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Euler Polynomials Approach to the System of Nonlinear Fractional Differential Equations

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Abstract. This paper is concerned with the application of Euler polynomials in solving a system of nonlinear fractional differential equations (SNFDE). For this purpose, an operational matrix of fractional integration is designed for Euler polynomials. Together with collocation method, this matrix simplifies the main problem to a set of algebraic equations. Error analysis is also investigated. Numerical examples illustrate the impression of the method.

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Key Words: Fractional integration, System of nonlinear fractional differential equations, Euler operational matrix, Collocation method.

1. Introduction

Due to the connection of fractional calculus with the systems involving memory and after-effects, a lot of natural phenomena are formulated by SNFDE. Some topics related to this fact are fractional model for Hepatitis C virus infection [2], fractional chaotic system [3], fractional SIR epidemic model [6], fractional HIV infection model [7], fractional predator-prey model [12], fractional financial system [22], unsteady rotational flow of a second grade fluid [29], and also rotational motions of fractional Oldroyd-B fluids between circular cylinders [32].

One should mention that the available methods in the literature which focus on the solutions of SNFDE are listed as: Homotopy perturbation method [1, 21], Multi-stage Bernstein polynomials method [4], fractional generalized Laguerre functions method [9], Legendre wavelets method [10], differential transform method [15], Laplace transform method [16], variational iteration method [17], Adomian decomposition method [19], Chebyshev approach and fractional finite difference [20], Haar wavelets approach [27], and fractional natural decomposition method [28].

In this study, we consider the general form of a SNFDE as follows

$$\begin{cases} D_*^{\nu_1} y_1(t) = f_1\left(t, y_1(t), \ldots, y_d(t)\right), \\ D_*^{\nu_2} y_2(t) = f_2\left(t, y_1(t), \ldots, y_d(t)\right), \\ \quad \vdots \quad \vdots \qquad\qquad \vdots \\ D_*^{\nu_d} y_d(t) = f_d\left(t, y_1(t), \ldots, y_d(t)\right), \end{cases} \tag{1. 1}$$

with the initial conditions

$$y_l^{(j)}(0) = \lambda_{l\,j}, \tag{1. 2}$$

in which $D_*^{\nu_l}(\cdot)$ are the Caputo's type derivative of order $\nu_l > 0$; $\lambda_{l\,j}$ are given real constants; f_l denote the known real functions; $y_l(t)$ are the unknown functions; $l = 1, 2, \ldots, d$; $j = 0, 1, \ldots, p_l - 1$; $p_l - 1 < \nu_l \leq p_l$; $p_l, d \in \mathbb{N}$.

Euler polynomials are a family of non-orthogonal polynomials with various utilizations in number and combinational theories [25, 30, 34]. They also appear in the statistical physics as well as in semi-classical approximations to the quantum probability distributions [8]. Lately, Euler polynomials have been applied successfully for the numerical solutions of generalized pantograph equations [18], systems of linear Volterra integral equations with the variable coefficients [23], and also systems of linear Fredholm integro-differential equations [24]. This paper proceeds two main aims:

(i) Constructing the operational matrix of fractional integration for Euler polynomials.
(ii) Providing an instrumental approach by this operational matrix and collocation technique to obtain the solution of (1. 1) under the initial conditions (1. 2).

Although Euler polynomials do not constitute orthogonal basis, but they possess operational matrices of derivation and integration. Here, we construct the operational matrix of fractional integration of these polynomials explicitly. To the best of our knowledge, this operational matrix is new. The proposed method considers $D_*^{\nu_l} y_l(t)$ for $l = 1, 2, \ldots, d$ as the elements of Euler polynomials with unknown coefficients. By using the constructed operational matrix of fractional integration, it converts the problem to a nonlinear system of algebraic equations. After solving the new system, the solution of (1. 1) is identified.

The remainder of this paper is organized as follows: In Section 2, basic definitions and concepts used further in this work are given. In Section 3, Euler expansion of a real function is described and operational matrix of fractional integration of Euler polynomials is formed. Section 4 is devoted to the implementation of numerical method. Also, the error analysis of suggested method is investigated. The validity of method is demonstrated through some examples in Section 5. At the end, a conclusion is drawn in Section 6.

2. Preliminaries

For the convenience of the reader, we repeat some relevant materials of the fractional calculus [13, 26] and Euler polynomials [5, 11, 14, 31].

Definition 2.1. *The Riemann-Liouville's fractional-order integration for the function g on $L^1[a,b]$ is defined as follows*

$$I^{\nu}g(t)=\begin{cases}\frac{1}{\Gamma(\nu)}\int_0^t(t-\tau)^{\nu-1}g(\tau)d\tau, & \nu>0,\\ g(t), & \nu=0.\end{cases} \tag{2. 3}$$

For $\nu>0$, (2. 3) can also be written as

$$I^{\nu}g(t)=\frac{1}{\Gamma(\nu)}t^{\nu-1}\star g(t),$$

where $t^{\nu-1}\star g(t)$ is the convolution product of $t^{\nu-1}$ and $g(t)$.

Remark 2.2. *For the Riemann-Liouville fractional integral, we get*

$$I^{\nu}t^{k}=\frac{\Gamma(k+1)}{\Gamma(\nu+k+1)}t^{\nu+k},\ \ k>-1. \tag{2. 4}$$

Definition 2.3. *The Caputo's type derivative of order $\nu>0$ is defined in the following*

$$D_*^{\nu}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\nu-1}g^{(n)}(\tau)d\tau,\ \ n-1\le\nu<n,$$

where $t>0$ and n is an integer.

