Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(7)(2019) pp. 89-98

On the Stability Analysis of Non–Linear Hammerstein Impulsive Integro–Dynamic System on Time Scales with Delay

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Received: 28 September, 2018 / Accepted: 20 February, 2019 / Published online: 01 May, 2019

Abstract. This paper presents the Ulam's type stability results of non–linear Hammerstein impulsive integro–dynamic system on time scales with delay, by using fixed point method. In order to overcome difficulties arises in our considered model, we pose some conditions along with Lipschitz condition.

AMS (MOS) Subject Classification Codes: 34N05; 34G20; 35B35; 45J05
Key Words: Ulam's stability, Hammerstein integral, time scale, impulses, integro–dynamic system, delay system.

1. Introduction

In 1940, Ulam gave a famous talk before a mathematical seminar at the university of Wisconsin [21, 22]. He raised a question in the matter of stability of homomorphisms. His question was answered by Hyers [9] for the case of Banach spaces, by using direct method. So this interesting stability concept, initiated by Ulam and Hyers, was named as Hyers–Ulam stability. In 1978, Rassias [17] extended Hyers–Ulam stability concept by introducing new function variables and this stability concept was named as Hyers–Ulam–Rassias stability. For more details and discussions on Hyers–Ulam types stability, see [10–13, 15, 18–20, 23–27, 29–38].

The theory of time scale analysis has been rising fast and has acknowledged a lot of interest. The pioneer of this theory was Hilger [8]. He introduced this theory in 1988, in his PhD thesis. For further details on time scale, see [1–7, 14, 16, 19, 20, 28, 34, 36].

Agarwal *et al.* [1], in 2014, discussed some results about the stability of linear impulsive Volterra integro–dynamic system on time scales. Then Zada *et al.* [36] extended the stability results of [1] to non–linear impulsive Volterra integro–delay dynamic system on time scales. As we studied, no one has checked the Hyers–Ulam stability of non–linear Hammerstein impulsive integro–delay dynamic systems on time scales. So motivated by the work done in [36], for the first time, using fixed point method, we obtain Hyers–Ulam

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stability and Hyers–Ulam–Rassias stability of non–linear Hammerstein impulsive integro–delay dynamic system of the form:

$$\begin{cases} \omega^{\Delta}(t) = \mathcal{M}(t)\omega(t) + \mathbb{G}(t, \omega(t), \omega(p(t))) \int_{t_0}^{t} g(t,s)\mathbb{H}(s, \omega(s), \omega(p(s)))\Delta s, \\ t \in {T_S}' = {T_S}^0 \backslash \{t_1, t_2, \cdots, t_m\}, \\ \Gamma\omega(t_k) = \omega(t_k^+) - \omega(t_k^-) = \Upsilon_k(\omega(t_k^-)),\ k = \overline{1,\ m} \\ \omega(t) = \alpha(t),\ t \in [t_0 - \lambda, t_0]_{T_S}, \\ \omega(t_0) = \alpha(t_0) = \omega_0, \end{cases} \tag{1. 1}$$

where $\lambda > 0$, $\overline{1,\ m}$ denotes $1, 2, \cdots, m$, the $m \times m$ regressive square matrix $\mathcal{M}(t)$ is piecewise continuous on ${T_S}^0 := [t_0, t_f]_{T_S}, t_f > s > t_0 \geq 0$ and $\mathbb{G} : {T_S}^0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbb{H} : {T_S}^0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\Upsilon_k : \mathbb{R}^n \to \mathbb{R}^n$, $\alpha : [t_0 - \lambda, t_0]_{T_S} \to \mathbb{R}$, the kernal $g : {T_S}^0 \times {T_S}^0 \to \mathbb{R}^n$ are continuous functions. Also the right and left side limits, respectively $\omega(t_k^+) = \lim_{\tau \to 0^+} \omega(t_k + \tau)$ and $\omega(t_k^-) = \lim_{\tau \to 0^-} \omega(t_k - \tau)$ of $\omega(t)$ at t_k satisfies $t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = t_f < +\infty$. Moreover, $p : {T_S}^0 \to {T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}$ with $p(t) \leq t$, is a continuous delay function.

