^{\sqrt{\xi/2}} s^\xi e^{-s^2} ds} \epsilon^{-1/4}. \tag{4. 27}$$

Proof of Theorem 1. On noting that much of the preceding expression is a constant in ϵ that depends on ξ, we find the following expression for draining time

$$T_d(\epsilon) = K_1(\xi)\epsilon^{-1/4} \quad \text{with} \quad K_1(\xi) := \frac{V \int\limits_0^{\sqrt{\xi/2}} s^{\xi-\frac{1}{2}} e^{-s^2} ds}{\int\limits_0^{\sqrt{\xi/2}} s^\xi e^{-s^2} ds}. \tag{4. 28}$$

Since $T_d(\epsilon)$ is a monotonically decreasing function of the parameter ϵ, the supremum and infimum occur at the limiting cases. Hence,

$$\inf T_d(\epsilon) = \lim_{\epsilon\to\infty} K_1(\xi)\epsilon^{-1/4} = 0, \quad \sup T_d(\epsilon) = \lim_{\epsilon\to 0} K_1(\xi)\epsilon^{-1/4} = \infty \tag{4. 29}$$

and

$$\mathcal{R}(T_d) = (0, \infty). \tag{4. 30}$$

This completes the proof of the theorem. □

We now explain the origin of the family defined by (4. 26). It is formed by the monotonically increasing part of the curve (4. 17), i.e. we restrict the domain $[0, y_0]$ to $[0, y_*]$ with the appropriately chosen y_*. Geometrically, we are "cutting" the top from the tank considered in Subsection 4.1. As a result, we prove that the asymptotic representation (4. 23) for the draining time is actually *exact* for the constructed family, and this implies Theorem 1. So, we consider the family of tanks $\mathcal{T}_{\epsilon,*}$ defined by the cross-section area (4. 25) and find the critical point y_* of $A(y)$ as follows

$$\frac{d}{dy}A(y)\Big|_{y=y_*} = C_{\epsilon,*}\,\frac{d}{dy}y^\xi e^{-y^2/\epsilon}\Big|_{y=y_*} = C_{\epsilon,*}\,y_*^{\xi-1}e^{-y_*^2/\epsilon}\Big(\xi - \frac{2y_*^2}{\epsilon}\Big) = 0, \tag{4. 31}$$

so that the expression (4. 26) for y_* holds. We further proceed as in Subsection 4.1 and find similarly to (4. 21)

$$
\begin{aligned}
T(\epsilon) &= \int_0^{y_*} \frac{A(y)}{\sqrt{y}} dy = \int_0^{\sqrt{\xi\epsilon/2}} \frac{y^\xi C_{\epsilon,*} e^{-y^2/\epsilon}}{\sqrt{y}} dy = C_{\epsilon,*} \int_0^{\sqrt{\xi\epsilon/2}} y^{\xi-1/2} e^{-y^2/\epsilon} dy \\
&= C_\epsilon^* \int_0^{\sqrt{\xi/2}} (s\sqrt{\epsilon})^{\xi-1/2} e^{-s^2} \sqrt{\epsilon} ds = C_\epsilon^* \epsilon^{(\xi+1/2)/2} \int_0^{\sqrt{\xi/2}} s^{\xi-\frac{1}{2}} e^{-s^2} ds.
\end{aligned} \tag{4. 32}
$$

Substituting $y_0 = y_*$ into the volume constraint (2. 6) or (4. 18) yields

$$
C_{\epsilon,*} = \frac{V}{\int_0^{\sqrt{\xi\epsilon/2}} y^\xi e^{-y^2/\epsilon} dy} = \frac{V}{\epsilon^{(\xi+1)/2} \int_0^{\sqrt{\xi/2}} s^\xi e^{-s^2} ds}. \tag{4. 33}
$$

Excluding the constant $C_{\epsilon,*}$ from (4. 32) and (4. 33) we find the representation (4. 27) for the draining time

$$
T_d(\epsilon) = \frac{V \epsilon^{(\xi+1/2)/2} \int_0^{\sqrt{\xi/2}} s^{\xi-\frac{1}{2}} e^{-s^2} ds}{\epsilon^{(\xi+1)/2} \int_0^{\sqrt{\xi/2}} s^\xi e^{-s^2} ds} = \frac{V \int_0^{\sqrt{\xi/2}} s^{\xi-\frac{1}{2}} e^{-s^2} ds}{\int_0^{\sqrt{\xi/2}} s^\xi e^{-s^2} ds} \epsilon^{-1/4}
$$

which allows to prove Theorem 1 (see the beginning of this section).

