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Bounds for Probabilities of the Generalized Distribution Defined by Generalized Polylogarithm

Abiodun Tinuoye Oladipo
Department of Pure and Applied Mathematics,
Ladoke Akintola University of Technology Ogbomoso,
P. M. B. 4000, Ogbomoso Nigeria.
Email: atoladipo@lautech.edu.ng

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Abstract. The paper investigated the polynomials whose coefficients are generalized distribution. Convolution via generalized polylogarithm and subordination methods were employed to obtain the upper bounds for the first few coefficients of the class defined. Furthermore, relevant connections to Fekete-Szego classical theorem were established, particularly in conic region. Conclusively, consequences of various choices of parameters involved were pointed out. The results further established geometric properties of the generalized distribution associated with univalent functions.

AMS (MOS) Subject Classification Codes: 30C45
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1. Introduction

Let us denote by A the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1. 1}$$

which are analytic and univalent in the open unit disk $U = \{z : z \in C, |z| < 1\}$, and let $Y \in A$ consist of univalent functions in U normalized with $f(0) = f'(z) - 1 = 0$.
Suppose f and g are analytic in U, then f is said to be subordinate to g written as $f(z) \prec g(z)$, if there exists a function $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$.
Recently, Porwal [20] investigated generalized distribution and its geometric properties associated with univalent functions.

Let S denote the sum of the convergent series of the form

$$S = \sum_{n=0}^{\infty} a_n,$$

where $a_n \geq 0$ for all $n \in N$. The generalized discrete probability distribution whose probability mass function is given as

$$p(n) = \frac{a_n}{S},\ n = 0, 1, 2..,$$

$p(n)$ is the probability mass function because $p(n) \geq 0$ and $\sum_n p_n = 1$.
Furthermore, let

$$\psi(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then from $S = \sum_{n=0}^{\infty} a_n$ series ψ is convergent for both $|x| < 1$ and $x = 1$.
He provided information on various definitions and their derivations. For example:
(1). If X is a discrete random variable that takes values $x_1, x_2, ...$ associated with probabilities $p_1, p_2, ...$ then the expected X denoted by $E(X)$ is defined as

$$E(X) = \sum_{n=1}^{\infty} p_n x_n$$

. (2). The moment of a discrete probability distribution (r^{th}) about $x = 0$ is defined by

$$\mu_r' = E(X^r)$$

where μ_1' is the mean of the distribution and the variance is given as

$$\mu_2' - (\mu_1')^2.$$

(3). Moment about the origin is given as

$$Mean = \mu_1' = \frac{\psi'}{S},$$

$$Variance = \mu_2' - (\mu_1')^2 = \frac{1}{S}\left[\psi''(1) + \psi'(1) - \frac{(\psi'(1))^2}{S}\right].$$

The moment generating function of a random variable X is denoted by $M_X(t)$ and defined by

$$M_X(t) = E(e^{Xt})$$

and the moment generating function of generalized discrete probability is given as

$$M_X(t) = \frac{\psi(e^t)}{S}$$

For special values of a_n various well known discrete probability distributions such as Yule-Simon distribution, Logarithmic distribution, Poisson distribution, Binomial distribution, Beta-Binomial distribution, Zeta distribution, Geometric distribution and Bernoulli distribution can be obtained (see for detail [20]).

Of particular interest is the polynomial whose coefficients are probabilities of the generalized distribution introduced and investigated in [20] and it is defined by

$$K_{\psi}(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{S} z^n, \tag{1. 2}$$

where $S = \sum_{n=0}^{\infty} a_n$.
For $f(z)$ defined in (1.1) and h with the series form $h(z) = z + c_2 z^2 + ...$, the convolution of f and h denoted by $f * h$ is given as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_k c_k z^k = (h * f)(z). \tag{1. 3}$$

Suppose $|z| < 1$ and there exist $p \geq 2$, the classical polylogarithm $L_{i_\gamma}(z)$ of Leibniz with Bernoulli in 1696, is defined by the absolutely convergent series

$$L_{i_\gamma}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}.$$

Several other mathematicians have investigated polylogarithm function in various perspective (see for detail [18]). For the reason not known to the present author, studies on polylogarithm stopped for many periods of time, not until recently when its investigation gathered momentum again which is likely to be associated to its importance and application in many fields of endeavour, see for applications in [1,14].
Recently, Al-Shaqsi and Darus [22] generalized Ruscheweyh and Salagean derivative operators, using polylogarithm functions on function of the form (1).
Let f belongs to A, the generalized polylogarithms $D_{\lambda}^{m} f(z) : A \to A$

