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Some Generalized Distance and Similarity Measures for Picture Hesitant Fuzzy Sets and Their Applications in Building Material Recognition and Multi-Attribute Decision Making

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Abstract. The aim of this article is to develop some distance measures for newly defined framework of picture hesitant fuzzy set (PHFS). A PHFS is a picture fuzzy set (PFS) having membership, abstinence and non-membership grades in the form of hesitant fuzzy numbers (HFNs). These distance measures include generalized picture hesitant distance measure, generalized picture hesitant normalizer distance measure, generalized picture hesitant weighted distance measure and generalized picture hesitant normalizer weighted distance measure along with some other distance measures. A comparison of developed distance measures is established with existing distance measures and their advantages are discussed.

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1. Introduction

The notion of fuzzy set (FS) was developed by Zadeh [60] in 1965 opening a new area of interest for researchers. Zadeh's model of FS defined the membership of an element to a set in terms of a characteristic function on a unit interval and therefore described the uncertain events in a unique way. Zadeh's work was followed extensively by researchers as in [1] FS theory applied in medical diagnosis, [11,12] provide a way of handling the decision making problems, [20] proposed fuzzy soft set in BCI-Modules [18] applied fuzzy relations to solve clusters and [5] provide information on optimal control using fuzzy techniques. Some interesting application of FS theory are discussed in [6,7]. Atanassov [8] improved the idea of FS proposed by Zadeh and developed the theory of intuitionistic fuzzy sets (IFSs). An IFS generalizes the model of FS by describing the non-membership degree of an element along with membership degree and proved to be optimal in dealing with uncertain events, especially events having uncertainty of yes or no types. Some basic work on IFSs can be found in [17,22,29,30] and for some interesting applications of IFS theory we refer to [31,36,35,13,14,15] etc. The framework of IFS developed by Atanassov handled problems having uncertainty strongly but there was a problem the decision makers faced while dealing with voting situation where one may have more than two types of openions as one may vote in favor or remain abstain or vote against or refused to vote. Such type of circumstances could not be modeled using ordinary IFS creating a motivation for Cuong as he developed a new fuzzy model known as PFS [16] which is defined in terms of four characteristic functions denoting the membership, abstinence, non-membership and refusal degree of an element to a set. For some developments on PFSs we refer to [51,52,47,48,56]. The concept of hesitant fuzzy set (HFS) proposed by Torra [61] is also a generalization of FS providing the membership of an element in terms of a finite subset of unit interval [0,1]. HFS theory have been greatly applied to many challenging problems. Some work on HFSs could be found in [42,43,38,9,10,32]. It is common that combing two or more structures provided flexibility always in the history of FS theory. Some example can be found in [33,34,39,49,56,57] etc. Motivated by the work of [33,34,39,49,53,54,56,57,58], in [59], the theory of picture hesitant fuzzy sets is developed as a combination of PFS and HFS. The aim of this article is to develop some new distance measures for the newly defined framework of PHFSs. Motivated by the work of [43] some new distance measures are proposed in this article in the framework of PHFSs as a generalization of existing DMs. These distance measures could be very useful in pattern recognition problems. For some relevant work, one may refer to [2,3,4,19,21,23-28,37,40,41,44-46,50,55]. This article is organized as follows: Section one is based on a historical background of FS, IFS, PFS, HFS and related notions pointing towards the limitation of existing literature and the generalization of new concepts. In section two, some basic notions are defined providing a base for proposed work. Section three consists of a number of distance measures developed for PHFSs and the generalization of defined distance measures are proved using some remarks and examples. Some advantages of proposed work and concluding points are added at the end of the article.

2. NOTATION AND PRELIMINARIES

In this section, some basic results of IFSs, PFSs, HFSs and PHFSs are studied along with the basic concepts of similarity and distance measures.

2.1. **Definition [8].** Let X be a set. Then a IFS is having the shape $Z = \{< I(x), J(x) >: x \in X\}$ where $I : X \to [0,1]$ and $J : X \to [0,1]$ are the degree of membership and non-membership degree of x in Z respectively, provided that $0 \leq I(x) + J(x) \leq 1$. Further, $R(x) = 1 - (I(x) + J(x))$ is termed as hesitancy degree of x in X.

2.2. **Definition [11].** Let X be a set. Then a PFS is having the shape $Q = \{< I(x), J(x), K(x) >: x \in X\}$ where $I : X \to [0,1]$, $J : X \to [0,1]$ and $K : X \to [0,1]$ are the degree of membership, abstinence and non-membership degree of x in Q respectively, provided that $0 \leq I(x) + J(x) + K(x) \leq 1$. Further, $R(x) = 1 - (I(x) + J(x) + K(x))$ is termed as refusal degree of x in X.

2.3. **Definition [36].** Let X be a set. Then a HFS on X is a mapping S that gives us few elements of $[0,1]$ against each $x \in X$. i.e $H = \{< x, I(x) >: I(x)$is a finite subset of $[0,1] \forall x \in X\}$. Moreover $I(x)$ is called hesitant fuzzy number (HFN).

2.4. **Definition [49].** Let X be a set. Then a PHFS is having the shape $P = \{< I(x), J(x), K(x) >: x\}$ where I, J, K are HFNs denoting the degree membership, abstinence/neutral and non-membership degree of x in P respectively, provided that $0 \leq sup(I(x)) + sup(J(x)) + sup(K(x)) \leq 1$. Further, $R(x) = 1 - (sup(I(x)) + sup(J(x)) + sup\,(K(x)))$ is term as refusal degree of x in P.

2.5. **Definition [49].** Let $P = (I, J, K), P_1 = (I_1, J_1, K_1)$ and $P_2 = (I_2, J_2, K_2)$ be the three PHFNs. Then

[1] $P_1 \cup P_2 = (max(I_1(x), I_2(x)), min(J_1(x), J_2(x)), min(K_1(x), K_2(x)))$
[2] $P_1 \cap P_2 = (min(I_1(x), I_2(x)), min(J_1(x), J_2(x)), max(K_1(x), K_2(x)))$
[3] $P^c = (K(x), J(x), I(x))$

2.6. **Definition [49].** Let P, Q be two PHFSs on X. Then $d(P, Q)$ is called a distance measure satisfying the following conditions:

[1] $0 \leq d(P, Q) \leq 1$
[2] $d(P, Q) = 0 \Leftrightarrow P = Q$
[3] $d(P, Q) = d(Q, P)$

2.7. **Definition [49].** Let P, Q be two PHFSs on X. Then $S(P, Q)$ is called a similarity measure satisfying the following conditions:

[1] $0 \leq S(P, Q) \leq 1$
[2] $S(P, Q) = 0 \Leftrightarrow P = Q$
[3] $S(P, Q) = S(Q, P)$

3. Distance Measure for Picture Hesitant Fuzzy Set

In this section, we proposed several distance measures for PHFSs. Note that $PHFS(X)$ denote the set of all PHFS on X in this manuscript.

