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Neighborhood Properties for k-Uniformly Starlike Functions

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Abstract. In this note, we define the class $S_p(\alpha, k)$ and introduce and investigate coefficient estimates, neighborhood property for functions in the class $S_p(\alpha, k)$. In addition we provide conditions such that the confluent hypergeometric function, belongs to $S_p(\alpha, k)$.

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1. Introduction

We show the set of all holomorphic functions g in the unit disk $E = \{z : |z| < 1\}$ which are

$$g(z) = z + \sum_{s=2}^{\infty} a_s z^s, \tag{1. 1}$$

with $\mathcal{A}$ and let S be the subclass of $\mathcal{A}$ consisting of univalent functions. Suppose that T be the subclass of S which are in the form

$$g(z) = z - \sum_{s=2}^{\infty} a_s z^s \tag{1. 2}$$

satisfies the conditions $a_s \geq 0$ $(s = 2, 3, ...)$ with $\sum_{s=2}^{\infty} a_s < 1$.

Also suppose that $S^*(\alpha)$ be the famous subclass of S which are starlike of order α. Indeed $h \in S^*(\alpha)$ is equivalent to $Re(zh'(z)/h(z)) > \alpha$ in E. This subclass has so long history in geometric function theory (for example see [3, 4, 5, 7, 10]).

Let $a, b, c \in \mathbb{C}$ (the set of all complex numbers), such that $c \neq 0, -1, -2, ...$ It is well known that the answer of the ordinary equation

$$(1 - z)z\varphi''(z) + [c - z(a + b + 1)]\varphi'(z) - ab\varphi(z) = 0$$

is

$$F(a, b, c; z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s (1)_s} z^s$$

and the function $g(z) = zF(a, b, c; z), z \in E$, is called hypergeometric function. We note that $(a)_0 = 1$ for $a \neq 0$ and $(a)_s = a(a+1)(a+2)...(a+s-1)$.

The hypergeometric function plays an important role in various fields. We refer to [9, 11, 12] and references therein for more details about this function.

Finally, for $-1 \leq \alpha \leq 1$ and $k \geq 0$, we introduce a subclass $S_p(\alpha, k)$ of starlike functions in the following way

$$S_p(\alpha, k) = \{g \in S : Re\left(\frac{zg'(z)}{g(z)}\right) \geq k\left|\frac{zg'(z)}{g(z)} - 1\right| + \alpha, z \in E\}. \qquad (1.\ 3)$$

This class is very famous and important in univalent function theory and relevant subclasses of it have been obtained by many authors such as ([8, 13]). We note that the case $k = 0$ reduce to starlike functions of order α and the case $k = 1$ reduce to uniformly starlike functions of order α. We also let

$$TS_p(\alpha, k) = T \cap S_p(\alpha, k) \qquad and \qquad TS^*(\alpha) = T \cap S^*(\alpha).$$

Lemma 1.1. *Let $0 \leq \alpha < 1, k \geq 0$ and $\beta \in \mathbb{R}$. Then $Re(w) > k|w - t| + \alpha$ is equivalent to $Re[w(1 + ke^{i\beta}) - kte^{i\beta}] > \alpha$ where w and t are arbitrary complex numbers.*

Lemma 1.2. *Let $k \geq 0$ and $t \in \mathbb{C}$. Then $Re(t) > k$ is equivalent to $|t - (1 + k)| < |t + (1 - k)|$.*

2. Coefficient Bounds

In this section we introduce an inequality that provide a necessary and sufficient Coefficient for functions in the class $TS_p(\alpha, k)$.

Theorem 2.1. *Let $-1 \leq \alpha \leq 1, k \geq 0$ and $g \in TS_p(\alpha, k)$ be in the form (1. 2). Then we have*

$$\sum_{s=2}^{\infty}(s(1-k) + k - \alpha)a_s \leq 1 - \alpha. \qquad (2.\ 4)$$

Proof. Let $g \in TS_p(\alpha, k)$ be in the form (1. 2). By putting $w = \frac{zg'(z)}{g(z)}$ in (1. 3) and by lemma 1.1, we obtain $Re(w(1 + ke^{i\beta}) - ke^{i\beta}) \geq \alpha$ or

$$Re(\frac{(1 + ke^{i\beta})(1 - \sum_{s=2}^{\infty} sa_s z^{s-1}) - (ke^{i\beta} + \alpha)(1 - \sum_{s=2}^{\infty} a_s z^{s-1})}{1 - \sum_{s=2}^{\infty} a_s z^{s-1}}) \geq 0$$