We now briefly discuss the dual problem to Problem I.

Problem II. *Given the draining time $T > 0$, find the range $\mathcal{R}(V)$ of the functional V.*

We use the same family $\mathcal{T}_{\epsilon,*}$ of tanks as in 4. 17 , $A(y) := \hat{C}_\epsilon\, y^\xi \hat{e}^{-y^2/\epsilon}; y \in [0, y_*],\ y_* = \sqrt{\xi\epsilon/2}$, though the factor $\hat{C}_\epsilon \neq C_\epsilon$.

Theorem 2. *Given the draining time $T > 0$, the range of the volumes of the family $\mathcal{T}_{\epsilon,*}$ of tanks satisfies $\mathcal{R}(V) = (0, \infty)$.*

We do not give the proof which is similar to one of Theorem 1. It appears that

$$
\sup T_d(\epsilon) = \lim_{\epsilon\to\infty} K_2(\xi)\epsilon^{1/4} = \infty, \quad \inf T_d(\epsilon) = \lim_{\epsilon\to 0} K_2(\xi)\epsilon^{1/4} = 0 \tag{4. 34}
$$

and

$$
\mathcal{R}(V) = (0, \infty) \tag{4. 35}
$$

with some function $K_2(\xi)$ independent of ϵ.

5. Optimal Tanks When Torricelli's Law is Not Valid

We briefly discuss here two physical models for which our use of Torricelli's law is invalid, and hence, the preceding optimization methods must be changed.

Model 1. We consider the optimization of the draining time of a granular solid through an orifice. We use the formula for the flow rate W of the granular solid through an orifice at the bottom $W = \zeta \rho_B g^{1/2}(S - kd)^{5/2}$ (see [9], Ch. 7, pp. 105–106), where ζ and k are positive constants, d is the diameter of a grain, ρ_B is the effective density of the granulate, and S is the area of the orifice. Hence, we observe that the flow of a granular solid does not depend on the height and Torricelli's law does not apply. The draining time is derived as follows. The volume which drains from the orifice in a given time interval dt can be equated to the flow rate W. Thus, we have $dV/dt = -W$. This leads to the following expression for the draining time $T_d = V/W = V/\zeta \rho_B g^{1/2}(S - kd)^{5/2}$, which, for the given volume V, is a constant. Since the draining time is independent of height and form of a tank, the optimization problem is trivial.

Model 2. The authors of [7] describe a model that takes into account the effect of the pressure head on the area of the orifice of a draining tank of liquid. They show that the area of an orifice increases linearly with pressure as described by the equation $S(y) = S_0 + my$ where S_0 is the area of the orifice when there is no pressure and m is a positive (small) constant of the model. The authors further apply Torricelli's law and introduce a dimensionless constant known as the coefficient of discharge C_d to obtain the volume flow rate out of the tank. We rewrite their equation in terms of the infinitesimal element of volume

$$dV = -C_d(S_0 + my)\sqrt{2gy}\, dt = A(y)dy. \tag{5. 36}$$

Rearranging terms and integrating yields

$$C_d\sqrt{2g}\int_0^{t_d} dt = \int_0^{y_0} \frac{A(y)}{(S_0 + my)\sqrt{y}}\, dy. \tag{5. 37}$$

Unlike in the previous sections, it is natural to define the draining time to be $\hat{T}_d := C_d\sqrt{2g}\, t_d$. We find

$$\hat{T}_d = \int_0^{y_0} \frac{A(y)dy}{(S_0 + my)\sqrt{y}}. \tag{5. 38}$$

Theorem 3. $\mathcal{R}(\hat{T}_d) = (0, \infty)$.

The proof is based on the inequalities

$$\frac{1}{S_0 + my_0}\int_0^{y_0} \frac{A(y)dy}{\sqrt{y}} \le \int_0^{y_0} \frac{A(y)dy}{(S_0 + my)\sqrt{y}} = \hat{T}_d \le \frac{1}{S_0}\int_0^{y_0} \frac{A(y)dy}{\sqrt{y}}\,. \tag{5. 39}$$

Here, the right and left sides are proportional to the draining time T_d (see Section 4). By Theorem 1, and the squeeze theorem, the functional $\hat{T}_d$ has the same range as the corresponding functional in Section 4. $\square$

Remark 4. In [11] Torricelli's law is rederived with the help of Bernoulli's principle for the case of unsteady flow from a cylindrical tank having the cross section S_0 through the orifice S is found in terms of a hypergeometric function. In our model, we assume that $S_0/S \gg 1$. The leading term of the series for the time found in [11] coincides with the draining time (3. 14). We leave the optimization problem for this model for the future project.