3.1. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized picture hesitant distance measure (GPHDM) is of the form:

$$d_{gphdm}(P, Q) = \left(\sum_{j=1}^{m} \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) \right)^{1/\daleth} \quad (1)$$

Obviously, the following properties hold true for GPHDM:

[1] $0 \leq d_{gphdm}(P, Q) \leq 1$
[2] $d_{gphdm}(P, Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gphdm}(P, Q) = d_{gphdm}(Q, P)$
[4] $d_{gphdm}(P, Q) + d_{gphdm}(Q, R) \geq d_{gphdm}(P, R)$

Proof. It is easy to see that $d_{gphdm}(P,Q)$ satisfies the conditions $(1-3)$. We have only to prove (4) for $d_{gphdm}(P,Q)$. Let $P \subset Q \subset R$ then $I_P^{o(i)}(x_j) \leq I_Q^{o(i)}(x_j) \leq I_R^{o(i)}(x_j), J_P^{o(i)}(x_j) \leq J_Q^{o(i)}(x_j) \leq J_R^{o(i)}(x_j)$ and $K_P^{o(i)}(x_j) \leq K_Q^{o(i)}(x_j) \leq K_R^{o(i)}(x_j)$ $\forall x_j \in X$. It follows that

$$\begin{pmatrix} ((I_P^{o(i)}(x_j)-I_Q^{o(i)}(x_j))^{\daleth}+ \\ ((J_P^{o(i)}(x_j)-J_Q^{o(i)}(x_j))^{\daleth}+ \\ ((K_P^{o(i)}(x_j)-K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix} \geq \begin{pmatrix} ((I_P^{o(i)}(x_j)-I_R^{o(i)}(x_j))^{\daleth}+ \\ ((J_P^{o(i)}(x_j)-J_R^{o(i)}(x_j))^{\daleth}+ \\ ((K_P^{o(i)}(x_j)-K_R^{o(i)}(x_j))^{\daleth} \end{pmatrix}$$

$$\left(\sum_{j=1}^{m}\left(\frac{1}{3Z_{x_j}}\sum_{i=1}^{Z_{x_j}}\begin{pmatrix} ((I_P^{o(i)}(x_j)-I_Q^{o(i)}(x_j))^{\daleth}+ \\ ((J_P^{o(i)}(x_j)-J_Q^{o(i)}(x_j))^{\daleth}+ \\ ((K_P^{o(i)}(x_j)-K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix}\right)\right)^{1/\daleth} \geq$$

$$\left(\sum_{j=1}^{m}\left(\frac{1}{3Z_{x_j}}\sum_{i=1}^{Z_{x_j}}\begin{pmatrix} ((I_P^{o(i)}(x_j)-I_R^{o(i)}(x_j))^{\daleth}+ \\ ((J_P^{o(i)}(x_j)-J_R^{o(i)}(x_j))^{\daleth}+ \\ ((K_P^{o(i)}(x_j)-K_R^{o(i)}(x_j))^{\daleth} \end{pmatrix}\right)\right)^{1/\daleth}$$

$$d_{gphdm}(P,Q) \geq d_{gphdm}(P,R)$$

similarly

$$d_{gphdm}(Q,R) \geq d_{gphdm}(P,R)$$

then, we combined the above two inequality such that

$$d_{gphdm}(P,Q) + d_{gphdm}(Q,R) \geq d_{gphdm}(P,R)$$

□

3.2. **Definition.** Let $P,Q,R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized picture hesitant normalizer distance measure (GPHNDM) is of the form:

$$d_{gphndm}(P,Q) = \left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{3Z_{x_j}}\sum_{i=1}^{Z_{x_j}}\begin{pmatrix} ((I_P^{o(i)}(x_j)-I_Q^{o(i)}(x_j))^{\daleth}+ \\ ((J_P^{o(i)}(x_j)-J_Q^{o(i)}(x_j))^{\daleth}+ \\ ((K_P^{o(i)}(x_j)-K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix}\right)\right)^{1/\daleth} \quad (2)$$

Obviously, the following properties hold true for GPHNDM:
[1] $0 \leq d_{gphndm}(P,Q) \leq 1$
[2] $d_{gphndm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gphndm}(P,Q) = d_{gphndm}(Q,P)$
[4] $d_{gphndm}(P,Q) + d_{gphndm}(Q,R) \geq d_{gphndm}(P,R)$

Proof. Straightforward. □

3.3. **Remark.** If we place $\gimel = 1$. Then $Eq.(1)$ and $Eq.(2)$ are known as generalized picture hesitant hamming distance measure (GPHHDM) and generalized picture hesitant normalizer hamming distance measure (GPHNHDM). Similarly, if we place $\gimel = 2$. Then $Eq.(1)$ and $Eq.(2)$ are known as generalized picture hesitant Euclidean distance measure (GPHEDM) and generalized picture hesitant normalizer Euclidean distance measure (GPHNEDM). If we take $J = 0$. Then $Eq.(1)$ and $Eq.(2)$ are known as generalized intuitionistic hesitant distance measure (GIHDM) and generalized intuitionistic hesitant normalizer distance measure (GIHNDM) [46].

3.4. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\gimel > 0$. The generalized picture hesitant normalizer weighted distance measure (GPHNWDM) is of the form:

$$d_{gphnwdm}(P,Q) = \left(\sum_{j=1}^{m} w_j \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\gimel} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\gimel} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\gimel} \end{array} \right) \right) \right)^{1/\gimel} \quad (3)$$

Obviously, the following properties hold true for GPHNDM:
[1] $0 \leq d_{gphnwdm}(P,Q) \leq 1$
[2] $d_{gphnwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gphnwdm}(P,Q) = d_{gphnwdm}(Q,P)$
[4] $d_{gphnwdm}(P,Q) + d_{gphnwdm}(Q,R) \geq d_{gphnwdm}(P,R)$
Where $w_j (j = 1, 2, .., m)$ is weighted such that $\sum_{j=1}^{m} w_j = 1$

Proof. Straightforward. □

3.5. **Remark.** If we place $\gimel = 1$. Then $Eq.(3)$ are known as generalized picture hesitant normalizer weighted hamming distance measure (GPHNWHDM). Similarly, we if place $\gimel = 2$. Then $Eq.(3)$ are known as generalized picture hesitant normalizer weighted Euclidean distance measure (GPHNWEDM). If we take $J = 0$. Then $Eq.(3)$ are known as generalized intuitionistic normalizer weighted distance measure (GIHNWDM).