2. Preliminaries

Any non–empty arbitrary closed subset of real numbers is called time scale, which is denoted by T_S. The forward jump operator $\Theta : T_S \to T_S$ is defined as:

$$\Theta(s) = \inf\{t \in T_S : t > s\}.$$

The derived form of a time scale T_S, denoted by ${T_S}^z$, is defined as:

$${T_S}^z = \begin{cases} T_S \backslash (\rho(\sup T_S), \sup T_S], & \text{if } \sup T_S < \infty, \\ T_S, & \text{if } \sup T_S = \infty. \end{cases}$$

The delta derivative and $\Delta-$integral of $H : T_S \to \mathbb{R}$ are respectively defined as

$$H^{\Delta}(t) = \lim_{s \to t,\ s \neq \Theta(t)} \frac{H(\Theta(t)) - H(s)}{\Theta(t) - s},\ t \in {T_S}^z,\ \int_a^b H(t)\Delta t = h(b) - h(a),\ \forall\, a, b \in T_S,$$

where $h^{\Delta} = H$ on ${T_S}^z$.

The equation $\xi^{\Delta}(t) = \mathcal{M}(t)\xi(t),\ \xi(t_0) = \xi_0,\ t \in {T_S}^0$ has general solution called fundamental matrix denoted by $\Psi_{\mathcal{M}}(t, t_0)$.

3. Basic concepts and remarks

Let $\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n)$ (resp. $P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n)$) be the Banach space of continuous functions (resp. the Banach space of piecewise continuous functions) with the norm
$\|\omega\| = \sup_{t \in {T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}} \|\omega(t)\|$. Also, we denote $P\mathbb{C}^1({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n) = \{\omega \in P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n) : \omega^{\Delta} \in P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n)\}$, the Banach space with norm $\|\omega\|_1 = \max\{\|\omega\|, \|\omega^{\Delta}\|\}$.

Consider the following inequalities in the sequel:

$$\begin{cases} \left\|\zeta^{\Delta}(s)-\mathcal{M}(s)\zeta(s)-\mathbb{G}(s,\zeta(s),\zeta(p(s)))\displaystyle\int_{s_0}^{s} g(s,t)\mathbb{H}(t,\zeta(t),\zeta(p(t)))\Delta t\right\| \le \epsilon; \\ \quad s\in {T_S}', \\ \left\|\Gamma\zeta(s_k)-\Upsilon_k(\zeta(s_k^-))\right\| \le \epsilon,\ k=\overline{1,\ m} \end{cases} \tag{3. 2}$$

$$\begin{cases} \left\|\zeta^{\Delta}(s)-\mathcal{M}(s)\zeta(s)-\mathbb{G}(s,\zeta(s),\zeta(p(s)))\displaystyle\int_{s_0}^{s} g(s,t)\mathbb{H}(t,\zeta(t),\zeta(p(t)))\Delta t\right\| \le \varphi(s); \\ \quad s\in {T_S}', \\ \left\|\Gamma\zeta(s_k)-\Upsilon_k(\zeta(s_k^-))\right\| \le \kappa,\ k=\overline{1,\ m} \end{cases} \tag{3. 3}$$

where $\varphi : {T_S}^0 \cup [t_0-\lambda, t_0]_{T_S} \to \mathbb{R}^+$ is right dense continuous and increasing.

Definition 3.1. *Equation* (1. 1) *is said to be Hyers–Ulam stable on* ${T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}$ *if for every* $\zeta \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ *satisfying* (3. 2)*, there exists* $\zeta_0 \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ *of* (1. 1) *with* $\|\zeta_0(s)-\zeta(s)\| \le C\epsilon,\ C>0,\ \forall\, s \in {T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}$.

Definition 3.2. *Equation* (1. 1) *is said to be Hyers–Ulam–Rassias stable on* ${T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}$ *if for every* $\zeta \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ *that satisfies* (3. 3)*, there exists* $\zeta_0 \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ *of* (1. 1) *with* $\|\zeta_0(s)-\zeta(s)\| \le C\varphi(s)$*,* $C>0,\ \forall\, s \in {T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}$.