Remark 2.4. *Caputo's integral operator for $g\in L^1[a,b]$ has the useful property below*

$$I^{\nu}D_*^{\nu}g(t)=g(t)-\sum_{i=0}^{n-1}g^{(i)}\left(0^{+}\right)\frac{t^{i}}{i!},\ \ n-1\le\nu<n. \tag{2. 5}$$

Definition 2.5. *The Euler polynomials $E_m(t)$ are defined for $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ by the following generating function*

$$\frac{2e^{xt}}{e^{x}+1}=\sum_{m=0}^{\infty}E_m(t)\frac{x^m}{m!}.$$

Remark 2.6. *The Euler polynomials satisfy in the identity below*

$$E_m(t)=\frac{1}{m+1}\sum_{k=1}^{m+1}(2-2^{k+1})\binom{m+1}{k}B_k(0)t^{m+1-k}, \tag{2. 6}$$

in which $B_k(t)$ are the Bernoulli polynomials of degree k specified by

$$\sum_{k=0}^{m}\binom{m+1}{k}B_k(t)=(m+1)t^m.$$

In addition, for $m,m'\in\mathbb{N}_0$, we have

$$\int_0^1E_m(t)E_{m'}(t)dt=\frac{4(-1)^m(2^{m+m'+2}-1)m!m'!}{(m+m'+2)!}B_{m+m'+2}(0). \tag{2. 7}$$

3. APPROXIMATION AND OPERATIONAL MATRIX

Function approximation. Assume that $\mathbf{E}(t) = [E_0(t), E_1(t), \ldots, E_M(t)]^T$, in which for $0 \leq j \leq M$, $E_j(t)$ allude to the Euler polynomials of degree j, and

$$\boldsymbol{\mathcal{E}} = span\{E_0(t), E_1(t), \ldots, E_M(t)\} \subset L^2[0,1]. \tag{3. 8}$$

Consider that $g(t)$ be an arbitrary element in $L^2[0,1]$. Since $\boldsymbol{\mathcal{E}}$ is a finite dimensional vector space, $g(t)$ has the best approximation such as $g_b(t) \in \boldsymbol{\mathcal{E}}$, namely

$$\forall z(t) \in \boldsymbol{\mathcal{E}} : \|g(t) - g_b(t)\|_2 \leq \|g(t) - z(t)\|_2.$$

Because of $g_b(t) \in \boldsymbol{\mathcal{E}}$, there exist the unique coefficients a_k such that

$$g(t) \approx g_b(t) = \sum_{k=0}^{M} a_k E_k(t) = \mathbf{A}^T \mathbf{E}(t), \tag{3. 9}$$

where $\mathbf{A} = [a_0, a_1, \ldots, a_M]^T$.

In order to determine the coefficients a_k, assume that

$$g_j = \int_0^1 g(t) E_j(t) dt; \;\; j = 0, 1, \ldots, M. \tag{3. 10}$$

From (3. 9) and (3. 10),

$$g_j = \sum_{k=0}^{M} a_k \int_0^1 E_k(t) E_j(t) dt = \sum_{k=0}^{M} a_k \theta_{k,j}, \tag{3. 11}$$

where $\theta_{k,j} = \int_0^1 E_k(t)E_j(t)dt$ can be uncovered by (2. 7). In this way, matrix representation of (3. 11) is

$$\mathbf{G} = \boldsymbol{\Theta}^T \mathbf{A},$$

$$\mathbf{G} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_M \end{bmatrix}, \; \boldsymbol{\Theta} = \begin{bmatrix} \theta_{0,0} & \theta_{0,1} & \ldots & \theta_{0,M} \\ \theta_{1,0} & \theta_{1,1} & \ldots & \theta_{1,M} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{M,0} & \theta_{M,1} & \ldots & \theta_{M,M} \end{bmatrix}, \; \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix}.$$

Subsequently,

$$\mathbf{A} = (\boldsymbol{\Theta}^T)^{-1} \mathbf{G}.$$

Operational matrix of fractional integration. The fractional integration of $\mathbf{E}(t)$ may be approximated as

$$I^{\nu} \mathbf{E}(t) \approx \boldsymbol{\mathcal{Q}}^{(\nu)} \mathbf{E}(t), \tag{3. 12}$$

where $\boldsymbol{\mathcal{Q}}^{(\nu)}$ is the operational matrix of Riemann-Liouville's fractional integration. The size of this matrix is $(M+1) \times (M+1)$.

By applying (2. 4) and (2. 6), one can write for every $m = 0, 1, \ldots, M$,

$$\begin{aligned} I^\nu E_m(t) &= \frac{1}{m+1}\sum_{k=1}^{m+1}(2-2^{k+1})\binom{m+1}{k}B_k(0)I^\nu t^{m+1-k} \\ &= \frac{1}{m+1}\sum_{k=1}^{m+1}(2-2^{k+1})\binom{m+1}{k}B_k(0)\frac{\Gamma(m-k+2)t^{m-k+\nu+1}}{\Gamma(m-k+\nu+2)}. \end{aligned}$$