3.6. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\gimel > 0$. The generalized picture hesitant Hausdorff distance measure (GPHHDM) is of the form:

$$d_{gphhdm}(P,Q) = \left(\frac{1}{3} \sum_{i=1}^{Z_{x_j}} \max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\gimel} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\gimel} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\gimel} \end{array} \right) \right)^{1/\gimel} \quad (4)$$

Obviously, the following properties hold true for GPHHDM:
[1] $0 \leq d_{gphhdm}(P,Q) \leq 1$
[2] $d_{gphhdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gphhdm}(P,Q) = d_{gphhdm}(Q,P)$
[4] $d_{gphhdm}(P,Q) + d_{gphhdm}(Q,R) \geq d_{gphhdm}(P,R)$

Proof. It is easy to see that $d_{gphhdm}(P,Q)$ satisfies the conditions $(1-3)$. We have only to prove (4) for $d_{gphhdm}(P,Q)$. Let $P \subset Q \subset R$ then $I_P^{o(i)}(x_j) \leq I_Q^{o(i)}(x_j) \leq I_R^{o(i)}(x_j), J_P^{o(i)}(x_j) \leq J_Q^{o(i)}(x_j) \leq J_R^{o(i)}(x_j)$ and $K_P^{o(i)}(x_j) \leq K_Q^{o(i)}(x_j) \leq K_R^{o(i)}(x_j)$ $\forall x_j \in X$. It follows that

$$\begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{ℷ} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{ℷ} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{ℷ} \end{pmatrix} \geq \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_R^{o(i)}(x_j))^{ℷ} + \\ ((J_P^{o(i)}(x_j) - J_R^{o(i)}(x_j))^{ℷ} + \\ ((K_P^{o(i)}(x_j) - K_R^{o(i)}(x_j))^{ℷ} \end{pmatrix}$$

$$\left(\frac{1}{3} \sum_{i=1}^{Z_{x_j}} \max_i \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{ℷ} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{ℷ} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{ℷ} \end{pmatrix} \right)^{1/ℷ} \geq$$

$$\left(\frac{1}{3} \sum_{i=1}^{Z_{x_j}} \max_i \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_R^{o(i)}(x_j))^{ℷ} + \\ ((J_P^{o(i)}(x_j) - J_R^{o(i)}(x_j))^{ℷ} + \\ ((K_P^{o(i)}(x_j) - K_R^{o(i)}(x_j))^{ℷ} \end{pmatrix} \right)^{1/ℷ}$$

$$d_{gphhdm}(P,Q) \geq d_{gphhdm}(P,R)$$

similarly

$$d_{gphhdm}(Q,R) \geq d_{gphhdm}(P,R)$$

then, we combined the above two inequality such that

$$d_{gphhdm}(P,Q) + d_{gphhdm}(Q,R) \geq d_{gphhdm}(P,R)$$

□

3.7. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $ℷ > 0$. The generalized picture hesitant Hausdorff weighted distance measure (GPHHWDM) is of the form:

$$d_{gphhwdm}(P,Q) = \left(\frac{1}{3} \sum_{i=1}^{Z_{x_j}} w_j \max_i \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{ℷ} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{ℷ} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{ℷ} \end{pmatrix} \right)^{1/ℷ} \quad (5)$$

Obviously, the following properties hold true for GPHHWDM:
[1] $0 \leq d_{gphhwdm}(P,Q) \leq 1$
[2] $d_{gphhwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gphhwdm}(P,Q) = d_{gphhwdm}(Q,P)$
[4] $d_{gphhwdm}(P,Q) + d_{gphhwdm}(Q,R) \geq d_{gphhwdm}(P,R)$

Proof. Straightforward. □

3.8. **Remark.** If we place $\daleth = 1$. Then $Eq.(4)$ and $Eq.(5)$ are known as generalized picture hesitant Hausdorff hamming distance measure (GPHHHDM) and generalized picture hesitant Hausdorff weighted hamming distance measure (GPHHWHDM). Similarly, if we place $\daleth = 2$. Then $Eq.(1)$ and $Eq.(2)$ are known as generalized picture hesitant Hausdorff Euclidean distance measure (GPHHEDM) and generalized picture hesitant Hausdorff weighted Euclidean distance measure (GPHHWEDM). If we take $J = 0$. Then $Eq.(4)$ and $Eq.(5)$ are known as generalized intuitionistic hesitant distance measure (GIHDM) and generalized intuitionistic hesitant weighted distance measure (GIHWDM).

3.9. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized hybrid picture hesitant weighted distance measure (GHPHWDM) is of the form:

$$d_{ghphwdm}(P,Q) = \left(\sum_{j=1}^{m} \frac{w_j}{2} \left(\begin{array}{c} \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) + \\ \left(\frac{1}{3} \sum_{i=1}^{Z_{x_j}} \max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) \end{array} \right) \right)^{1/\daleth}$$

Obviously the above Eq. (6), the following properties hold true for GHPHWDM:
[1] $0 \leq d_{ghphwdm}(P,Q) \leq 1$
[2] $d_{ghphwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{ghphwdm}(P,Q) = d_{ghphwdm}(Q,P)$
[4] $d_{ghphwdm}(P,Q) + d_{ghphwdm}(Q,R) \geq d_{ghphwdm}(P,R)$

Proof. Straightforward. □

3.10. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized hybrid picture hesitant normelizer weighted distance measure (GHPHNWDM) is of the form:

$$d_{ghphnwdm}(P,Q) = \left(\frac{1}{m} \sum_{j=1}^{m} \frac{w_j}{2} \left(\begin{array}{c} \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) + \\ \left(\frac{1}{3} \sum_{i=1}^{Z_{x_j}} \max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) \end{array} \right) \right)^{1/\daleth}$$

Obviously the above is called Eq. (7), the following properties hold true for GHPHNWDM:
[1] $0 \leq d_{ghphnwdm}(P,Q) \leq 1$
[2] $d_{ghphnwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{ghphnwdm}(P,Q) = d_{ghphnwdm}(Q,P)$
[4] $d_{ghphnwdm}(P,Q) + d_{ghphnwdm}(Q,R) \geq d_{ghphnwdm}(P,R)$

Proof. Straightforward. □

3.11. **Remark.** If we place $\daleth = 1$. Then $Eq.(6)$ and $Eq.(7)$ are known as generalized hybrid picture hesitant weighted hamming distance measure (GHPHWHDM) and generalized hybrid picture hesitant normalizer weighted hamming distance measure (GHPHNWHDM). Similarly, if we place $\daleth = 2$. Then $Eq.(6)$ and $Eq.(7)$ are known as generalized hybrid picture hesitant weighted Euclidean distance measure (GHPHWEDM) and generalized hybrid picture hesitant normalizer weighted Euclidean distance measure (GHPHNWEDM). If we take $J = 0$. Then $Eq.(6)$ and $Eq.(7)$ are known as generalized hybrid intuitionistic hesitant weighted distance measure (GHIHWDM) and generalized hybrid intuitionistic hesitant normalizer weighted distance measure (GHIHNWDM).