Remark 3.3. *A function* $\zeta \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ *satisfies* (3. 2) *if and only if there exist* $f \in P\mathbb{C}({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ *and a finite sequence* f_k *such that* $\|f(t)\| \le \epsilon$*,* $\forall\, t \in {T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}$*,* $\|f_k\| \le \epsilon,\ \forall\, k=\overline{1,\ m}$*,*

$$\begin{cases} \zeta^{\Delta}(t)=\mathcal{M}(t)\zeta(t)+\mathbb{G}(t,\zeta(t),\zeta(p(t)))\displaystyle\int_{t_0}^{t} g(t,s)\mathbb{H}(s,\zeta(s),\zeta(p(s)))\Delta s+f(t), \\ \zeta(t_0)=\zeta_0,\ t\in {T_S}', \\ \Gamma\zeta(t_k)=\Upsilon_k(\zeta(t_k^-))+f_k. \end{cases} \tag{3. 4}$$

Lemma 3.4. *Every solution* $\zeta \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ *of* (3. 2) *also satisfies*

$$\begin{cases} \Bigg\| \zeta(t)-\zeta(t_0)-\Psi_{\mathcal{M}}(t,t_0)\zeta_0-\displaystyle\sum_{j=1}^{m}\Upsilon(\zeta(t_j^-)) \\ -\displaystyle\int_{t_0}^{t}\Psi_{\mathcal{M}}(t,\Theta(s))\mathbb{G}(s,\zeta(s),\zeta(p(s)))\int_{s_0}^{s} g(s,r)\mathbb{H}(r,\zeta(r),\zeta(p(r)))\Delta r\Delta s\Bigg\| \le (m \\ +C(t_f-t_0))\epsilon, \end{cases}$$

for $t \in (t_k, t_{k+1}] \subset {T_S}^0$, where C is the bound of fundamental matrix $\Psi_{\mathcal{M}}(t,\Theta(s))$.

Proof: If $\zeta \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ satisfies (3. 2), so by Remark 3.3, the solution of equation (3. 4) is given as

$$\zeta(t) = \zeta(t_0) + \Psi_{\mathcal{M}}(t,t_0)\zeta_0 + \sum_{j=1}^{m}\Upsilon(\zeta(t_j^-)) + \sum_{i=1}^{m} f_i$$
$$+ \int_{t_0}^{t} \Psi_{\mathcal{M}}(t,\Theta(s))\mathbb{G}(s,\zeta(s),\zeta(p(s))) \int_{s_0}^{s} g(s,r)\mathbb{H}(r,\zeta(r),\zeta(p(r)))\Delta r\Delta s$$
$$+ \int_{t_0}^{t} \Psi_{\mathcal{M}}(t,\Theta(s)) f(s)\Delta s.$$

So,

$$\begin{aligned}
&\Bigg\|\zeta(t) - \zeta(t_0) - \Psi_{\mathcal{M}}(t,t_0)\zeta_0 - \sum_{j=1}^{m}\Upsilon(\zeta(t_j^-)) \\
&- \int_{t_0}^{t} \Psi_{\mathcal{M}}(t,\Theta(s))\mathbb{G}(s,\zeta(s),\zeta(p(s))) \int_{s_0}^{s} g(s,r)\mathbb{H}(r,\zeta(r),\zeta(p(r)))\Delta r\Delta s\Bigg\| \\
\leq\ & \int_{t_0}^{t} \|\Psi_{\mathcal{M}}(t,\Theta(s))\|\|f(s)\|\Delta s + \sum_{i=1}^{m}\|f_i\| \\
\leq\ & (m + C(t-t_0))\epsilon \\
\leq\ & (m + C(t_f - t_0))\epsilon.
\end{aligned}$$

Similar remarks also holds for (3. 3).