Summarily,

$$I^\nu E_m(t) = \sum_{k=1}^{m+1}\omega_{m,k}^{(\nu)} t^{m-k+\nu+1}, \tag{3. 13}$$

where

$$\omega_{m,k}^{(\nu)} = \frac{m!(2-2^{k+1})B_k(0)}{k!\Gamma(m-k+\nu+2)}.$$

Besides this, let $t^{m-k+\nu+1}$ be expanded into $(M+1)$ terms of Euler polynomials as

$$t^{m-k+\nu+1} \approx \sum_{j=0}^{M} r_{k,j}E_j(t). \tag{3. 14}$$

The procedure of obtaining $r_{k,j}$ has been explained in the previous subsection. Placing (3. 14) into (3. 13), one gains

$$\begin{aligned} I^\nu E_m(t) &\approx \sum_{k=1}^{m+1}\omega_{m,k}^{(\nu)}\sum_{j=0}^{M} r_{k,j}E_j(t) \\ &= \sum_{j=0}^{M}\left(\sum_{k=1}^{m+1}\gamma_{m,j,k}^{(\nu)}\right)E_j(t), \end{aligned} \tag{3. 15}$$

so that $\gamma_{m,j,k}^{(\nu)} = \omega_{m,k}^{(\nu)} r_{k,j}$. Obviously, for $m = 0, 1, \ldots, M$, (3. 15) can be rewritten in the form of

$$I^\nu E_m(t) \approx \left[\sum_{k=1}^{m+1}\gamma_{m,0,k}^{(\nu)}, \sum_{k=1}^{m+1}\gamma_{m,1,k}^{(\nu)}, \ldots, \sum_{k=1}^{m+1}\gamma_{m,M,k}^{(\nu)}\right]\mathbf{E}(t).$$

Consequently,

$$\boldsymbol{\mathcal{Q}}^{(\nu)} = \begin{bmatrix} \gamma_{0,0,1}^{(\nu)} & \gamma_{0,1,1}^{(\nu)} & \cdots & \gamma_{0,M,1}^{(\nu)} \\ \sum_{k=1}^{2}\gamma_{1,0,k}^{(\nu)} & \sum_{k=1}^{2}\gamma_{1,1,k}^{(\nu)} & \cdots & \sum_{k=1}^{2}\gamma_{1,M,k}^{(\nu)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{M+1}\gamma_{M,0,k}^{(\nu)} & \sum_{k=1}^{M+1}\gamma_{M,1,k}^{(\nu)} & \cdots & \sum_{k=1}^{M+1}\gamma_{M,M,k}^{(\nu)} \end{bmatrix}.$$

4. Method of solution and Error analysis

Method of solution. For the implementation of method, we first expanded the fractional derivative terms of (1. 1) as a linear combination of $\mathbf{E}(t)$ entries. That is,

$$D_*^{\nu_l} y_l(t) \approx \mathbf{C}_l^T \mathbf{E}(t), \tag{4. 16}$$

in which $\mathbf{C}_l^T = [c_{l0}, c_{l1}, \ldots, c_{lM}]$ for every $l = 1, 2, \ldots, d$.

Integrating both sides of (4. 16) and considering (2. 5) and (3. 13),

$$y_l(t) \approx \mathbf{C}_l^T \boldsymbol{\mathcal{Q}}^{(\nu_l)} \mathbf{E}(t) + \sum_{j=0}^{p_l - 1} \lambda_{l_j} \frac{t^j}{j!}. \tag{4. 17}$$

Substituting (4. 16) and (4. 17) into (1. 1) and collocating the resulting system at the points $t \in \{\frac{i}{M} : i = 0, 1, \ldots, M\}$, a nonlinear system of $d \times (M+1)$ algebraic equations with $d \times (M+1)$ unknowns is obtained. We solve this algebraic system by *fsolve* function of MATLAB software with the initial guess $\mathbf{C}_l^T = [0, 0, \ldots, 0]_{1\times(M+1)}$ for $l = 1, \ldots, d$. After $\mathbf{C}_l^T$ are designated, the solution of (1. 1) with the initial conditions (1. 2) can be assigned by (4. 17).

Error analysis. Before saying main result, we need two lemmas.

Lemma 4.1. *Let the function $g : [t_0, 1] \to \mathbb{R}$ be $M + 1$ times continuously differentiable for $0 \le t_0 < 1$, $g \in C^{M+1}[t_0, 1]$, and $\boldsymbol{\mathcal{E}}$ be in the form of* (3. 8). *If $g_b(t)$ introduced in* (3. 9) *be the best approximation to g, then the error bound is declared as follows:*

$$\|g(t) - g_b(t)\|_2 \le \frac{\sqrt{2} N T^{\frac{2M+3}{2}}}{(M+1)! \sqrt{2M+3}}, \tag{4. 18}$$

in which $N = \max_{t\in[t_0,1]} \left|g^{(M+1)}(t)\right|$ and $T = \max\{1 - t_0, t_0\}$.

Proof. Suppose that $\tilde{g}(t)$ is an arbitrary approximation of $g(t)$. We select this approximation in the form of Taylor series of $g(t)$. Clearly,

$$\tilde{g}(t) = g(t_0) + g'(t_0)(t - t_0) + g''(t_0)\frac{(t-t_0)^2}{2!} + \cdots + g^{(M)}(t_0)\frac{(t-t_0)^M}{M!}.$$

Therefore, there exists an $\eta \in (t_0, 1)$ such that

$$|g(t) - \tilde{g}(t)| = \left| g^{(M+1)}(\eta) \frac{(t-t_0)^{(M+1)}}{(M+1)!} \right|.$$

Since $g_b(t)$ is the best approximation of g,

$$\begin{aligned}\|g(t)-g_b(t)\|_2^2 \le \|g(t)-\tilde{g}(t)\|_2^2 &= \int_0^1 |g(t)-\tilde{g}(t)|^2 dt \\ &= \int_0^1 \left| g^{(M+1)}(\eta)\frac{(t-t_0)^{(M+1)}}{(M+1)!}\right|^2 dt \\ &\le \frac{N^2}{((M+1)!)^2}\int_0^1 (t-t_0)^{(2M+2)}dt \\ &\le \frac{2N^2T^{2M+3}}{((M+1)!)^2(2M+3)}.\end{aligned}$$

□

A result of (4. 18) is that if $M \to \infty$ then $g_b(t) \to g(t)$ in $L^2[0,1]$.