$$D_{\lambda}^{m} f(z) = z + \sum_{n=2}^{\infty} \frac{n^m (n + \lambda - 1)!}{\lambda!(n-1)!} z^n \tag{1. 4}$$

where $m \in N_0 = \{0, 1, 2, ..\}, z \in U$. It is obvious that the derivative operator $D_{\lambda}^{m} f(z)$ engages two derivative operators. If $\lambda = 0$, the operator reduces to Salagean differential operator and if $m = 0$ the operator reduces to Ruscheweyh operator (see for more properties of polylogarithm in [18]).
Motivated by the works in [17,19,20,22], we investigate bounds for the class defined in conic domain. The principal significance of the sharp bounds of the coefficients is the information about geometric properties of the functions. For instance, the sharp bounds of the second coefficient of normalized univalent functions readily yields the growth and distortion bounds. Also, sharp bounds of the coefficient functional $|a_3 - \mu a_2^2|$ helps in the investigation of univalence of analytic function. Additionally, apart from $n - th$ coefficients, bounds were used to determine the extreme points of the classes of analytic functions; (see also [3]).
Furthermore, by applying the concept of convolution defined in (1.3) to (1.2) and using a function of the form (1.4) yields

$$D_{\lambda}^{m} * K_{\psi}(z) = z + \sum_{n=2}^{\infty} \frac{n^m (n + \lambda - 1)!}{\lambda!(n-1)!} \frac{a_{n-1}}{S} z^n \tag{1. 5}$$

where $m, \lambda \in N \cup \{0\}$ and $S = \sum_{n=0}^{\infty} a_n$
The present work is designed as follows: section one contained the introduction and short literature review as background information to the present investigation, section 2 contains lemma, definition and objective while section 3 contains our main results for the present investigation; and lastly conclusion and acknowledgment followed.

2. Lemma and Definition

Let P denote the analytic functions and let $p \in P$ with $Re\ p(z) > 0$ and $p(z) = 1 + p_1 z + ...$ in U; if there exist $k \in [0, \infty)$ and $p_k \in P$, such that $p \prec p_k$ in U, where the function p_k maps the unit disk conformally onto the region Ω_k then we have

$$\partial\Omega_k = \left\{u + iv : u^2 = k^2(u-1)^2 + k^2v^2\right\}.$$

The detail of functions in conic region can be found in the literatures [5,6-13,21].
Definition 2.1 Let $k \in [0, \infty), \quad \alpha \in [0, 1),\ m, \lambda \in N \cup \{0\},\ b \neq 0$, the function $f \in A$ is in the class $\psi S_\lambda^m(b, p_k)$ if

$$1 + \frac{1}{b}\left(\left(\frac{z(D_\lambda^m K_\psi(z))'}{D_\lambda^m K_\psi(z)}\right) - 1\right) \prec p_k(z) \qquad (z \in U),$$

and $f \in A$ is in the class $\psi C_\lambda^m(b, \psi(z))$ if

$$\frac{1}{b}\left(\frac{z(D_\lambda^m K_\psi(z))'}{D_\lambda^m K_\psi(z)}\right) \prec \phi(z) - 1 \qquad (z \in U).$$

Lemma 2.1 Let $w(z) = w_1 z + w_2 z^2 + ... \in \Omega$ be so that $|w(z)| < 1$ in U. If ρ is a complex number, then

$$\left|w_2 + tw_1^2\right| \leq max(1, |\rho|).$$

The sharp inequality for the functions $w(z) = z$ or $w(z) = z^2$, (see also [2,4,15,16]).
In this work the author investigates the first few early coefficients of the defined subclasses and its relevant connection to Fekete-Szego functional associated with polylogarithms in conic domain using subordination principle.