If $z \in E$ is real and tends to 1^- through reals, then we have

$$Re(1 - \alpha + \sum_{s=2}^{\infty}(\alpha - s)a_s + ke^{i\beta}\sum_{s=2}^{\infty}(1 - s)a_s) \geq 0.$$

Therefore

$$1 - \alpha - \sum_{s=2}^{\infty}(s - \alpha)a_s + k\sum_{s=2}^{\infty}(s - 1)a_s \geq 0.$$

□

Theorem 2.2. *Let $k \geq 0$ and $g \in T$ be an analytic function of the form (1. 2). Then the following condition is sufficient for g to be in the class $S_p(\alpha, k)$.*

$$\sum_{s=2}^{\infty}(k(s-1)+s-\alpha)|a_s| \leq 1 \tag{2. 5}$$

if $-1 \leq \alpha < 0$ and

$$\sum_{s=2}^{\infty}(k(s-1)+s-\alpha)|a_s| \leq 1-\alpha \tag{2. 6}$$

if $0 \leq \alpha \leq 1$.

Proof. By lemma 1.1, we note that the condition (1. 3) is equivalent to $Re(w(1+ke^{i\beta})-(\alpha+ke^{i\beta})) \geq 0$ where $w = \frac{zg'(z)}{g(z)}$. So by lemma 1.2, it is sufficient to show that $A \geq B$ where

$$\begin{aligned} A &= |1+w(1+ke^{i\beta})-(\alpha+ke^{i\beta})| \\ &= \left|\frac{(z-\sum_{s=2}^{\infty}a_s z^s)+(1+ke^{i\beta})(z-\sum_{s=2}^{\infty}sa_s z^s)-(ke^{i\beta}+\alpha)(z-\sum_{s=2}^{\infty}a_s z^s)}{z-\sum_{s=2}^{\infty}a_s z^s}\right| \end{aligned}$$

and

$$\begin{aligned} B &= |1-w(1+ke^{i\beta})+\alpha+ke^{i\beta}| \\ &= \left|\frac{z-\sum_{s=2}^{\infty}a_s z^s-(1+ke^{i\beta})(z-\sum_{s=2}^{\infty}sa_s z^s)+(ke^{i\beta}+\alpha)(z-\sum_{s=2}^{\infty}a_s z^s)}{z-\sum_{s=2}^{\infty}a_s z^s}\right|. \end{aligned}$$

Let $M = 1/|1-\sum_{s=2}^{\infty}a_s z^{s-1}|$. Therefore

$$A \geq M(|2-\alpha| - \sum_{s=2}^{\infty}(k(s-1)+|\alpha-(s+1)|)|a_s|) \tag{2. 7}$$

and

$$B \leq M(|\alpha| + \sum_{s=2}^{\infty}(k(s-1)+|s-(1+\alpha)|)|a_s|). \tag{2. 8}$$

So by the hypothesis, if $-1 \leq \alpha < 0$, then by (2. 7) and (2. 8)

$$A - B \geq 2M(1-\sum_{s=2}^{\infty}(k(s-1)+s-\alpha)|a_s|).$$

The last expression is non-negative by (2. 5) and so g belongs to the class $S_p(\alpha, k)$. Also if $0 \leq \alpha \leq 1$, then by (2. 7) and (2. 8) we obtain

$$A - B \geq 2M(1-\alpha-\sum_{s=2}^{\infty}(k(s-1)+s-\alpha)|a_s|).$$

The last expression is non-negative by (2. 6) and so $g \in S_p(\alpha, k)$. □

The case $k = 0$ in two previous theorems leads to

Corollary 2.3. *Let* $g(z) = z - \sum_{s=2}^{\infty} a_s z^s \in T$ *and* $0 \leq \alpha \leq 1$. *Then* $g \in S^*(\alpha)$ *if and only if* $\sum_{s=2}^{\infty}(s-\alpha)a_s \leq 1-\alpha$.

Theorem 2.4. *Let* $-1 \leq \alpha \leq 1, 0 \leq k < 1$ *and let* $g_1(z) = z$,

$$g_s(z) = z - \frac{1-\alpha}{s(1-k)+k-\alpha} z^s, s \geq 2.$$

If $g \in TS_p(\alpha, k)$ *then we have* $g(z) = \sum_{s=1}^{\infty} \lambda_s g_s(z)$ *where* $\lambda_s \geq 0$ *and* $\sum_{s=1}^{\infty} \lambda_s = 1$.