4. Main Results

Before proving our result on Hyers–Ulam stability for equation (1. 1), we assume the following conditions:
$(\mathbf{C1})$ $\mathbb{H} : {T_S}^0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with the Lipschitz condition $\big\|\mathbb{H}(t,x_1,x_2) - \mathbb{H}(t,y_1,y_2)\big\| \leq \sum_{i=1}^{2} L\|x_i - y_i\|$, $L > 0$, for all $t \in {T_S}^0$ and $x_i, y_i \in \mathbb{R}^n$, $i \in \{1,2\}$;
$(\mathbf{C2})$ $\Upsilon_k : \mathbb{R}^n \to \mathbb{R}^n$ is such that $\|\Upsilon_k(x_1) - \Upsilon_k(x_2)\| \leq M_k\|x_1 - x_2\|$, $M_k > 0$, for all $k \in \{1,2,\ldots,m\}$ and $x_1, x_2 \in \mathbb{R}^n$, $i \in \{1,2\}$;
$(\mathbf{C3})$ For some positive constants $C,\ \delta$ and τ, we have $\|\Psi_{\mathcal{M}}(t,\Theta(s))\| \leq C$, $\|\mathbb{G}(s,\omega_1(s),\omega_1(p(s))) - \mathbb{G}(s,\omega_2(s),\omega_2(p(s)))\| \leq \delta$, $\|g(t,s)\| \leq \tau$ for every $t,s \in {T_S}^0$;
$(\mathbf{C4})$ $\left(\sum_{j=1}^{m} M_j + 2\int_{t_0}^{t}\int_{s_0}^{s} C\delta\tau L\Delta r\Delta s\right) < 1$;
$(\mathbf{C5})$ $\varphi : {T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S} \to \mathbb{R}^+$ is right dense continuous and increasing such that

$$\int_{t_0}^{t} \varphi(s)\Delta s \leq \rho\varphi(t),\ \rho > 0.$$

Theorem 4.1. *If conditions* $(\mathbf{C1}) - (\mathbf{C4})$ *hold, then equation* (1. 1) *has unique solution in* $P\mathbb{C}^1({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n)$.

Proof. Consider an operator $\Lambda : P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n) \to P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n)$ by

$$(\Lambda\omega)(t) = \begin{cases} \alpha(t),\ t \in [t_0 - \lambda, t_0]_{T_S}, \\ \alpha(t_0) + \Psi_{\mathcal{M}}(t, t_0)\omega_0 \\ + \displaystyle\int_{t_0}^{t} \Psi_{\mathcal{M}}(t, \Theta(s))\mathbb{G}(s, \omega(s), \omega(p(s))) \int_{s_0}^{s} g(s, r)\mathbb{H}(r, \omega(r), \omega(p(r)))\Delta r\Delta s, \\ t \in (t_0, t_1], \\ \alpha(t_0) + \displaystyle\sum_{j=1}^{i} \Upsilon_j(\omega(t_j^-)) + \Psi_{\mathcal{M}}(t, t_0)\omega_0 \\ + \displaystyle\int_{t_0}^{t} \Psi_{\mathcal{M}}(t, \Theta(s))\mathbb{G}(s, \omega(s), \omega(p(s))) \int_{s_0}^{s} g(s, r)\mathbb{H}(r, \omega(r), \omega(p(r)))\Delta r\Delta s, \\ t \in (t_i, t_{i+1}],\ i = \overline{1,\ m}. \end{cases} \tag{4. 5}$$

We see that for any $\omega_1, \omega_2 \in P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^n)$ and for all $t \in [t_0 - \lambda, t_0]_{T_S}$, we have $\|(\Lambda\omega_1)(t) - (\Lambda\omega_2)(t)\| = 0$. For $t \in (t_m, t_{m+1}]$, simple calculation shows that