Lemma 4.2. [33] *(Young's convolution inequality) Assume that g is in $L^p(\mathbb{R}^n)$ and h is in $L^q(\mathbb{R}^n)$ and*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1,$$

with $1 \le p,q,r < \infty$. Then,

$$\|g \star h\|_r \le \|g\|_p \|h\|_q, \tag{4. 19}$$

in which $\star$ denotes convolution of two functions, L^p is Lebesgue space, and $\|\cdot\|_p$ refers to the usual L^p norm.

It is notable that an important outcome of (4. 19) is

$$\|g \star h\|_2 \le \|g\|_1 \|h\|_2. \tag{4. 20}$$

Recall (1. 1) with conditions (1. 2). For every $l = 1,\dots,d$, suppose that $\hat{y}_l(t)$ be approximate solution of $y_l(t)$ with a given M. Moreover, let us define the residual error functions as

$$res(y_l(t)) = D_*^{\nu_l}\hat{y}_l(t) - f_l\left(t, \hat{y}_1(t), \dots, \hat{y}_d(t)\right). \tag{4. 21}$$

We state the main result of error analysis in the theorem below.

Theorem 4.3. *For $l = 1,\dots,d$, consider $y_l(t)$ and $\hat{y}_l(t)$ are the exact and approximate solutions of (1. 1) with the conditions (1. 2), respectively. Also, let $D_*^{\nu_l}y_l(t) : [t_0,1] \to \mathbb{R}$ be $M+1$ times continuously differentiable for $0 \le t_0 < 1$, $D_*^{\nu_l}y_l(t) \in C^{M+1}[t_0,1]$. Furthermore, let the functions f_l satisfy in the Lipschitz condition with the Lipschitz constants μ_l. Then,*

$$\|res(y_l(t))\|_2 \le \frac{\sqrt{2}N_l T^{\frac{2M+3}{2}}}{(M+1)!\sqrt{2M+3}} + \left(\sum_{l=1}^{d}\frac{\sqrt{2}\mu_l N_l}{\Gamma(\nu_l+1)}\right)\frac{T^{\frac{2M+3}{2}}}{(M+1)!\sqrt{2M+3}},$$

where $T = \max\{1-t_0, t_0\}$ and $N_l = \max_{t\in[t_0,1]}\left|D_*^{\nu_l+M+1}y_l(t)\right|$ for each $l = 1,\dots,d$.

Proof. Using (4. 21) and (1. 1),

$$\begin{aligned}
&\|res(y_l(t))\|_2\\
&= \|D_*^{\nu_l}\hat{y}_l(t) - f_l\left(t,\hat{y}_1(t),\dots,\hat{y}_d(t)\right) - D_*^{\nu_l}y_l(t) + f_l\left(t,y_1(t),\dots,y_d(t)\right)\|_2\\
&\le \|D_*^{\nu_l}y_l(t) - D_*^{\nu_l}\hat{y}_l(t)\|_2\\
&+ \|f_l\left(t,y_1(t),\dots,y_d(t)\right) - f_l\left(t,\hat{y}_1(t),\dots,\hat{y}_d(t)\right)\|_2.
\end{aligned}$$

Since f_l satisfy in the Lipschitz conditions with Lipschitz constants μ_l,

$$\|res(y_l(t))\|_2 \le \|D_*^{\nu_l}y_l(t) - D_*^{\nu_l}\hat{y}_l(t)\|_2 + \sum_{l=1}^{d}\mu_l\|y_l(t)-\hat{y}_l(t)\|_2. \tag{4. 22}$$

Now, employing (4. 16) and (4. 18) entails

$$\|D_*^{\nu_l}y_l(t) - D_*^{\nu_l}\hat{y}_l(t)\|_2 \le \frac{\sqrt{2}N_lT^{\frac{2M+3}{2}}}{(M+1)!\sqrt{2M+3}}. \tag{4. 23}$$

On the other hand,

$$\begin{aligned}
\|y_l(t)-\hat{y}_l(t)\|_2 &= \|I^{\nu_l}\left(D_*^{\nu_l}y_l(t) - D_*^{\nu_l}\hat{y}_l(t)\right)\|_2\\
&= \left\|\frac{1}{\Gamma(\nu_l)}t^{(\nu_l-1)}\star\left(D_*^{\nu_l}y_l(t) - D_*^{\nu_l}\hat{y}_l(t)\right)\right\|_2.
\end{aligned} \tag{4. 24}$$

Now, by applying (4. 20) for (4. 24), we conclude

$$\|y_l(t)-\hat{y}_l(t)\|_2 \le \frac{1}{\Gamma(\nu_l)}\left\|t^{(\nu_l-1)}\right\|_1\left\|D_*^{\nu_l}y_l(t) - D_*^{\nu_l}\hat{y}_l(t)\right\|_2. \tag{4. 25}$$