3. Bounds for Generalized Distribution

Here the first few coefficient bounds for the class defined in Definition 2.1. is considered.
Theorem 3.1 Let $k \in [0, \infty), \quad 0 \leq \alpha < 1,\ m, \lambda \in N \cup \{0\}\ b \neq 0.$ If f given by (1.1) belongs to $\psi S_\lambda^m(b, p_k)$ then

$$\left|\frac{a_1}{S}\right| \leq \frac{|b|p_1}{2^m(\lambda+1)}$$

$$\left|\frac{a_2}{S}\right| \leq \frac{|b|p_1}{3^m(\lambda+2)(\lambda+1)} max\left\{1, \left|\frac{p_2}{p_1} + bp_1\right|\right\}$$

$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \frac{|b|p_1}{3^m(\lambda+2)(\lambda+1)} max\left\{1, \left|\frac{p_2}{p_1} + \frac{2^{2m}(\lambda+1) - 3^m\mu(\lambda+2)}{2^{2m}(\lambda+1)} bp_1\right|\right\}$$

Proof. If $f \in \psi S_q^b(p_k)$, and $w \in \Omega$ such that

$$1+\frac{1}{b}\left(\left(\frac{z(D_\lambda^m K_\psi(m,z))'}{D_\lambda^m K_\psi(m,z)}\right)-1\right)=p_k(w(z)). \tag{3. 6}$$

It is observed that

$$2^m(\lambda+1)\frac{a_1}{S}z^2+3^m(\lambda+2)(\lambda+1)\frac{a_2}{S}z^{+}...$$

$$=b\left[p_1w_1z^2+(p_1w_2+p_2w_1^2)z^3+...2^m(\lambda+1)\frac{a_1}{S}p_1w_1z^3+...\right]. \tag{3. 7}$$

Hence by (3.6) and (3.7) we have

$$\frac{a_1}{S}=\frac{bp_1w_1}{2^m(\lambda+1)} \tag{3. 8}$$

and

$$\frac{a_2}{S}=\frac{bp_1}{3^m(\lambda+2)(\lambda+1)}\left(w_2+\left(\frac{p_2}{p_1}+bp_1\right)w_1^2\right). \tag{3. 9}$$

By (3.8) and (3.9) we have

$$\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}=\frac{bp_1}{3^m(\lambda+2)(\lambda+1)}\left(w_2+tw_1^2\right)$$

where

$$\rho=\frac{p_2}{p_1}+\frac{2^{2m}(\lambda+1)-3^m\mu(\lambda+2)}{2^{2m}(\lambda+1)}bp_1.$$

The desired result is obtained by applying Lemma 2.1

Corolllary 3.1 $k\in[0,\infty),\ 0\le\alpha<1,\ m=0\ b\neq 0$ in Theorem 3.1 and if $f\in\psi S_\lambda^0(b,p_k)$ then

$$\left|\frac{a_1}{S}\right|\le\frac{|b|p_1}{(\lambda+1)}$$

$$\left|\frac{a_2}{S}\right|\le\frac{|b|p_1}{(\lambda+2)(\lambda+1)}max\left\{1,\left|\frac{p_2}{p_1}+bp_1\right|\right\}$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right|\le\frac{|b|p_1}{(\lambda+2)(\lambda+1)}max\left\{1,\left|\frac{p_2}{p_1}+\frac{\lambda(1-\mu)+(1-2\mu)}{(\lambda+1)}bp_1\right|\right\}$$

Corollary 3.2 For $k\in[0,\infty),\ 0\le\alpha<1,\ \lambda=0\ b\neq 0$ in Theorem 3.1, and if $f\in\psi S_0^m(b,p_k)$ then

$$\left|\frac{a_1}{S}\right|\le\frac{|b|p_1}{2^m}$$

$$\left|\frac{a_2}{S}\right|\le\frac{|b|p_1}{2\times 3^m}max\left\{1,\left|\frac{p_2}{p_1}+bp_1\right|\right\}$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right|\le\frac{|b|p_1}{2\times 3^m}max\left\{1,\left|\frac{p_2}{p_1}+\frac{2^{2m}-2\times 3^m\mu}{2^{2m}}bp_1\right|\right\}$$

Corollary 3.3 Let $k \in [0,\infty),\ 0 \le \alpha < 1,\ m = 0, \lambda = 0\ b \neq 0$ and if $f \in \psi S_0^0(b, p_k)$ then

$$\left|\frac{a_1}{S}\right| \le |b|p_1$$

$$\left|\frac{a_2}{S}\right| \le \frac{|b|p_1}{2} max\left\{1, \left|\frac{p_2}{p_1} + bp_1\right|\right\}$$

$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \le \frac{|b|p_1}{2} max\left\{1, \left|\frac{p_2}{p_1} + (1-2\mu)bp_1\right|\right\}$$