We are fined the previous work of distance measure for discrete. The all elements under integral is continues. The weights $w(x) \in [0, 1]$ of $x \in X = [l, p]$ and $\int_1^p w(x)dx = 1$

Then we proposed the following definitions.

3.12. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized continuous picture hesitant weighted distance measure (GCPHWDM) is of the form:

$$d_{gcphwdm}(P,Q) = \left(\int_1^p w(x) \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) dx \right)^{1/\daleth}$$

Obviously the above is called Eq. (8), the following properties hold true for GCPHWDM:

[1] $0 \leq d_{gcphwdm}(P,Q) \leq 1$
[2] $d_{gcphwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gcphwdm}(P,Q) = d_{gcphwdm}(Q,P)$
[4] $d_{gcphwdm}(P,Q) + d_{gcphwdm}(Q,R) \geq d_{gcphwdm}(P,R)$.

Proof. It is easy to see that $d_{gcphwdm}(P,Q)$ satisfies the conditions $(1-3)$. We have only to prove (4) for $d_{gcphwdm}(P,Q)$. Let $P \subset Q \subset R$ then $I_P^{o(i)}(x_j) \leq I_Q^{o(i)}(x_j) \leq I_R^{o(i)}(x_j), J_P^{o(i)}(x_j) \leq J_Q^{o(i)}(x_j) \leq J_R^{o(i)}(x_j)$ and $K_P^{o(i)}(x_j) \leq K_Q^{o(i)}(x_j) \leq K_R^{o(i)}(x_j)$ $\forall x_j \in X$. It follows that

$$\left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \geq \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_R^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_R^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_R^{o(i)}(x_j))^{\daleth} \end{array} \right)$$

$$\left(\int_1^p w(x) \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) dx \right)^{1/\daleth} \geq$$

$$\left(\int_1^p w(x) \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_R^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_R^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_R^{o(i)}(x_j))^{\daleth} \end{array} \right) \right) dx \right)^{1/\daleth}$$

$$d_{gcphwdm}(P,Q) \geq d_{gcphwdm}(P,R)$$

similarly

$$d_{gcphwdm}(Q,R) \geq d_{gcphwdm}(P,R)$$

then, we combined the above two inequality such that

$$d_{gcphwdm}(P,Q) + d_{gcphwdm}(Q,R) \geq d_{gcphwdm}(P,R)$$

□

3.13. **Remark.** If we place $\daleth = 1$. Then $Eq.(8)$ are known as generalized continuous picture hesitant weighted hamming distance measure (GCPHWHDM). Similarly, if we place $\daleth = 2$. Then $Eq.(8)$ are known as generalized continuous picture hesitant weighted Euclidean distance measure (GCPHWEDM).

If we consider $w(x) = \frac{1}{(p-l)}$, then the $Def.(15)$ is converted into $Def.(16)$.

3.14. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized continuous picture hesitant normalizer weighted distance measure (GCPHNWDM) is of the form:

$$d_{gcphnwdm}(P,Q) = \left(\frac{1}{p-l} \int_1^p \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix} \right) dx \right)^{1/\daleth} \quad (9)$$

Obviously, the following properties hold true for GCPHNWDM:

[1] $0 \leq d_{gcphnwdm}(P,Q) \leq 1$
[2] $d_{gcphnwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gcphnwdm}(P,Q) = d_{gcphnwdm}(Q,P)$
[4] $d_{gcphnwdm}(P,Q) + d_{gcphnwdm}(Q,R) \geq d_{gcphnwdm}(P,R)$

Proof. It is easy to see that $d_{gcphnwdm}(P,Q)$ satisfies the conditions $(1-3)$. We have only to prove (4) for $d_{gcphnwdm}(P,Q)$. Let $P \subset Q \subset R$ then $I_P^{o(i)}(x_j) \leq I_Q^{o(i)}(x_j) \leq I_R^{o(i)}(x_j), J_P^{o(i)}(x_j) \leq J_Q^{o(i)}(x_j) \leq J_R^{o(i)}(x_j)$ and $K_P^{o(i)}(x_j) \leq K_Q^{o(i)}(x_j) \leq K_R^{o(i)}(x_j)$ $\forall x_j \in X$. It follows that

$$\begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix} \geq \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_R^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_R^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_R^{o(i)}(x_j))^{\daleth} \end{pmatrix}$$

$$\left(\frac{1}{p-l} \int_1^p \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix} \right) dx \right)^{1/\daleth} \geq$$

$$\left(\frac{1}{p-l} \int_1^p \left(\frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_R^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_R^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_R^{o(i)}(x_j))^{\daleth} \end{pmatrix} \right) dx \right)^{1/\daleth}$$

$$d_{gcphnwdm}(P,Q) \geq d_{gcphnwdm}(P,R)$$

similarly

$$d_{gcphnwdm}(Q,R) \geq d_{gcphnwdm}(P,R)$$

Then, we combined the above two inequality such that

$$d_{gcphnwdm}(P,Q) + d_{gcphnwdm}(Q,R) \geq d_{gcphnwdm}(P,R)$$

□

3.15. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized continuous picture hesitant Hausdorff weighted distance measure (GCPHHWDM) is of the form:

$$d_{gcphhwdm}(P,Q) = \left(\frac{1}{3} \int_1^p w(x) \max_i \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix} dx \right)^{1/\daleth} \quad (10)$$

Obviously, the following properties hold true for GCPHHWDM:
[1] $0 \leq d_{gcphhwdm}(P,Q) \leq 1$
[2] $d_{gcphhwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gcphhwdm}(P,Q) = d_{gcphhwdm}(Q,P)$
[4] $d_{gcphhwdm}(P,Q) + d_{gcphhwdm}(Q,R) \geq d_{gcphhwdm}(P,R)$

Proof. Straightforward. □

3.16. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized continuous picture hesitant housdroff normalizer weighted distance measure (GCPHHNWDM) is of the form:

$$d_{gcphhnwdm}(P,Q) = \left(\frac{1}{3\,(p-l)} \int_1^p \max_i \begin{pmatrix} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{pmatrix} dx \right)^{1/\daleth} \quad (11)$$

Obviously, the following properties hold true for GCPHHNWDM:
[1] $0 \leq d_{gcphhnwdm}(P,Q) \leq 1$
[2] $d_{gcphhnwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gcphhnwdm}(P,Q) = d_{gcphhnwdm}(Q,P)$
[4] $d_{gcphhnwdm}(P,Q) + d_{gcphhnwdm}(Q,R) \geq d_{gcphhnwdm}(P,R)$

Proof. Straightforward. □

3.17. **Remark.** If we place $\daleth = 1$. Then $Eq.(9), Eq.(10)$ and $Eq.(11)$ are known as generalized continuous picture hesitant normalizer weighted hamming distance measure (GCPHNWHDM), generalized continuous picture hesitant Hausdorff weighted hamming distance measure (GCPHHWHDM) and generalized continuous picture hesitant Hausdorff normalizer weighted hamming distance measure (GCPHHNWHDM). Similarly, if we place $\daleth = 2$. Then $Eq.(9), Eq.(10)$ and $Eq.(11)$ are known as generalized continuous picture hesitant normalizer weighted Euclidean distance measure (GCPHNWEDM), generalized continuous picture hesitant Hausdorff weighted Euclidean distance measure (GCPHHWEDM) and generalized continuous picture hesitant Hausdorff normalizer weighted Euclidean distance measure (GCPHHNWEDM). If we take the $J = 0$. Then $Eq.(9), Eq.(10)$ and $Eq.(11)$ are known as generalized continuous intuitionistic hesitant normalizer weighted distance measure (GCIHNWDM), generalized continuous intuitionistic hesitant Hausdorff weighted distance measure (GCIHHWDM) and generalized continuous intuitionistic hesitant Hausdorff normalizer weighted distance measure (GCIHHNWDM).