$$\begin{aligned} & \|(\Lambda\omega_1)(t) - (\Lambda\omega_2)(t)\| \leq \sum_{j=1}^{m} \|\Upsilon_j(\omega_1(t_j^-)) - \Upsilon_j(\omega_2(t_j^-))\| \\ & + \int_{t_0}^{t} \|\Psi_{\mathcal{M}}(t, \Theta(s))\| \Big\| \Big(\mathbb{G}(s, \omega_1(s), \omega_1(p(s))) \\ & - \mathbb{G}(s, \omega_2(s), \omega_2(p(s))) \Big) \Big\| \int_{s_0}^{s} \|g(s, r)\| \Big\| \mathbb{H}(r, \omega_1(r), \omega_1(p(r))) \\ & - \mathbb{H}(r, \omega_2(r), \omega_2(p(r))) \Big\| \Delta r\Delta s \\ \leq\ & \sum_{j=1}^{m} M_j \|\omega_1(t_j^-) - \omega_2(t_j^-)\| + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L \|\omega_1(r) - \omega_2(r)\| \Delta r\Delta s \\ & + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L \|\omega_1(p(r)) - \omega_2(p(r))\| \Delta r\Delta s \\ \leq\ & \sum_{j=1}^{m} M_j \sup_{t \in {T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}} \|\omega_1(t) - \omega_2(t)\| \\ & + 2 \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L \sup_{t \in {T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}} \|\omega_1(t) - \omega_2(t)\| \Delta r\Delta s \\ \leq\ & \sum_{j=1}^{m} M_j \|\omega_1 - \omega_2\| + 2\|\omega_1 - \omega_2\| \int_{t_0}^{t} \int_{s_0}^{s} C\delta\tau L \Delta r\Delta s \\ \leq\ & \|\omega_1 - \omega_2\| \Big(\sum_{j=1}^{m} M_j + 2 \int_{t_0}^{t} \int_{s_0}^{s} C\delta\tau L \Delta r\Delta s \Big). \end{aligned}$$

From $(\mathbf{C_4})$, Λ is contractive and so it is a Picard operator on $P\mathbb{C}({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$. The operator Λ has a unique fixed point which is the unique solution of (1. 1) (from (4. 5)) in $P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$. □

Theorem 4.2. *If conditions* $(\mathbf{C1})-(\mathbf{C4})$ *hold, then equation* (1. 1) *has Hyers–Ulam stability on* ${T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}$.

Proof. Let $\zeta \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ satisfies (3. 2). The unique solution $\omega \in P\mathbb{C}^1({T_S}^0 \cup [t_0-\lambda, t_0]_{T_S}, \mathbb{R}^n)$ of the dynamic equation

$$\begin{cases} \omega^{\Delta}(t) = \mathcal{M}(t)\omega(t) + \mathbb{G}(t,\omega(t),\omega(p(t)))\displaystyle\int_{t_0}^{t} g(t,s)\mathbb{H}(s,\omega(s),\omega(p(s)))\Delta s, \\ t \in {T_S}' = {T_S}^0 \backslash \{t_1, t_2, \cdots, t_m\}, \\ \Gamma\omega(t_k) = \omega(t_k^+) - \omega(t_k^-) = \Upsilon_k(\omega(t_k^-)),\ k = \overline{1,\ m}, \\ \omega(t) = \zeta(t),\ t \in [t_0-\lambda, t_0]_{T_S}, \\ \omega(t_0) = \zeta(t_0) = \omega_0, \end{cases}$$

is

$$\omega(t) = \begin{cases} \zeta(t),\ t \in [t_0-\lambda, t_0]_{T_S}, \\ \zeta(t_0) + \Psi_{\mathcal{M}}(t,t_0)\omega_0 \\ \quad + \displaystyle\int_{t_0}^{t} \Psi_{\mathcal{M}}(t,\Theta(s))\mathbb{G}(s,\omega(s),\omega(p(s)))\int_{s_0}^{s} g(s,r)\mathbb{H}(r,\omega(r),\omega(p(r)))\Delta r\Delta s, \\ t \in (t_0, t_1], \\ \zeta(t_0) + \displaystyle\sum_{j=1}^{i} \Upsilon_j(\omega(t_j^-)) + \Psi_{\mathcal{M}}(t,t_0)\omega_0 \\ \quad + \displaystyle\int_{t_0}^{t} \Psi_{\mathcal{M}}(t,\Theta(s))\mathbb{G}(s,\omega(s),\omega(p(s)))\int_{s_0}^{s} g(s,r)\mathbb{H}(r,\omega(r),\omega(p(r)))\Delta r\Delta s, \\ t \in (t_i, t_{i+1}],\ i = \overline{1,\ m}. \end{cases}$$