6. Similar Optimization Problem on the Star Graphs

Part I. We firstly consider a set $\{\mathcal{T}_k\}_{k=1}^n$ of identically shaped and oriented tanks filled to the same height where for every $i \in \{1, ..., n-1\}$, tank $\mathcal{T}_i$ drains into tank $\mathcal{T}_{i+1}$ (see Fig. 4). It is easy to see that if liquid starts leaking from all tanks simultaneously, the amount of liquid in the tanks will look, qualitatively, like on Fig. 5 since every tank $\mathcal{T}_k,\ k \geq 2$ is losing the same amount of the liquid as it is gaining from the previous tank $\mathcal{T}_{k-1}$ until the previous tank is empty. These tanks may be viewed as a directed path graph P_n (see Fig. 4b). Having this analogy in mind we now ignore the physical nature of the original model and consider the star graph $K_{1,n},\ n > 1$ that we call the *star* for brevity (see Fig. 6). We denote $J := \{1, ..., n\}$, equip every edge $e_j,\ j \in J$ with the coordinates $\{y :\ 0 \leq y \leq y_j\}$, so that all edges have the common vertex $y_j = 0,\ j \in J$. In terms of the original physical model, *liquid is draining through an orifice located at* $y = 0$.

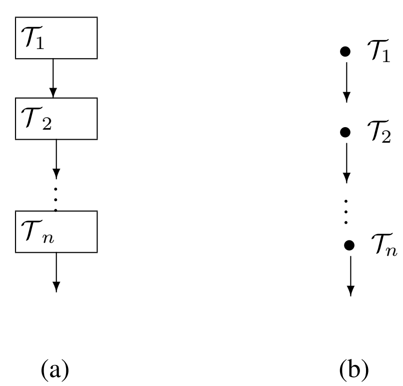

FIGURE 4. (a) Draining from a set of identically shaped and oriented tanks $\mathcal{T}_k,\ k = 1, 2, ..., n$. (b) Draining from the graph P_n.

Though we do not present a physical model similar to our original model that could be described by this problem, we use the words "tank", "orifice", "draining time", etc. for brevity. On each edge of this graph, we introduce the function $A_j(y) : [0, y_j] \to \mathbf{R}_+$ of the class $\mathcal{F}$ (see Definition 1). We further introduce the class of vector-functions $\mathcal{A} := (A_1(y), ..., A_n(y))$ and two functionals on this class, the "volume" and "draining time"

$$V[\mathcal{A}] := \sum_J \int_0^{y_j} A_j(y)dy, \quad T^*[\mathcal{A}] := \sum_J \int_0^{y_j} \frac{A_j(y)}{\sqrt{y}}dy. \tag{6. 40}$$

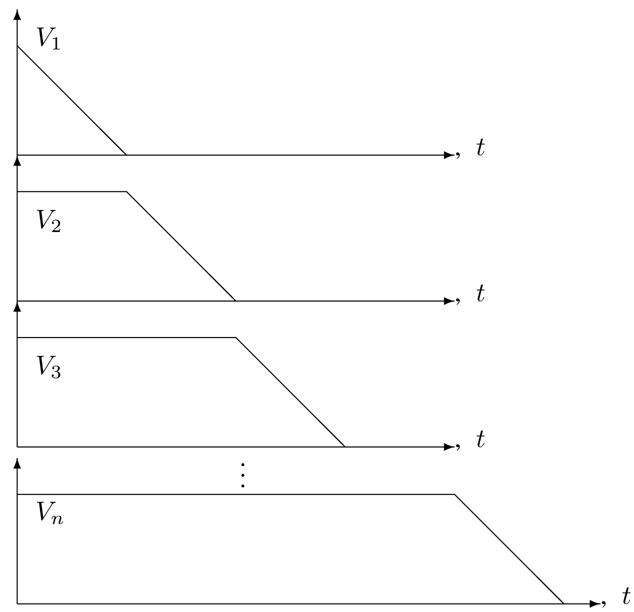

FIGURE 5. Draining from the graph P_n. V_k is the volume of liquid in the tank $\mathcal{T}_k,\ k = 1, 2, ..., n$ as a function of time t.