3.18. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized hybrid continuous picture hesitant weighted distance measure (GHCPHWDM) is of the form: $\mathrm{d}_{ghcphwdm}(P,Q) =$

$$\left(\frac{1}{2}\int_1^p w(x) \left(\begin{array}{c} \frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) + \\ \frac{1}{3}\max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \end{array} \right) dx \right)^{1/\daleth}$$

Obviously the above is called Eq. (12), the following properties hold true for GHCPHWDM:

[1] $0 \leq d_{ghcphwdm}(P,Q) \leq 1$
[2] $d_{ghcphwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{ghcphwdm}(P,Q) = d_{ghcphwdm}(Q,P)$
[4] $d_{ghcphwdm}(P,Q) + d_{ghcphwdm}(Q,R) \geq d_{ghcphwdm}(P,R)$.

Proof. Straightforward. □

3.19. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\daleth > 0$. The generalized hybrid continuous picture hesitant normalizer weighted distance measure (GHCPHNWDM) is of the form: $\mathrm{d}_{ghcphnwdm}(P,Q) =$

$$\left(\frac{1}{2\,(p-l)}\int_1^p \left(\begin{array}{c} \frac{1}{3Z_{x_j}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) + \\ \frac{1}{3}\max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_j) - I_Q^{o(i)}(x_j))^{\daleth} + \\ ((J_P^{o(i)}(x_j) - J_Q^{o(i)}(x_j))^{\daleth} + \\ ((K_P^{o(i)}(x_j) - K_Q^{o(i)}(x_j))^{\daleth} \end{array} \right) \end{array} \right) dx \right)^{1/\daleth}$$

Obviously the above is called Eq. (13), the following properties hold true for GHCPHNWDM:

[1] $0 \leq d_{ghcphnwdm}(P,Q) \leq 1$
[2] $d_{ghcphnwdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{ghcphnwdm}(P,Q) = d_{ghcphnwdm}(Q,P)$
[4] $d_{ghcphnwdm}(P,Q) + d_{ghcphnwdm}(Q,R) \geq d_{ghcphnwdm}(P,R)$.

Proof. Straightforward. □

3.20. **Remark.** If we place $\gimel = 1$. Then $Eq.(12)$ and $Eq.(13)$ are known as generalized hybrid continuous picture hesitant weighted hamming distance measure (GHCPHWHDM), generalized hybrid continuous picture hesitant normalizer weighted hamming distance measure (GHCPHNWHDM). Similarly, if we place $\gimel = 2$. Then $Eq.(12)$ and $Eq.(13)$ are known as generalized hybrid continuous picture hesitant weighted hamming distance measure (GHCPHWHDM), generalized hybrid continuous picture hesitant normalizer weighted hamming distance measure (GHCPHNWHDM). If we take $J = 0$. Then $Eq.(12)$ and $Eq.(13)$ are known as generalized hybrid continuous intuitionistic hesitant weighted distance measure (GHCIHWDM), generalized hybrid continuous intuition istic hesitant normalizer weighted distance measure (GHCIHNWDM).

3.21. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\gimel > 0$. The generalized picture hesitant ordered weighted distance measure (GPHOWDM) is of the form:

$$d_{gphowdm}(P,Q) = \left(\sum_{j=1}^{m} w_j \left(\frac{1}{3Z_{x_{\circ(j)}}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)}) - I_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j)}) - J_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j)}) - K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array} \right) \right) \right)^{1/\gimel}$$

Obviously the above is called Eq. (14), the following properties hold true for GPHOWDM:
[1] $0 \leq d_{gphowdm}(P,Q) \leq 1$
[2] $d_{gphowdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gphowdm}(P,Q) = d_{gphowdm}(Q,P)$
[4] $d_{gphowdm}(P,Q) + d_{gphowdm}(Q,R) \geq d_{gphowdm}(P,R)$

Proof. Straightforward. □

3.22. **Remark.** If we place $\gimel = 1$. Then $Eq.(14)$ are known as generalized picture hesitant ordered weighted hamming distance measure (GPHOWHDM). Similarly, if we place $\gimel = 2$. Then $Eq.(14)$ are known as generalized picture hesitant ordered weighted Euclidean distance measure (GPHOWEDM). If we take $J = 0$. Then $Eq.(14)$ are known as generalized intuitionistic hesitant ordered weighted distance measure (GIHOWDM). The follows is holds obviously:

$$\frac{1}{3Z_{x_{\circ(j+1)}}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j+1)}) - I_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j+1)}) - J_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j+1)}) - K_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} \end{array} \right) \geq$$

$$\frac{1}{3Z_{x_{\circ(j)}}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)}) - I_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j)}) - J_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j)}) - K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array} \right)$$

3.23. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\gimel > 0$. The generalized picture hesitant Hausdorff ordered weighted distance measure (GPHHOWDM) is of the form:

$$d_{gphhowdm}(P,Q) = \left(\frac{1}{3} \sum_{j=1}^{m} w_j \left(\max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)}) - I_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j)}) - J_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j)}) - K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array} \right) \right) \right)^{1/\gimel} \quad (15)$$

Obviously, the following properties hold true for GPHHOWDM:
[1] $0 \leq d_{gphhowdm}(P,Q) \leq 1$
[2] $d_{gphhowdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{gphhowdm}(P,Q) = d_{gphhowdm}(Q,P)$
[4] $d_{gphhowdm}(P,Q) + d_{gphhowdm}(Q,R) \geq d_{gphhowdm}(P,R)$