Proof. Let $g \in TS_p(\alpha, k)$ has the form $z - \sum_{s=2}^{\infty} a_s z^s$. By Theorem 2.1 we obtain

$$\sum_{s=2}^{\infty} \frac{s(1-k)+k-\alpha}{1-\alpha} a_s \leq 1$$

and so

$$a_s \leq \frac{1-\alpha}{s(1-k)+k-\alpha}, \qquad s \geq 2.$$

Therefore we can set $\lambda_s = \frac{s(1-k)+k-\alpha}{1-\alpha} a_s$ for $s = 2, 3, ...$ and $\lambda_1 = 1 - \sum_{s=2}^{\infty} \lambda_s$. Thus, $0 \leq \lambda_s \leq 1$ for each $s \in \mathbb{N}$ and $\sum_{s=1}^{\infty} \lambda_s = 1$. Also $g(z)$ has the form

$$\begin{aligned} g(z) &= z - \sum_{s=2}^{\infty} a_s z^s = z - \sum_{s=2}^{\infty} \frac{\lambda_s(1-\alpha)}{s(1-k)+k-\alpha} z^s \\ &= \lambda_1 z + \sum_{s=2}^{\infty} \lambda_s (z - \frac{1-\alpha}{s(1-k)+k-\alpha} z^s) \\ &= \sum_{s=1}^{\infty} \lambda_s g_s(z). \end{aligned}$$

□

Theorem 2.5. *Let* $0 \leq \alpha \leq 1$ *and* $0 \leq k < 1$. *Also let* $a, b \in \mathbb{C} - \{0\}$ *and* $c > |a| + |b| + 1$. *Then the condition*

$$\frac{\Gamma(c-|a|-|b|-1)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}((1+k)|ab| + (1-\alpha)(c-|a|-|b|-1)) \leq 2(1-\alpha) \quad (2.\ 9)$$

is sufficient for the function $zF(a, b, c; z)$ *belongs to* $S_p(\alpha, k)$.

Proof. Set $zF(a, b, c; z)$. By Theorem 2.2, we need to show that

$$N := \sum_{s=2}^{\infty} [s(1+k) - (k+\alpha)] \left| \frac{(a)_{s-1}(b)_{s-1}}{(c)_{s-1}(1)_{s-1}} \right| \leq 1-\alpha.$$

According to $|(a)_s| \leq (|a|)_s$, we observe that

$$\begin{aligned} N &\leq \sum_{s=2}^{\infty} [s(1+k) - (k+\alpha)] \frac{(|a|)_{s-1}(|b|)_{s-1}}{(c)_{s-1}(1)_{s-1}} \\ &= (1+k) \sum_{s=1}^{\infty} \frac{(s+1)(|a|)_s(|b|)_s}{(c)_s(1)_s} - (k+\alpha) \sum_{s=1}^{\infty} \frac{(|a|)_s(|b|)_s}{(c)_s(1)_s} \end{aligned}$$

$$= (1+k)\sum_{s=1}^{\infty}\frac{(|a|)_s(|b|)_s}{(c)_s(1)_{s-1}} + (1-\alpha)\sum_{s=1}^{\infty}\frac{(|a|)_s(|b|)_s}{(c)_s(1)_s}$$
$$= \frac{|ab|}{c}(1+k)\sum_{s=0}^{\infty}\frac{(1+|a|)_s(1+|b|)_s}{(1+c)_s(1)_s} + (1-\alpha)\sum_{s=1}^{\infty}\frac{(|a|)_s(|b|)_s}{(c)_s(1)_s}$$
$$= \frac{|ab|}{c}(1+k)F(1+|a|,1+|b|,1+c;1) + (1-\alpha)(F(|a|,|b|,c;1)-1)$$
$$= \frac{|ab|}{c}(1+k)\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} + (1-\alpha)\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$-(1-\alpha).$$

Therefore according to (2. 9), N is less than $1-\alpha$. □

In the following section we investigate neighborhood property of the functions belongs to the class $TS_p(\alpha, k)$. We remark that this property was introduced by Goodman [6] and Ruscheweyh [14]. See also [1], [2] and [15].