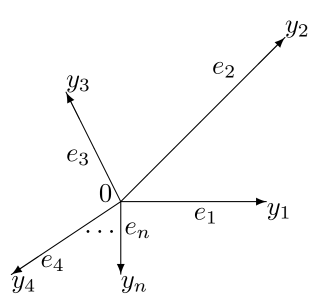

FIGURE 6. Draining from the star graph

Here and below we use the abbreviation $\sum_J := \sum_{j \in J}$. Note that $T^*[\mathcal{A}]$ represents the total draining time of the set of tanks. We consider the following problem.

Problem III. *Given the volume $V > 0$, find the range $\mathcal{R}(T)$ of the functional T^*.*

Theorem 4. $\mathcal{R}(T^*) = (0, \infty)$.

Proof. We proceed as in Section 4.1. We specify the functions

$$A_j(y) := C_j\, y^{\xi_j} e^{-y^2/\epsilon}, \quad y \in [0, y_j], \quad j \in J \tag{6. 41}$$

with some positive parameters ξ_j, C_j, ϵ and $y_j := \sqrt{\xi_j \epsilon/2}$ and find, in a similar manner to the proof of Theorem 1

$$V[\mathcal{A}] = V = \sum_J C_j \epsilon^{(\xi_j+1)/2}\, v_j, \quad v_j := \int_0^{\sqrt{\xi_j/2}} s^{\xi_j} e^{-s^2} ds \tag{6. 42}$$

and

$$T^*[\mathcal{A}] = \sum_J C_j \epsilon^{(\xi+1/2)/2}\, \tau_j, \quad \tau_j := \int_0^{\sqrt{\xi_j/2}} s^{\xi_j - 1/2} e^{-s^2} ds. \tag{6. 43}$$

We let

$$a_j := C_j \epsilon^{(\xi+1)/2}\, \frac{v_j}{V}, \quad \text{so that} \quad \sum_J a_j = 1, \tag{6. 44}$$

and

$$p_j := \frac{\tau_j}{v_j}, \quad \text{so that} \quad \epsilon^{1/4} T^*[\mathcal{A}] = \sum_J p_j\, a_j. \tag{6. 45}$$

Furthermore, we have

$$\min_j p_j = \min_j p_j \sum_J a_j \le \sum_J p_j\, a_j \le \max_j p_j \sum_J a_j = \max_j p_j. \tag{6. 46}$$

We conclude that

$$\min_j p_j \epsilon^{-1/4} \le T^*[\mathcal{A}] \le \max_j p_j \epsilon^{-1/4}. \tag{6. 47}$$

The quantities p_j are positive and do not depend on ϵ. We observe that the inequalities (6. 47) are similar to the relation (4. 28). □

Remark 5. It may be seen that the common reason for Theorems 1–4 to be valid is the presence, due to Torricelli's law, of the factor $\sqrt{y}$ in the denominator of the integrals for the draining time. It may be checked though that these theorems are still valid if we (formally) change this factor for y^β with any exponent $\beta > 0$. Moreover, in Problem III we may assume that exponents β are different for different edges.

Remark 6. For an arbitrary acyclic graph, the "orifices" are the support or interior vertices, so that every "tank" (the edge) has either one "orifice" or two "orifices". So, we proceed in a slightly different fashion. For the "tanks" with one "orifice", we know that the "draining time" may be arbitrarily small or arbitrarily large. We now consider an edge with two support (or interior) vertices, introduce the local coordinate $y \in [-y_0, y_0]$ on it and introduce the "draining time" as follows (see *Remark 5*)

$$T := \int_{-y_0}^{y_0} \frac{A(y)}{[(y_0 - y)(y_0 + y)]^{1/4}}\, dy. \tag{6. 48}$$

If we take $A(y) = C_\epsilon(y_0^2 - y^2)^{\xi/2}\, e^{-\frac{y_0^2-y^2}{\epsilon}}$ with some constant C_ϵ, exclude C_ϵ from the formulas for T and V and make substitution $y = y_0 \sin\theta$, we find

$$T = K_3(\xi)\epsilon^{-1/4}, \quad \text{where} \quad K_3(\xi) := \frac{\int_{-\pi/2}^{\pi/2} \cos^{\xi+1/2}\theta \cdot e^{-\frac{y_0^2}{\epsilon}\sin^2\theta}\, d\theta}{\int_{-\pi/2}^{\pi/2} \cos^{\xi+1}\theta \cdot e^{-\frac{y_0^2}{\epsilon}\sin^2\theta}\, d\theta}. \qquad (6.\ 49)$$

We observe that for $y_0 = \sqrt{\epsilon}$, the "draining time" has the same range $\mathcal{R}(T_d) = (0, \infty)$.