Proof. Straightforward. □

3.24. **Remark.** If we place $\gimel = 1$. Then $Eq.(15)$ are known as generalized picture hesitant Hausdorff ordered weighted hamming distance measure (GPHHOWHDM). Similarly, if we place $\gimel = 2$. Then $Eq.(15)$ are known as generalized picture hesitant Hausdorff ordered weighted Euclidean distance measure (GPHHOWEDM). If we take $J = 0$. Then $Eq.(15)$ are known as generalized intuitionistic hesitant Hausdorff ordered weighted distance measure (GIHHOWDM). The follows is holds obviously:

$$\max_i \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j+1)}) - I_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j+1)}) - J_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j+1)}) - K_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} \end{array} \right) \geq$$

$$\max_i \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)}) - I_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j)}) - J_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j)}) - K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array} \right)$$

3.25. **Definition.** Let $P, Q, R \in PHFS(X)$ where X is any set and $\gimel > 0$. The generalized hybrid picture hesitant ordered weighted distance measure (GHPHOWDM) is of the form:

$$d_{ghphowdm}(P,Q) = \left(\sum_{j=1}^{m} \frac{w_j}{2} \left(\begin{array}{c} \frac{1}{3Z_{x_{\circ(j)}}} \sum_{i=1}^{Z_{x_j}} \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)}) - I_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j)}) - J_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j)}) - K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array} \right) + \\ \frac{1}{3} \max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)}) - I_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((J_P^{o(i)}(x_{\circ(j)}) - J_Q^{o(i)}(x_{\circ(j)}))^{\gimel} + \\ ((K_P^{o(i)}(x_{\circ(j)}) - K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array} \right) \end{array} \right) \right)^{1/\gimel}$$

Obviously the above is Eq. (16), the following properties hold true for GHPHOWDM:
[1] $0 \leq d_{ghphowdm}(P,Q) \leq 1$
[2] $d_{ghphowdm}(P,Q) = 0 \Leftrightarrow P = Q$
[3] $d_{ghphowdm}(P,Q) = d_{ghphowdm}(Q,P)$
[4] $d_{ghphowdm}(P,Q) + d_{ghphowdm}(Q,R) \geq d_{ghphowdm}(P,R)$

Proof. Straightforward. □

3.26. **Remark.** If we place $\gimel = 1$. Then $Eq.(16)$ are known as generalized hybrid picture hesitant ordered weighted hamming distance measure (GHPHOWHDM). Similarly, if we place $\gimel = 2$. Then $Eq.(16)$ are known as generalized hybrid picture hesitant ordered weighted Euclidean distance measure (GHPHOWEDM). If we take $J = 0$. Then $Eq.(16)$ are known as generalized hybrid intuitionistic hesitant ordered weighted distance measure (GHIHOWDM). The follows is holds obviously:

$$\frac{1}{2}\left(\begin{array}{c} \frac{1}{3Z_{x_{\circ(j)}}}\sum_{i=1}^{Z_{x_j}}\left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j+1)})-I_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel}+ \\ ((J_P^{o(i)}(x_{\circ(j+1)})-J_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel}+ \\ ((K_P^{o(i)}(x_{\circ(j+1)})-K_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} \end{array}\right)+ \\ \frac{1}{3}\max_i \sum_{i=1}^{Z_{x_j}}\left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j+1)})-I_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel}+ \\ ((J_P^{o(i)}(x_{\circ(j+1)})-J_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel}+ \\ ((K_P^{o(i)}(x_{\circ(j+1)})-K_Q^{o(i)}(x_{\circ(j+1)}))^{\gimel} \end{array}\right) \end{array}\right)\geq$$

$$\frac{1}{2}\left(\begin{array}{c} \frac{1}{3Z_{x_{\circ(j)}}}\sum_{i=1}^{Z_{x_j}}\left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)})-I_Q^{o(i)}(x_{\circ(j)}))^{\gimel}+ \\ ((J_P^{o(i)}(x_{\circ(j)})-J_Q^{o(i)}(x_{\circ(j)}))^{\gimel}+ \\ ((K_P^{o(i)}(x_{\circ(j)})-K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array}\right)+ \\ \frac{1}{3}\max_i \left(\begin{array}{c} ((I_P^{o(i)}(x_{\circ(j)})-I_Q^{o(i)}(x_{\circ(j)}))^{\gimel}+ \\ ((J_P^{o(i)}(x_{\circ(j)})-J_Q^{o(i)}(x_{\circ(j)}))^{\gimel}+ \\ ((K_P^{o(i)}(x_{\circ(j)})-K_Q^{o(i)}(x_{\circ(j)}))^{\gimel} \end{array}\right) \end{array}\right)$$

4. APPLICATIONS

Distance and similarity measures have wide range of applications in pattern recognition and clustering that can be useful in many practical applications of engineering and other sciences. In this section, we are interested in applying the defined distance measures to a problem of pattern recognition and multi attributive decision making (MADM) i.e. we used the defined operators in building material recognition problems and MADM and discussed the results.

4.1. **Building Material Recognition.** In this type of problems, we need to identify the class of unknown building material using the distance measures for PHFSs. The detailed algorithm of this type of problem is given below.

4.1.1. *Algorithm.* **Step 1:** Get the information about the known building materials $P_i(i = 1, 2, 3, \dots, n)$ and the unknown building material P in form of PHFNs.

Step 2: Compute the distance measure of each P_i with P i.e. $d_{ghphowdm}$.

Step 3: Compute the similarity measure of each P_i with P by subtracting the distance measure form 1.

Step 4: Rank the degree of similarities of each P_i with P and P is classified.

The process is demonstrated with the help of an example below after algorithm.

Example 1: We assumed the example described in [44] where the class of an unknown building material is determined using similarity measures. We consider that three building materials which are represented by the PHFSs $P_i(i = 1, 2, 3)$ in the space attributes $X = \{x_1, x_2, x_3, x_4\}$. Let us consider the weight of $x_i(i = 1, 2, 3, 4)$ be $w = (0.3, 0.3, 0.2, 0.2)^T$. Now suppose an unknown building material P whose class is to be determined. For this purpose, the GHPHOWDM is used to identify the class of unknown building material P. The stepwise calculations are as follows:

Step 1: Information of the building material in the form of PHFNs is provided in $Table1$.