Also,

$$\left\|t^{(\nu_l-1)}\right\|_1 = \int_0^1\left|t^{(\nu_l-1)}\right|dt = \int_0^1 t^{(\nu_l-1)}dt = \frac{t^{\nu_l}}{\nu_l} \le \frac{1}{\nu_l}. \tag{4. 26}$$

Based on (4. 25) and (4. 26),

$$\|y_l(t)-\hat{y}_l(t)\|_2 \le \frac{1}{\Gamma(\nu_l+1)}\left\|D_*^{\nu_l}y_l(t) - D_*^{\nu_l}\hat{y}_l(t)\right\|_2. \tag{4. 27}$$

From (4. 23) and (4. 27), it is obvious that

$$\|y_l(t)-\hat{y}_l(t)\|_2 \le \frac{\sqrt{2}N_lT^{\frac{2M+3}{2}}}{\Gamma(\nu_l+1)(M+1)!\sqrt{2M+3}}. \tag{4. 28}$$

Ultimately, utilizing (4. 22), (4. 23) and (4. 28) completes the proof. □

According to Theorem 4.3, one can imply that $res(y_l(t)) \to 0$ in $L^2[0,1]$ as $M \to \infty$. In the other word, the accuracy of approximation is improvable by increasing sufficiently control parameter M.

5. Numerical examples

In this section, we evaluate three examples to indicate the efficiency of the proposed method. The computations are performed by computer programs written in MATLAB R2015a software on a 64-bit PC with 2.20 GHz processor and 8 GB memory. We report the results of applying the present method through several tables and figures.

Example 5.1. [15] *Consider the following SNFDE*

$$\begin{cases} D_*^{1.3} y_1(t) = y_1(t) + (y_2(t))^2, \\ D_*^{2.4} y_2(t) = y_1(t) + 5y_2(t), \end{cases} \tag{5. 29}$$

under the initial conditions

$$y_1(0) = 0,\ y_1'(0) = 1,\ y_2(0) = 0,\ y_2'(0) = 1,\ y_2''(0) = 1.$$

The exact solution of (5. 29) is unknown.

We implement the present method for $M = 4$. Figure 1 compares the graph of the present method with 4-terms of differential transform method (DTM) [15]. The calculations of two methods at some points are given in Table 1. Since the exact solution is not available, residual error function defined in (4. 21) is a good criterion to test the correctness of our method. Figure 2 demonstrates absolute residual errors of (5. 29) for $M = 4$, $M = 8$, and $M = 12$. The important point to mention here is that in order to get the best approximate solution of the equation, the truncation limit M must be chosen large enough. According to Figure 2, residual error for $M = 8$ is better than that for $M = 12$. This can be because of the effect of rounding errors or may be caused by an error in the experimental measure of the data. Another reason is that the example is more simple, and for smaller value of M, enables one to achieve an acceptable accuracy. This cannot completely represent the ability of presented method.

This is nice to compare the solutions of Euler polynomials with those of Legendre polynomials and Chebyshev polynomials. Table 2 represents

$$L^{\infty}(y_1(t)) = \max_{0\le t\le 1}\{|res(y_1(t))|\},$$
$$L^{\infty}(y_2(t)) = \max_{0\le t\le 1}\{|res(y_2(t))|\},$$

for aforesaid values of M.

Table 1: Numerical results of Example 5.1

t	$y_1(t)$		$y_2(t)$	
	DTM [15]	Present method	DTM [15]	Present method
0.1	0.102009	0.101160	0.105240	0.105234
0.3	0.329645	0.326609	0.355483	0.355413
0.5	0.614437	0.615760	0.686917	0.687835
0.7	0.996385	1.026678	1.146959	1.155808
0.9	1.529196	1.695088	1.797642	1.839826

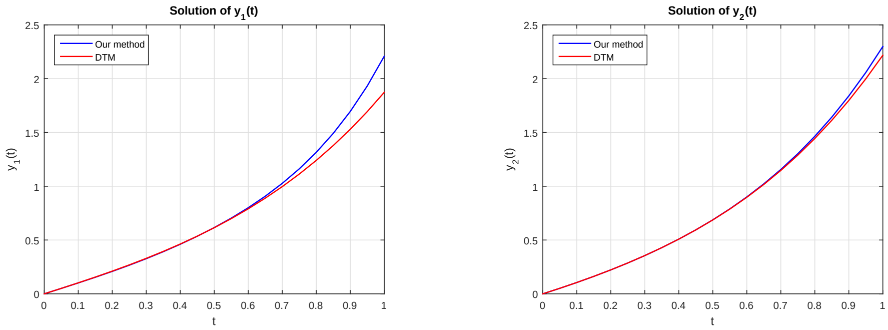

Figure 1: Our method ($M = 4$) and 4-terms of DTM [15] for Example 5.1

Table 2: A comparison with orthogonal polynomials for Example 5.1

Polynomial	$L^\infty(y_1(t))$			$L^\infty(y_2(t))$		
	$M = 4$	$M = 8$	$M = 12$	$M = 4$	$M = 8$	$M = 12$
Legendre	0.048837	5.57×10^{-5}	9.02×10^{-6}	0.013052	8.87×10^{-6}	9.93×10^{-6}
Chebyshev	0.048825	5.78×10^{-5}	2.03×10^{-4}	0.013055	6.44×10^{-6}	2.17×10^{-4}
Euler	0.048828	6.21×10^{-5}	5.17×10^{-4}	0.013049	1.11×10^{-5}	1.52×10^{-4}

Example 5.2. [28] *Consider the SNFDE in the form of*

$$\begin{cases} D_*^\alpha y_1(t) = y_1(t), \\ D_*^\beta y_2(t) = 2\left(y_1(t)\right)^2, \\ D_*^\gamma y_3(t) = 3y_1(t)y_2(t), \end{cases} \tag{5. 30}$$

subject to the initial conditions

$$y_1(0) = 1, \;\; y_2(0) = 1, \;\; y_3(0) = 0,$$

so that $0 < \alpha, \beta, \gamma \leq 1$. *The exact solutions of (5. 30) when* $\alpha = \beta = \gamma = 1$ *are* $y_1(t) = e^t$, $y_2(t) = e^{2t}$ *and* $y_3(t) = e^{3t} - 1$.