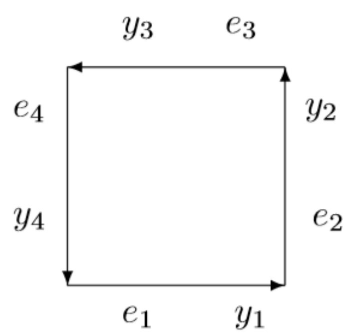

Figure 7: Draining for a Cycle Graph

Remark 7. The optimization problem for a graph that contains a cycle is more complicated. We explain this by considering a cycle that consists of four edges, $\{e_1, e_2, e_3, e_4\}$ that imitate the identical tanks with the agreement that the edge e_1 is draining into the edge e_2, e_2 is draining into e_3, e_3 is draining into e_4, and e_4 is draining into e_1 (see Fig. 7). We assume that the tanks have the same amount of liquid at the initial moment and start draining simultaneously. Using the basic equations (2. 1), (2. 2), in the appropriate system of units, we come to the Cauchy problem for the system of the differential equations

$$\begin{aligned} A(y_1)dy_1 &= -\sqrt{y_1} + \sqrt{y_4}, \\ A(y_2)dy_2 &= -\sqrt{y_2} + \sqrt{y_1}, \\ A(y_3)dy_3 &= -\sqrt{y_3} + \sqrt{y_2}, \\ A(y_4)dy_4 &= -\sqrt{y_4} + \sqrt{y_3}, \end{aligned}$$

with

$$y_1(0) = y_2(0) = y_3(0) = y_4(0) = 1.$$

The symmetry of the physical system implies that, for any given moment of time, the level of the liquid is the same in all tanks. Indeed, $y_1(t) = y_2(t) = y_3(t) = y_4(t) \equiv 1$ is the solution of this Cauchy problem. Hence, the optimization problem makes no sense for this graph.

Remark 8. If we split the "orifice" (located at the vertex $y = 0$) into n pieces, so that each of the edges is "draining" through its own "orifice", Theorem 4 still holds. But a modification of this model leads us to another optimization problem with variable size of the "orifices".

Part II. We now consider another optimization problem when the form of all "tanks" is given but the size of their "orifices" is not given. We need the following version of the Cauchy–Schwartz inequality.

Lemma 1. *For any positive numbers* $\{\tau_j\}_{j=1}^n$ *and positive numbers* $\{s_j\}_{j=1}^n$ *satisfying* $\sum_J s_j = 1$ *the inequality*

$$\left(\sum_J \sqrt{\tau_j}\right)^2 \leq \sum_J \frac{\tau_j}{s_j} \tag{6. 50}$$

holds. The equality holds iff the vectors $(\tau_1, ..., \tau_n)$ *and* $(s_1, ..., s_n)$ *are proportional.*

Proof. We have

$$\sum_J \sqrt{\tau_j} \leq \left(\sum_J s_j\right)^{1/2} \left(\sum_J \frac{\tau_j}{s_j}\right)^{1/2} = \left(\sum_J \frac{\tau_j}{s_j}\right)^{1/2}. \qquad \square$$

Introduce the set of all sequences of n positive numbers $\{s_j\}$, $j \in J$, such that

$$\sum_J s_j = 1, \quad \text{all} \quad s_j > 0. \tag{6. 51}$$

We denote the area of the orifice for the $j-$th edge to be $S_j := s_j S$, so that $\sum_J S_j = S > 0$ is given. The "real draining time" for this edge may be defined as follows (see Section 2)

$$t_{d,j} = \frac{1}{\sqrt{2g} S_j} \int_0^{y_j} \frac{A_j(y)}{\sqrt{y}}\, dy.$$

Previously, we scaled the time according to (2. 4). Since the size of the "orifices" may be different for different "tanks", we have to change the scaling (2. 4). We define "draining time" for the j^{th} edge as follows

$$T_j^{**} := S\sqrt{2g}\, t_{d,j} = \frac{1}{s_j} \int_0^{y_j} \frac{A_j(y)}{\sqrt{y}} dy \tag{6. 52}$$

and introduce the functional that we call "draining time for the star"

$$T^{**}[\mathcal{A}] := \sum_J T_j^{**} = \sum_J \frac{1}{s_j} \int_0^{y_j} \frac{A_j(y)}{\sqrt{y}} dy. \tag{6. 53}$$