Data	P_1	P_2	P_3	P
x_1	$\left(\{0.2,0.1\},\{0.3,0.4\},\{0.4,0.4\}\right)$	$\left(\{0.4,0.5\},\{0.3,0.3\},\{0.2,0.2\}\right)$	$\left(\{0.7,0.6\},\{0.1,0.1\},\{0.1,0.1\}\right)$	$\left(\{0.3,0.2\},\{0.3,0.3\},\{0.1,0.4\}\right)$
x_2	$\left(\{0.3,0.2\},\{0.1,0.5\},\{0.1,0.1\}\right)$	$\left(\{0.5,0.1\},\{0.1,0.3\},\{0.2,0.1\}\right)$	$\left(\{0.3,0.3\},\{0.2,0.2\},\{0.1,0.2\}\right)$	$\left(\{0.5,0.1\},\{0.5,0.1\},\{0,0\}\right)$
x_3	$\left(\{0.5,0.1\},\{0.3,0.3\},\{0.2,0.1\}\right)$	$\left(\{0.1,0.1\},\{0.2,0.2\},\{0.3,0.3\}\right)$	$\left(\{0.1,0.3\},\{0.3,0.1\},\{0.1,0.1\}\right)$	$\left(\{0.3,0.3\},\{0.1,0.4\},\{0.2,0.1\}\right)$
x_4	$\left(\{0.3,0.2\},\{0.2,0.2\},\{0.5,0.1\}\right)$	$\left(\{0.3,0.3\},\{0.2,0.2\},\{0.1,0.1\}\right)$	$\left(\{0.1,0.1\},\{0.3,0.1\},\{0.1,0.3\}\right)$	$\left(\{0.3,0.5\},\{0.2,0.1\},\{0.3,0.3\}\right)$

$Table1$(Information of building materials in the environment of PHFNs)

Step 2: Step two involves the calculation of distance measures of unknown material with the given materials. The values in $Table2$ are obtained using GHPHOWDM between P_i $(i = 1, 2, 3)$ and P where value of $3bb$ is set as 2.

Data	P_1	P_2	P_3
$P_{ghphowdm}(P_i, P)$	0.16	0.17	0.163

$Table2$(Distance measures of unknown building material with given materials)

Step 3: Step three involves the calculation of similarity measure of the unknown material with known materials. The calculations are listed in $Table3$.

Data	P_1	P_2	P_3
$P_{ghphowdm}(P_i, P)$	0.84	0.83	0.837

$Table3$(Similarity index of unknown building material with given materials)

In the above numerical results, clearly indicated that the unknown building material P has a similarity index of 0.84 with building material P_1 which is the greatest among all other similarity measures. Therefore, the unknown building material P is included to the class of building material P_1.

4.2. **Multi Attributive Decision Making.** In this subsection, we demonstrated the idea of multi attributive decision making (MADM) using the distance measures of PHFS. In such phenomenon, the selection of best candidates is carried out using distance measures of PHFSs. The detailed algorithm of the method is described below.

4.2.1. *Algorithm.* **Step 1:** Obtain information about some alternatives $P_i(i = 1, 2, 3, \ldots, n)$ under the attributes $C_i(i = 1, 2, \ldots, n)$ is the form of PHFNs and decision matrix is formal.

Step 2: In step two, we normalize the decision matrix exist any criteria of cost type.

Step 3: For an ideal alternative P^* define PHFN for each criterion as $C_j^* = (\{1\}, \{0\}, \{0\})$.

Step 4: The distance of information provided in $Table4$ are evaluated with ideal value of $(\{1\},\{0\},\{0\})$.

Step 5: The similarity value of information in **step 4** is calculated.

Step 6: Rank the similarity measures to get the best alternative.

Example 2: This example is adopted from [44], where the selection of a best strategy is carried out by a multinational company. The company need to select a strategy for its upcoming financial strategy. The company has possibly four strategies $\{P_1, P_2, P_3, P_4\}$ needs to be evaluated under four attributes $\{C_1, C_2, C_3, C_4\}$ wich are:

P_1: Investments in rural areas.

P_2: Investments in urban areas.

P_3: Investments in national markets.

P_4: Investments in international markets.

C_1, C_2, C_3 and C_4 are defined as growth analysis, risk analysis, political impact and social impact respectively. The weight vector of strategies is $w = (0.3, 0.3, 0.2, 0.2)^T$.

The step wise calculation are as follows:

Step 1: Formation of decision matrix

Data	C_1	C_2	C_3	C_4
P_1	$(\{0.2,0.4\},\{0.1,0.3\},\{0.3,0.1\})$	$(\{0.2,0.2\},\{0.1,0.1\},\{0.6,0.7\})$	$(\{0.4,0.1\},\{0.1,0.4\},\{0.2,0.2\})$	$(\{0.1,0.1\},\{0.2,0.2\},\{0.7,0.7\})$
P_2	$(\{0.1,0.1\},\{0.3,0.4\},\{0.2,0.5\})$	$(\{0.3,0.1\},\{0.1,0.3\},\{0.2,0.2\})$	$(\{0.1,0.3\},\{0.4,0.4\},\{0.3,0.3\})$	$(\{0.5,0.5\},\{0.1,0.3\},\{0.2,0.2\})$
P_3	$(\{0.3,0.3\},\{0.2,0.2\},\{0.1,0.1\})$	$(\{0.3,0.3\},\{0.4,0.4\},\{0.1,0.3\})$	$(\{0.1,0.2\},\{0.2,0.1\},\{0.7,0.7\})$	$(\{0.1,0.3\},\{0.5,0.1\},\{0.2,0.1\})$
P_4	$(\{0.1,0.1\},\{0.3,0.1\},\{0.6,0.5\})$	$(\{0.6,0.5\},\{0.3,0.1\},\{0.1,0.1\})$	$(\{0.1,0.1\},\{0.3,0.5\},\{0.4,0.4\})$	$(\{0.4,0.5\},\{0.3,0.3\},\{0.1,0.2\})$

$Table4$(matrix table of the data analysis)

Step 2: Normalization of data provided in $Table4$ for maximum profits

Data	C_1	C_2	C_3	C_4
P_1	$(\{0.3,0.1\},\{0.1,0.3\},\{0.2,0.4\})$	$(\{0.2,0.2\},\{0.1,0.1\},\{0.6,0.7\})$	$(\{0.4,0.1\},\{0.1,0.4\},\{0.2,0.2\})$	$(\{0.1,0.1\},\{0.2,0.2\},\{0.7,0.7\})$
P_2	$(\{0.2,0.5\},\{0.3,0.4\},\{0.1,0.1\})$	$(\{0.3,0.1\},\{0.1,0.3\},\{0.2,0.2\})$	$(\{0.1,0.3\},\{0.4,0.4\},\{0.3,0.3\})$	$(\{0.5,0.5\},\{0.1,0.3\},\{0.2,0.2\})$
P_3	$(\{0.1,0.1\},\{0.2,0.2\},\{0.3,0.3\})$	$(\{0.3,0.3\},\{0.4,0.4\},\{0.1,0.3\})$	$(\{0.1,0.2\},\{0.2,0.1\},\{0.7,0.7\})$	$(\{0.1,0.3\},\{0.5,0.1\},\{0.2,0.1\})$
P_4	$(\{0.6,0.5\},\{0.3,0.1\},\{0.1,0.1\})$	$(\{0.6,0.5\},\{0.3,0.1\},\{0.1,0.1\})$	$(\{0.1,0.1\},\{0.3,0.5\},\{0.4,0.4\})$	$(\{0.4,0.5\},\{0.3,0.3\},\{0.1,0.2\})$

$Table5$(using the **step 1**, update the matrix table of the data analysis)