Assume that $\alpha = \beta = \gamma$, and $M = 6$. Table 3 compares the results of computations (for $M = 6$) with those of fractional natural decomposition method (for $n = 6$) [28] and exact solution in the case of $\alpha = \beta = \gamma = 1$. Significantly, the present method is in a better agreement with the exact solution than method of [28].

Figure 3 reveals that solution of fractional-order system ($0 < \alpha, \beta, \gamma < 1$) closes to the solution of integer-order system ($\alpha = \beta = \gamma = 1$) whenever $\alpha = \beta = \gamma \to 1$. It also exposes that our method is well-adapted with exact solution of integer order system. The graph of absolute error in integer-order is seen in Figure 4 for $M = 4$, $M = 8$ and $M = 12$. In the case of integer order derivative, a comparison between absoluter errors of Euler polynomials and those of orthogonal polynomials (Legendre and Chebyshev) is

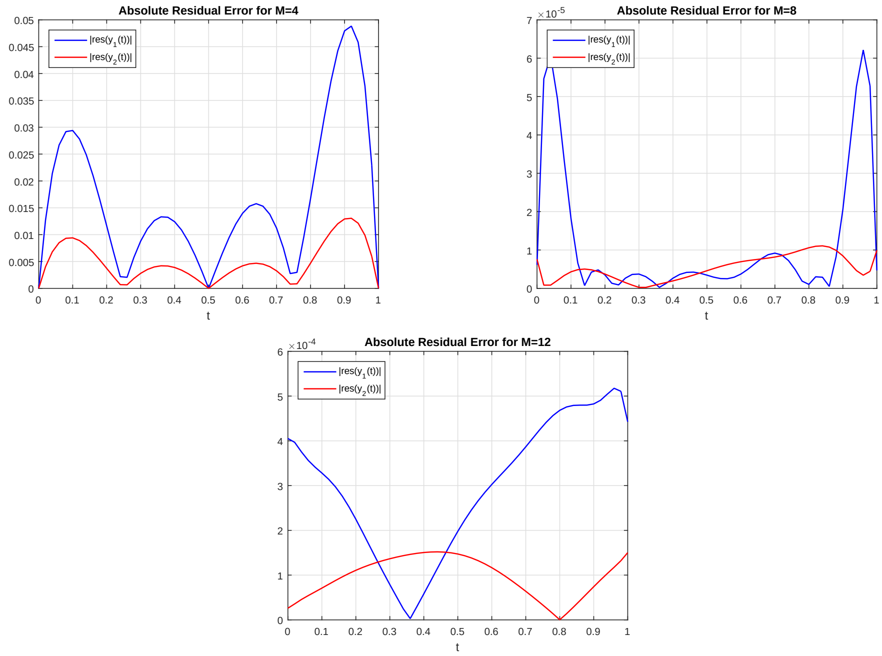

Figure 2: Absolute residual errors of (5. 29) for various M

presented through Figure 5 for $M = 4$, $M = 8$ and $M = 12$.

We do not access to exact solution when the order of derivative is less than 1. Hence, the computation of residual error will be helpful. Figure 6 illustrates the absolute residual errors for $M = 6$ and $M = 9$ when order of derivative is 0.75. Also, Figure 7 portrays the absolute residual errors for $M = 6$ and $M = 9$ when order is 0.95. From Theorem 4.3, value of M and order of derivative can both affect residual error function. Figures 6 and 7 also affirm this fact.

In Example 5.1 and Example 5.2, exact solution was not available for fractional derivatives. So, we decided to utilize residual error functions for those cases. In the next example, we assess suggested method by a fractional system whose exact solution is known.

Example 5.3. *Let us consider SNFDE as follows*

$$\begin{cases} D_*^{0.4} y_1(t) + y_1(t) y_2(t) = f_1(t), \\ D_*^{0.6} y_2(t) + y_1(t) y_2(t) = f_2(t), \end{cases} \tag{5. 31}$$