Corollary 3.4 Let $k \in [0,\infty),\ 0 \le \alpha < 1,\ m = 1, \lambda = 1\ b \neq 0$ and if $f \in \psi S_1^1(b, p_k)$ then

$$\left|\frac{a_1}{S}\right| \le \frac{|b|p_1}{4}$$

$$\left|\frac{a_2}{S}\right| \le \frac{|b|p_1}{2 \times 3^2} max\left\{1, \left|\frac{p_2}{p_1} + bp_1\right|\right\}$$

$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \le \frac{|b|p_1}{2 \times 3^2} max\left\{1, \left|\frac{p_2}{p_1} + \frac{2^3 - 3^2\mu}{2^3} bp_1\right|\right\}$$

Theorem 3.2. Let $k \in [0,\infty),\ 0 \le \alpha < 1,\ b \neq 0,\ m, \lambda \in N \cup \{0\}$ and let $p_k(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...,$. If $f \in \psi C_\lambda^m(b, \phi(z))$, then

$$\left|\frac{a_1}{S}\right| \le \frac{|b|c_1}{2^{m+1}(\lambda+1)}$$

$$\left|\frac{a_2}{S}\right| \le \frac{|b|c_1}{3^{m+1}(\lambda+2)(\lambda+1)} \left(c_1 + max\left\{2c_1, \left|bc_1^2\right| + 2|c_2|\right\}\right)$$

$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right|$$

$$\le \frac{2|b|c_1}{3^{m+1}(\lambda+2)(\lambda+1)} \left(c_1 + max\left\{2c_1, \left|\frac{2^{2(m+1)}(\lambda+1) - 3^{m+1}\mu(\lambda+2)}{2^{2(m+1)}(\lambda+1)}\right| |b|c_1 + 2|c_2|\right\}\right)$$

Proof: The method of proof is similar to that of Theorem 3.1 except that instead of using (3.6) we make use of

$$1 + \frac{1}{b}\left(\frac{z(D_\lambda^m K_\psi(z))'}{D_\lambda^m K_\psi(z)}\right) = (\phi(z) - 1)$$

Corollary 3.5 Let $k \in [0,\infty),\ 0 \le \alpha < 1,\ b \neq 0,\ m = 0$ and if $f \in \psi C_\lambda^0(\phi(z))$, then

$$\left|\frac{a_1}{S}\right| \le \frac{|b|c_1}{2(\lambda+1)}$$

$$\left|\frac{a_2}{S}\right| \le \frac{|b|c_1}{3(\lambda+2)(\lambda+1)} \left(c_1 + max\left\{2c_1, \left|bc_1^2\right| + 2|c_2|\right\}\right)$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right|\leq\frac{2|b|c_1}{3(\lambda+2)(\lambda+1)}\left(c_1+max\left\{2c_1,\left|\frac{\lambda(4-3\mu)-2(2-3\mu)}{4(\lambda+1)}\right||b|c_1+2|c_2|\right\}\right)$$

Corollary 3.6. Let $k\in[0,\infty),\ 0\leq\alpha<1,\ b\neq 0,\ \lambda=0$ and if $f\in\psi C_0^m(b,\phi(z))$, then

$$\left|\frac{a_1}{S}\right|\leq\frac{|b|c_1}{2^{m+1}}$$

$$\left|\frac{a_2}{S}\right|\leq\frac{|b|c_1}{3^{m+1}}\left(c_1+max\left\{2c_1,\left|bc_1^2\right|+2|c_2|\right\}\right)$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right|\leq\frac{|b|c_1}{3^{m+1}}\left(c_1+max\left\{2c_1,\left|\frac{2^{2m+1}-3^{m+1}\mu}{2^{2m+1}}\right||b|c_1+2|c_2|\right\}\right)$$

Corollary 3.7. Let $k\in[0,\infty),\ 0\leq\alpha<1,\ b\neq 0,\ m=0,\lambda=0$ and if $f\in\psi C_0^0(b,\phi(z))$, then

$$\left|\frac{a_1}{S}\right|\leq\frac{|b|c_1}{2}$$

$$\left|\frac{a_2}{S}\right|\leq\frac{|b|c_1}{3}\left(c_1+max\left\{2c_1,\left|bc_1^2\right|+2|c_2|\right\}\right)$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right|\leq\frac{|b|c_1}{3}\left(c_1+max\left\{2c_1,\left|\frac{2-3\mu}{2}\right||b|c_1+2|c_2|\right\}\right)$$

4. CONCLUSION

It is hereby concluded that with special values of parameters involved various interesting new results can be obtained. Also, result obtained to the best of our knowledge are new. Furthermore, the work establishes more relationship between Geometric Function Theory and Statistics.

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