Since for all $t \in [t_0-\lambda, t_0]_{T_S}$, $\|\zeta(t)-\omega(t)\| = 0$. For $t \in (t_m, t_{m+1}]$, using Lemma 3.4,

$$\begin{aligned} \|\zeta(t)-\omega(t)\| \le & \Bigg\|\zeta(t) - \zeta(t_0) - \Psi_{\mathcal{M}}(t,t_0)\zeta_0 - \sum_{j=1}^{m}\Upsilon(\zeta(t_j^-)) \\ & - \int_{t_0}^{t} \Psi_{\mathcal{M}}(t,\Theta(s))\mathbb{G}(s,\zeta(s),\zeta(p(s)))\int_{s_0}^{s} g(s,r)\mathbb{H}(r,\zeta(r),\zeta(p(r)))\Delta r\Delta s\Bigg\| \\ & + \sum_{j=1}^{m}\|\Upsilon_j(\zeta(t_j^-)) - \Upsilon_j(\omega(t_j^-))\| + \Bigg\|\int_{t_0}^{t}\Psi_{\mathcal{M}}(t,\Theta(s))\Big(\mathbb{G}(s,\zeta(s),\zeta(p(s))) \\ & - \mathbb{G}(s,\omega(s),\omega(p(s)))\Big)\int_{s_0}^{s} g(s,r)\Big(\mathbb{H}(r,\zeta(r),\zeta(p(r))) \\ & - \mathbb{H}(r,\omega(r),\omega(p(r)))\Big)\Delta r\Delta s\Bigg\| \end{aligned}$$

$$\begin{aligned}
&\leq (m + C(t_f - t_0))\epsilon + \sum_{j=1}^{m} M_j \|\zeta(t_j^-) - \omega(t_j^-)\| \\
&\quad + \int_{t_0}^{t} \|\Psi_{\mathcal{M}}(t,\Theta(s))\| \Bigg\| \Bigg(\mathbb{G}(s,\zeta(s),\zeta(p(s))) \\
&\quad - \mathbb{G}(s,\omega(s),\omega(p(s))) \Bigg) \Bigg\| \int_{s_0}^{s} \|g(s,r)\| \Bigg\| \Bigg(\mathbb{H}(r,\zeta(r),\zeta(p(r))) \\
&\quad - \mathbb{H}(r,\omega(r),\omega(p(r))) \Bigg) \Bigg\| \Delta r \Delta s \\
&\leq (m + C(t_f - t_0))\epsilon + \sum_{j=1}^{m} M_j \|\zeta(t_j^-) - \omega(t_j^-)\| + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L \|\zeta(r) \\
&\quad - \omega(r)\| \Delta r \Delta s + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L \|\zeta(p(r)) - \omega(p(r))\| \Delta r \Delta s.
\end{aligned}$$

Now we define an operator $T : P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^+) \to P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^+)$ and we will show that it is an increasing Picard operator on $P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^+)$,

$$(Tw)(t) = \begin{cases} 0, \ t \in [t_0 - \lambda, t_0]_{T_S}, \\ (t_f - t_0)\epsilon + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L w(r) \Delta r \Delta s \\ \ + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L w(p(r)) \Delta r \Delta s, \ t \in (t_0, t_1], \\ (i + t_f - t_0)\epsilon + \sum_{j=1}^{i} M_j w(t_j^-) + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L w(r) \Delta r \Delta s \\ \ + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L w(p(r)) \Delta r \Delta s, \ t \in (t_i, t_{i+1}], \ i = \overline{1, \ m}. \end{cases} \tag{4. 6}$$

For $t \in (t_m, t_{m+1}]$, following the same steps as in Theorem 4.1, we get

$$\|(Tw_1)(t) - (Tw_2)(t)\| \leq \|w_1 - w_2\| \Bigg(\sum_{j=1}^{m} M_j + 2 \int_{t_0}^{t} \int_{s_0}^{s} C\delta\tau L \Delta r \Delta s \Bigg).$$

From (**C4**), T is contractive on $P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^+)$. By using Banach contraction principle, T is Picard operator with unique fixed point $w^* \in P\mathbb{C}({T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}, \mathbb{R}^+)$ i.e.

$$\begin{aligned}
w^*(t) &= (m + C(t_f - t_0))\epsilon + \sum_{j=1}^{m} M_j w^*(t_j^-) + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L w^*(r) \Delta r \Delta s \\
&\quad + \int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L w^*(p(r)) \Delta r \Delta s.
\end{aligned}$$