with initial conditions

$$y_1(0) = 0, \;\; y_2(0) = 0,$$

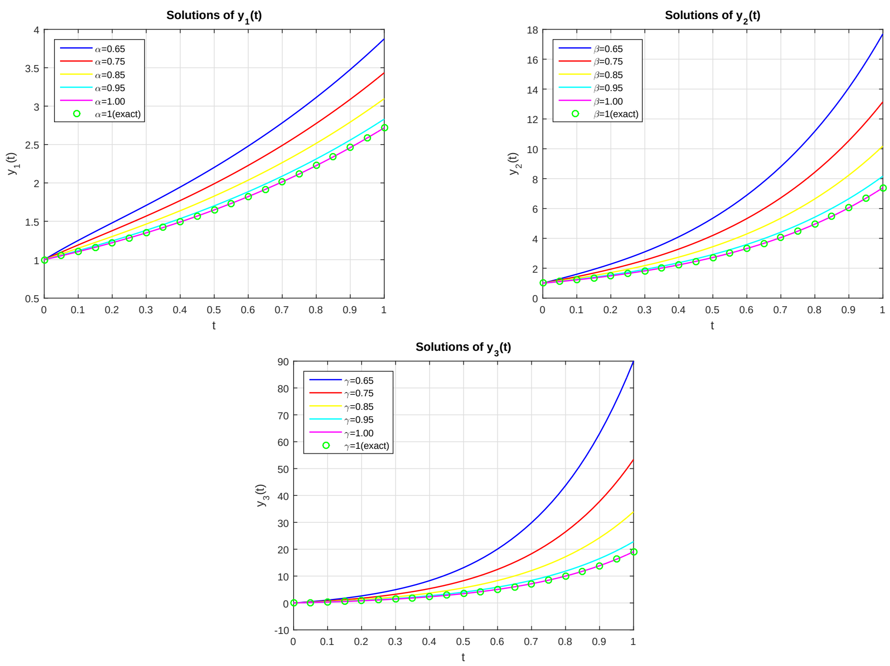

Figure 3: The numerical behaviour of Example 5.2 for $M = 6$

Table 3: Numerical results of $\alpha = \beta = \gamma = 1$ for Example 5.2

t	$y_1(t)$			$y_2(t)$			$y_3(t)$		
	[28]	Ours	Exact	[28]	Ours	Exact	[28]	Ours	Exact
0.2	1.2214	1.221403	1.221403	1.4917	1.491422	1.491825	0.8214	0.809923	0.822119
0.4	1.4917	1.491822	1.491825	2.2224	2.225148	2.225541	2.2944	2.308003	2.320117
0.6	1.8214	1.822116	1.822119	3.2944	3.319718	3.320117	4.8294	5.037109	5.049647
0.8	2.2224	2.225537	2.225541	4.8357	4.952622	4.953032	8.9664	10.01000	10.02318
1.0	2.7083	2.718277	2.718282	7.0000	7.388659	7.389056	15.375	19.07248	19.08554

where

$$\begin{cases} f_1(t) = t^8 - t^2 + \frac{24t^{3.6}}{\Gamma(4.6)} - \frac{t^{0.6}}{\Gamma(1.6)}, \\ f_2(t) = t^8 - t^2 + \frac{24t^{3.4}}{\Gamma(4.4)} + \frac{t^{0.4}}{\Gamma(1.4)}. \end{cases}$$

The exact solution of (5. 31) is $y_1(t) = t^4 - t$ *and* $y_2(t) = t^4 + t$.

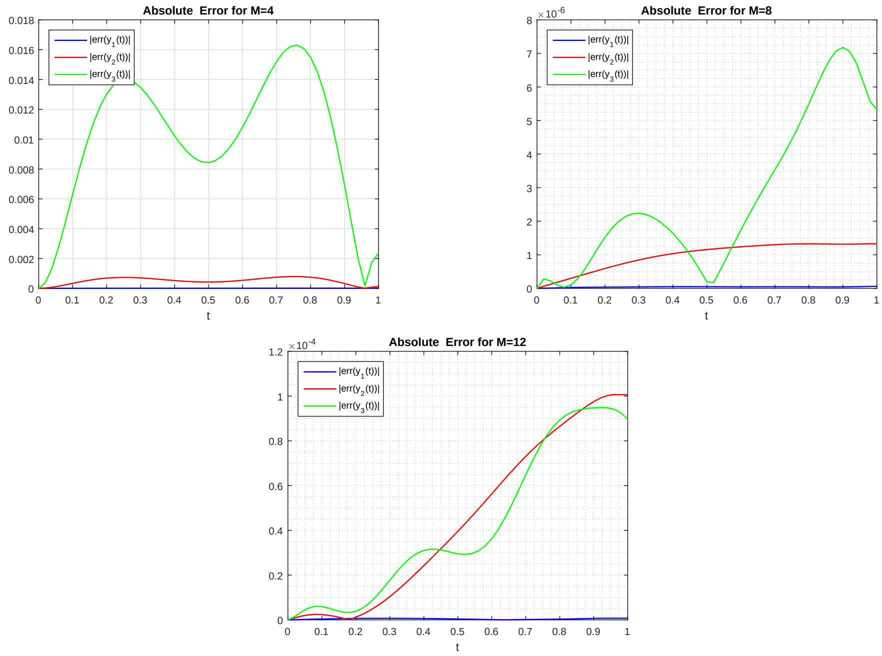

Figure 4: Absolute error of Example 5.2 for different M

Table 4 compares maximum values of absolute error for $M = 4$. Figure 8 demonstrates absolute error of (5. 31) for $M = 4$ and $M = 10$. One can see that the solutions for $M = 10$ provide more accurate results than those for $M = 4$.