3. Neighborhood property and Applications

For $\eta \geq 0$ and a function f belonging to $\mathcal{A}$ of the form (1. 1), we let (η, ρ)-neighborhood of f by

$$\mathcal{N}_\rho^\eta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{s=2}^{\infty} b_s z^s, \sum_{s=2}^{\infty} s^\eta |a_s - b_s| \leq \rho\}.$$

For $e(z) = z$, the identity function, we obtain

$$\mathcal{N}_\rho^\eta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{s=2}^{\infty} b_s z^s, \sum_{s=2}^{\infty} s^\eta |b_s| \leq \rho\}.$$

Theorem 3.1. *Let* $0 \leq k < \alpha \leq 1$ *and* $\eta \leq 1$. *Then* $TS_p(\alpha, k) \subseteq \mathcal{N}_\rho^\eta(e)$ *where* $\rho = \frac{2(1-\alpha)}{2-(\alpha+k)}$.

Proof. Let $g(z) = z - \sum_{s=2}^{\infty} a_s z^s$ be in a class $TS_p(\alpha, k)$. By Theorem 2.1 we obtain

$$\sum_{s=2}^{\infty}(s(1-k)+k-\alpha)a_s \leq 1-\alpha \tag{3. 10}$$

and so

$$\sum_{s=2}^{\infty} s^\eta a_s \leq \sum_{s=2}^{\infty} s a_s \leq \frac{1-\alpha}{1-k} + \frac{\alpha-k}{1-k}\sum_{s=2}^{\infty} a_s. \tag{3. 11}$$

On the other hand, from (3. 10), we implies that

$$\sum_{s=2}^{\infty} a_s \leq \frac{1-\alpha}{2(1-k)+k-\alpha}. \tag{3. 12}$$

Therefore by (3. 11) and (3. 12) we have

$$\sum_{s=2}^{\infty} s^{\eta} a_s \leq \frac{2(1-\alpha)}{2-(\alpha+k)}.$$

□

Corollary 3.2. *For $0 \leq \alpha \leq 1$ and $\eta \leq 1$, we have $TS^*(\alpha) \subseteq \mathcal{N}_{\rho}^{\eta}(e)$ where $\rho = \frac{2(1-\alpha)}{2-\alpha}$.*

The case $\eta = 1$ in Theorem 3.1 leads to

Corollary 3.3. *Let $g(z) = z - \sum_{s=2}^{\infty} a_s z^s \in T$ and $0 \leq k < \alpha \leq 1$. If $\frac{g(z)+\epsilon z}{1+\epsilon} \in S_p(\alpha, k)$, in which $\epsilon > 0$, then*

$$\sum_{s=2}^{\infty} s a_s \leq \frac{2(1-\alpha)(1+\epsilon)}{2-(k+\alpha)}$$

and equation is established for the following function

$$g(z) = z - \frac{(1-\alpha)(1+\epsilon)}{2-(k+\alpha)} z^2.$$

Theorem 3.4. *Let $g \in T$ and $0 \leq k < \alpha \leq 1$. If $\frac{g(z)+\epsilon z}{1+\epsilon} \in S_p(\alpha, k)$, in which $\epsilon > 0$, then $\mathcal{N}_{\beta}(g)$ is a subset of $S_p(\alpha, k)$ where*

$$\beta \leq \frac{1-\alpha}{1+k} - \frac{2(1-\alpha)(1+\epsilon)}{2-(k+\alpha)}. \tag{3. 13}$$

Proof. Let $g(z) = z - \sum_{s=2}^{\infty} a_s z^s$ and $f(z) = z + \sum_{s=2}^{\infty} b_s z^s \in \mathcal{N}_{\beta}(g)$. This means that we have $\sum_{s=2}^{\infty} s|a_s + b_s| \leq \beta$. So by the hypothesis and Corollary 3.3, we obtain

$$\begin{aligned} \sum_{s=2}^{\infty} s|b_s| &= \sum_{s=2}^{\infty} s|b_s + a_s - a_s| \\ &\leq \beta + \frac{2(1-\alpha)(1+\epsilon)}{2-(k+\alpha)} \\ &\leq \frac{1-\alpha}{1+k}. \end{aligned}$$

Therefore we have

$$\sum_{s=2}^{\infty} \frac{s(1+k)-(k+\alpha)}{1-\alpha} |b_s| \leq \frac{1+k}{1-\alpha} \sum_{s=2}^{\infty} s|b_s| \leq 1$$

and by Theorem 2.2, $f(z) \in S_p(\alpha, k)$. □

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