As w^* is increasing, so $w^*(p(s)) \leq w^*(s)$ and we get

$$w^*(t) \leq (m + C(t_f - t_0))\epsilon + \sum_{j=1}^{m} M_j w^*(t_j^-) + 2\int_{t_0}^{t} C\delta \int_{s_0}^{s} \tau L w^*(r)\Delta r \Delta s.$$

By applying Grönwall's inequality ([14], Lemma 2.1), we get

$$w^*(t) \leq (m + C(t_f - t_0))\epsilon \prod_{t_0 < t_j < t} (1 + M_j) e_P(t, t_0),$$

where $P = \int_{s_0}^{s} 2C\delta\tau L\Delta r$. By setting $w(t) = \|\zeta(t) - \omega(t)\|$ and from (4. 6), $w(t) \leq (Tw)(t)$ and hence by abstract Grönwall lemma, we get $w(t) \leq w^*(t)$, so

$$\|\zeta(t) - \omega(t)\| \leq (m + C(t_f - t_0))\epsilon \prod_{t_0 < t_j < t} (1 + M_j) e_P(t, t_0).$$

□

Similarly, we can prove the following theorem.

Theorem 4.3. *If conditions* $(\mathbf{C1}) - (\mathbf{C5})$ *hold, then* (1. 1) *has Hyers–Ulam–Rassias stability on* ${T_S}^0 \cup [t_0 - \lambda, t_0]_{T_S}$.

Remark 4.4. *Following the same procedure, results of Hyers–Ulam stability and Hyers–Ulam–Rassias stability of* (1. 1) *can be extended to noninstantaneous impulses of the form:*

$$\begin{cases} \omega^{\Delta}(t) = \mathcal{A}(t)\omega(t) + \mathbb{G}(t, \omega(t), \omega(p(t))) \displaystyle\int_{t_0}^{t} G(t,s)\mathbb{H}(s, \omega(s), \omega(p(s)))\Delta s, \\ t \in (s_i, t_{i+1}] \cap T_S, \ i = 0, \overline{1,\ m}, \\ \omega(t) = g_i(t, \omega(t), \omega(p(t))), \ t \in (t_i, s_i] \cap T_S, \ i = \overline{1,\ m}, \\ \omega(t) = \alpha(t), \ t \in [s_0 - \lambda, s_0] \cap T_S, \\ \omega(t_0) = \alpha(t_0) = \omega_0, \end{cases}$$

where $\lambda > 0$, $\mathcal{A}(t)$ *is a piecewise continuous regressive square matrix,* $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots t_m \leq s_m \leq t_m \leq t_{m+1} = t_f$ *are pre–fixed numbers,* $g_i : (t_i, s_i] \cap T_S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $i = \overline{1,\ m}$ *are continuous functions and* $\mathbb{G} : (s_i, t_{i+1}] \cap T_S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbb{H} : (s_i, t_{i+1}] \cap T_S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, *the kernal* $G : (s_i, t_{i+1}] \cap T_S \times (s_i, t_{i+1}] \cap T_S \to \mathbb{R}^n$, $i = 0, \overline{1,\ m}$, *are continuous functions. Also* $\alpha : [t_0 - \lambda, t_0]_{T_S} \to \mathbb{R}$ *is a history function. Moreover,* $h : [s_0 - \lambda, t_f] \cap T_S \to (s_i, t_{i+1}] \cap T_S$ *is a delay function with the consumption of continuity, additionally* $h(t) \leq t$.

5. Conclusion

In this paper, we established the Hyers–Ulam stability and Hyers–Ulam–Rassias stability of equation (1. 1) with the help of fixed point method together with abstract Grönwall lemma and Grönwall's inequality. When finding the exact solution is difficult, then the concept of Hyers–Ulam stability is very important i.e. our results are fruitful in approximation theory.

Authors Contributions: All the authors contributed equally to the writing of this paper. All the authors read and approved the final manuscript.
Acknowledgments: The authors express their sincere gratitude to the Editors and referees for the careful reading of the original manuscript and useful comments that helped to improve the presentation of the results.

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