Step 3: The ideal value of criterion is $C_j^* = (\{1\}, \{0\}, \{0\})$

Step 4: The distance of the information in **step 2** is determined with P^* and $C_j^* = (\{1,1\}, \{0,0\}, \{0,0\})$ as follows

$$\begin{aligned} d_{gphowdm}(P_1, P^*) &= 0.42 \\ d_{gphowdm}(P_2, P^*) &= 0.34 \\ d_{gphowdm}(P_3, P^*) &= 0.312 \\ d_{gphowdm}(P_4, P^*) &= 0.215 \end{aligned}$$

Step 5: The similarity measures of the data provided in **step 4** as determined as follows

$$\begin{aligned} S_{gphowdm}(P_1, P^*) &= 1 - 0.42 = 0.58 \\ S_{gphowdm}(P_2, P^*) &= 1 - 0.34 = 0.66 \\ S_{gphowdm}(P_3, P^*) &= 1 - 0.312 = 0.688 \\ S_{gphowdm}(P_4, P^*) &= 1 - 0.215 = 0.785 \end{aligned}$$

Steps 6: The similarity measures are ranked to get the best strategy and the ranking is as follows.

$$P_1 < P_2 < P_3 < P_4$$

The ranking shows that the similarity of P_4 and P^* has a greater value so it is the best policy. **Comparative Study and Advantages:** In this work, we studied some distance measures for PHFSs. These distance measures are the generalizations of distance measures proposed for HFSs in [29]. As PHFSs is a generalization of both PFSs and HFSs and could deal effectively in real life phenomena's. In Remarks (1-7) it is demonstrated under which conditions, the proposed distance measures become hamming and Euclidean distance measures. It is also discussed in Remarks (1-7) that the proposed work shifted to the environment of IHFSs if we assume $j = 0$ showing the worth of our work. By assuming $J = K = 0$ all the proposed work shifted to environment of HFSs proposed in [29]. However, keeping in mind the effectiveness of PHFSs the proposed work csould be more efficient in problems of engineering and other sciences. Now to show the superiority of our proposed work, we consider some information in the existing environments and the way how proposed work deal with them. First, consider the information in Table 6 in the form of IHFSs. The proposed DMs can handle this type of information by taking $J = 0$.

Data	P_1	P_2	P_3	P
x_1	$\begin{pmatrix} \{0.2, 0.1\}, \\ \{0.4, 0.4\} \end{pmatrix}$	$\begin{pmatrix} \{0.4, 0.5\}, \\ \{0.2, 0.2\} \end{pmatrix}$	$\begin{pmatrix} \{0.7, 0.6\}, \\ \{0.1, 0.1\} \end{pmatrix}$	$\begin{pmatrix} \{0.3, 0.2\}, \\ \{0.1, 0.4\} \end{pmatrix}$
x_2	$\begin{pmatrix} \{0.3, 0.2\}, \\ \{0.1, 0.1\} \end{pmatrix}$	$\begin{pmatrix} \{0.5, 0.1\}, \\ \{0.2, 0.1\} \end{pmatrix}$	$\begin{pmatrix} \{0.3, 0.3\}, \\ \{0.1, 0.2\} \end{pmatrix}$	$\begin{pmatrix} \{0.5, 0.1\}, \\ \{0, 0\} \end{pmatrix}$
x_3	$\begin{pmatrix} \{0.5, 0.1\}, \\ \{0.2, 0.1\} \end{pmatrix}$	$\begin{pmatrix} \{0.1, 0.1\}, \\ \{0.3, 0.3\} \end{pmatrix}$	$\begin{pmatrix} \{0.1, 0.3\}, \\ \{0.1, 0.1\} \end{pmatrix}$	$\begin{pmatrix} \{0.3, 0.3\}, \\ \{0.2, 0.1\} \end{pmatrix}$
x_4	$\begin{pmatrix} \{0.3, 0.2\}, \\ \{0.5, 0.1\} \end{pmatrix}$	$\begin{pmatrix} \{0.3, 0.3\}, \\ \{0.1, 0.1\} \end{pmatrix}$	$\begin{pmatrix} \{0.1, 0.1\}, \\ \{0.1, 0.3\} \end{pmatrix}$	$\begin{pmatrix} \{0.3, 0.5\}, \\ \{0.3, 0.3\} \end{pmatrix}$

Table6(Intuitionistic Hesitant Fuzzy Set) Now, if the information provided are in the form of HFSs as in $Table7$. Then the proposed DMs can handle this type of data by taking $J = K = 0$.

Data	P_1	P_2	P_3	P
x_1	$\{0.2, 0.1\}$	$\{0.4, 0.5\}$	$\{0.7, 0.6\}$	$\{0.3, 0.2\}$
x_2	$\{0.3, 0.2\}$	$\{0.5, 0.1\}$	$\{0.3, 0.3\}$	$\{0.5, 0.1\}$
x_3	$\{0.5, 0.1\}$	$\{0.1, 0.1\}$	$\{0.1, 0.3\}$	$\{0.3, 0.3\}$
x_4	$\{0.3, 0.2\}$	$\{0.3, 0.3\}$	$\{0.1, 0.1\}$	$\{0.3, 0.5\}$

$Table7$(Hesitant Fuzzy Set)

If the information provided are in the form of PFSs as shown in **Table 8**. Then the proposed DMs can handle this type of data where every membership grade can be considered as HFN.

Data	P_1	P_2	P_3	P
x_1	$\{0.2, 0.1, 0.3\}$	$\{0.4, 0.5, 0.1\}$	$\{0.7, 0.6, 0.1\}$	$\{0.3, 0.2, 0.4\}$
x_2	$\{0.3, 0.2, 0.1\}$	$\{0.5, 0.1, 0.3\}$	$\{0.3, 0.3, 0.4\}$	$\{0.5, 0.1, 0.4\}$
x_3	$\{0.5, 0.1, 0.4\}$	$\{0.1, 0.1, 0.7\}$	$\{0.1, 0.3, 0.5\}$	$\{0.3, 0.3, 0.1\}$
x_4	$\{0.3, 0.2, 0.3\}$	$\{0.3, 0.3, 0.3\}$	$\{0.1, 0.1, 0.8\}$	$\{0.3, 0.5, 0.1\}$

$Table8$(Picture Fuzzy sets)

5. Conclusion

This article is based on several distance measures for PHFSs. First, we described some basic notions along with the concept of PHFS. Then some DMs are developed for PHFSs including generalized picture hesitant distance measures (GPHDMs), generalized picture hesitant normalizer distance measures (GPHNDMs) and their extended forms. We also discussed the Euclidean, hamming and Hausdorff DMs in the environment of PHFSs. With the help of several remarks, it is discussed how the proposed work the work done proposed in the environments of IFSs and IHFSs etc. Further, the proposed work is applied to a pattern recognition and a MADM problem and the results are discussed. A comparative study is also established with existing concepts and the advantages of the proposed work are studied..

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