Table 4: A comparison for $M = 4$ corresponding to Example 5.3

Polynomial	$\max \lvert error(y_1) \rvert$	$\max \lvert error(y_2) \rvert$
Legendre	0.018791786	0.037283818
Chebyshev	0.018792741	0.037285112
Euler	0.018792298	0.037283562

6. Conclusion

The results of presented method disclosed that this method is very contributory. One of the advantages of the method was that the Euler coefficients of the solution can be found by using a computer code written in MATLAB. As it was seen, in some cases, Euler polynomials produced a bit better results than orthogonal polynomials (Legendre and Chebyshev). We also observed that for problems which their exact solutions are unknown, by evaluating

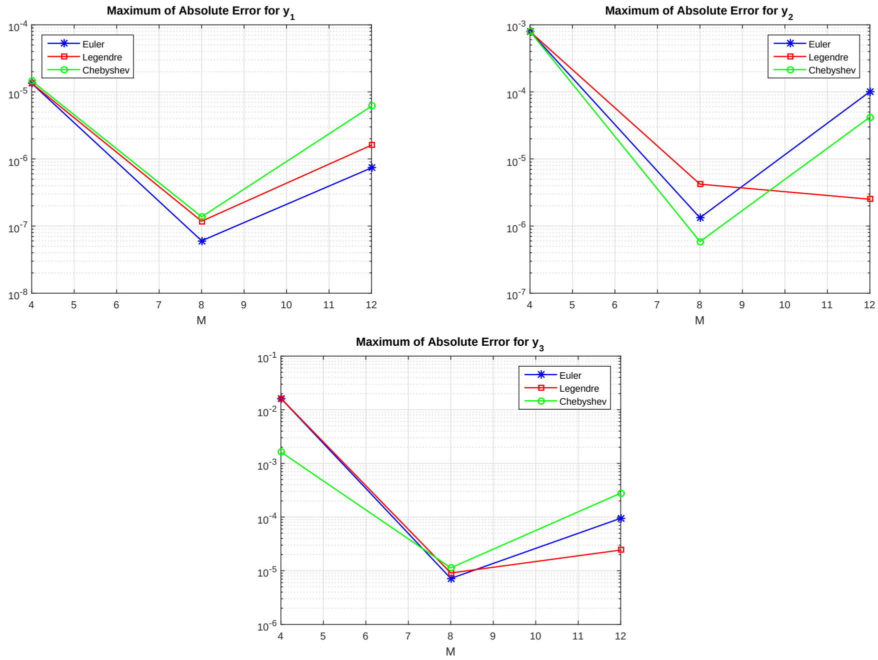

Figure 5: The maximum values of absolute errors for Example 5.2

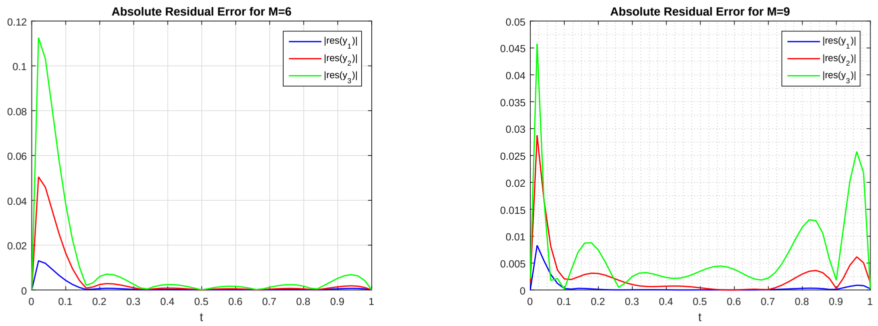

Figure 6: The absolute residual error of Example 5.2 when $\alpha = \beta = \gamma = 0.75$

residual error functions, one can control the error and choose a suitable value for M.

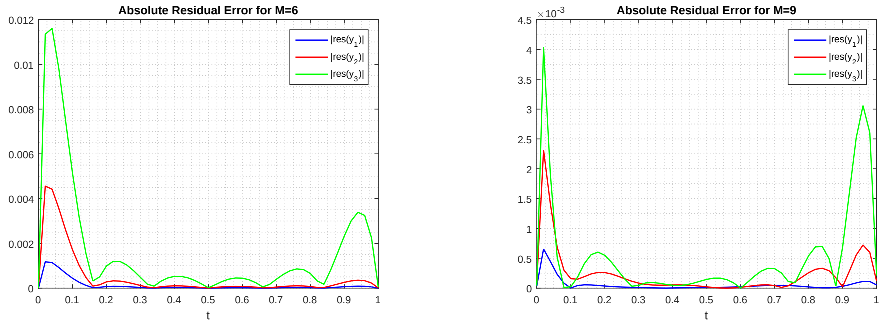

Figure 7: The absolute residual error of Example 5.2 when $\alpha = \beta = \gamma = 0.95$

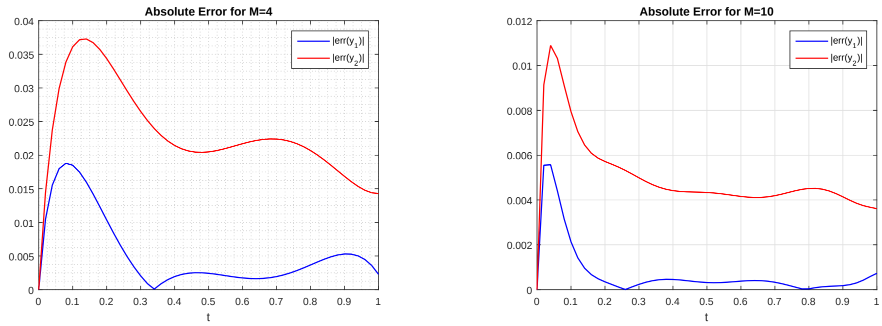

Figure 8: The absolute error of Example 5.3 for $M = 4$ and $M = 10$

This is to announce that the discussed method may also be developed to a system of nonlinear fractional integro-differential equations, but some modifications must be executed. This can be a subject for future research.

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