Problem IV. *Given the constraint* (6. 51)*, find the range* $\mathcal{R}(T^{**})$ *of the functional* T^{**}*.*

Theorem 5. *The range* $\mathcal{R}(T^{**})$ *of the "draining time for the star" subject to the constraint* (6. 51) *satisfies*

$$\mathcal{R}(T^{**}) = [m, \infty), \tag{6. 54}$$

where

$$m := \left(\sum_J \sqrt{\tau_j}\right)^2 \quad \text{with} \quad \tau_j := \int_0^{y_j} \frac{A_j(y)}{\sqrt{y}}\, dy, \quad j \in J. \tag{6. 55}$$

Proof. We find the "draining time" using (6. 53) and (6. 55),

$$T^{**}[\mathcal{A}] = \sum_{J} \frac{\tau_j}{s_j} := F(s_1, ..., s_n), \tag{6. 56}$$

so that we need to optimize the function $F(s_1, ..., s_n)$, subject to the constraint (6. 51). We use Lagrange multipliers method to find the critical point $(s_1^0, ..., s_n^0)$,

$$s_j^0 = \frac{\sqrt{\tau_j}}{\sum_J \sqrt{\tau_j}}.$$

Hence,

$$F(s_1^0, ..., s_n^0) = \left(\sum_J \sqrt{\tau_j} \right)^2. \tag{6. 57}$$

Lemma 1, along with (6. 57), implies that

$$\min_{s_j} T^{**}[\mathcal{A}] = \min_{s_j} F(s_1, ..., s_n) = \left(\sum_J \sqrt{\tau_j} \right)^2. \tag{6. 58}$$

It is clear that if we choose one of s_j to be arbitrarily small, then the function $F(s_1, ..., s_n)$ becomes arbitrarily large, so that $\sup_{\mathcal{A}} T^{**} = \infty$. □

Remark 9. We note that the range $\mathcal{R}(T^{**}) = [m, \infty) \neq (0, \infty)$ as we might expect based on all previous results of this paper. This is because the values of τ_j in (6. 55) are supposed to be given, unlike in all previous optimization problems. The range is bounded below since we can only make m as small as the given edges in the star allow. But it is not bounded above since we can make at least one s_j arbitrarily small.

7. Conclusion

We consider some isoperimetric problems motivated by the physical problem of finding time taken for a tank filled with liquid of the given volume V to drain from a small orifice at the bottom. For a tank, the draining time is derived from Torricelli's law. We find that this time has a range of $\mathcal{R}(T) = (0, \infty)$, i.e. for any given time $T \in (0, \infty)$, there exists a tank with this draining time. The proof is based on asymptotic analysis. We consider the dual problem and prove that for the given draining time T, the range of the corresponding values of the volume is similar, $\mathcal{R}(V) = (0, \infty)$. We consider two models for which Torricelli's law takes an atypical form and discuss the optimization problems for the draining time of each. We also consider optimization problems on star graphs motivated by the previous physical model.

Remark 10. The results of Theorem 1 have the following physical meaning. Torricelli's law results from Bernoulli's principle [6], which states that the sum of potential and kinetic energy per unit volume, and pressure is constant (the energy conservation law). At the top of the tank, the velocity of a cross sectional area is small (all energy is potential); at the bottom the height is taken to be zero and the velocity is maximal (all energy is kinetic). Hence, velocity will be faster when the liquid level is higher. The larger velocity (and hence, larger volume flow rate) in a tall tank results in a smaller draining time than that of a short, wide tank with the same volume. We conclude that the supremum of the draining time occurs for a short, wide tank and the infimum occurs for a tall, thin tank.

Remark 11. The assumption that the velocity of the exiting liquid be given by Torricelli's law is crucial for the physical model. Yet, liquid may not drain at all out of a long narrow tube due to the effects of capillary action (which we ignore from the very beginning). We also have to assume that the area of the orifice S is sufficiently smaller that $A(y)$, and that the velocity is small at the liquid's surface. We conclude that the result $\inf T_d(\epsilon) = 0$ (Section 4) is pure mathematical. The same observation might be made for the result in [10].

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