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Some New Fractional Integral Results Involving Convex Functions by Means of Generalized k-Fractional Conformable Integral

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Abstract. In the research paper, some new integral results regarding convex functions are authenticated by means of a new generalized fractional operators so-called generalized k-fractional conformable integral operators. We also deduce some other classical integral inequalities as particular cases for our results.

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1. Introduction

The inequalities involving integral operators have a fundamental role in differential equations concerning Mathematics and many fields of sciences. Furthermore, the investigation of fractional integral inequalities is also of great significance. Over the past two decades, a great development in this subject has been contributed by many researchers. A large bulk of work is available in the literature on the integral inequalities by involving fractional integral operators and the k-analogues of fractional integral operators, see [2, 3, 4, 5, 7, 11, 12, 13, 14, 15, 16, 19, 20, 33, 34] and references therein. The further details and information can be studied in [6]-[30]. Now, we present some existing results that have motivated our study. Let us begin by the work of Ngo et al. [28], in which the following result is established

$$\int_0^1 g^{\lambda+1}(y)dy \geq \int_0^1 y^{\lambda} g(y)dy \tag{1. 1}$$

and

$$\int_0^1 g^{\lambda+1}(y)dy \geq \int_0^1 yg^{\lambda}(y)dy, \tag{1. 2}$$

provided that $\lambda > 0$ and $g > 0$ is a continuous function on $0 \leq y \leq 1$ satisfying

$$\int_\tau^1 g(y)dy \geq \int_\tau^1 ydy, \tau \in [0, 1].$$

In [25], W. J. Liu. et. al. proved that

$$\int_a^b g^{\lambda+\delta}(y)dy \geq \int_a^b (y-a)^{\lambda} g^{\delta}(y)dy, \tag{1. 3}$$

where $\lambda > 0$, $\delta > 0$ and $g > 0$ is a continuous function on $a \leq y \leq b$ such that

$$\int_\tau^b g^{\gamma}(y)dy \geq \int_\tau^b (y-a)^{\gamma} dy; \gamma \in [a, b].$$

In [26], the following two theorems were presented by using the results (1. 1)-(1. 3)

1.1. **Theorem.** For two continuous positive functions g and h on $a \leq y \leq b$ where g is increasing on $[a, b]$ such that $g \leq h$ on $[a, b]$ and $\frac{g}{h}$ is decreasing, then for a convex function φ; $\varphi(0) = 0$, the relation

$$\frac{\int_a^b g(y)dy}{\int_a^b h(y)dy} \geq \frac{\int_a^b \varphi(g(y))dy}{\int_a^b \varphi(h(y))dy} \tag{1. 4}$$

holds. And

1.2. **Theorem.** For three continuous and positive functions g_1, g_2 and h on $a \leq y \leq b$ where g_1 and g_2 are increasing on $[a, b]$ such that $g_1 \leq h$ on $[a, b]$ and $\frac{g_1}{h}$ is decreasing, then for a convex function φ; $\varphi(0) = 0$, the relation

$$\frac{\int_a^b g_1(y)dy}{\int_a^b h(y)dy} \geq \frac{\int_a^b \varphi(g_1(y))g_2(y)dy}{\int_a^b \varphi(h(y))g_2(y)dy} \tag{1. 5}$$

holds. A considerable attention is given by many researchers in literature to (1. 1), (1. 2) and (1. 3). Numerous generalizations, extensions and variations have existed in the literature,(e.g. [8, 9, 17, 18, 24, 31]). The following studies [27, 29, 31, 32] and the references therein can be referred for details. In [11], Dahmani has proved the following integral inequalities by using the fractional Riemann-Liouville integral operator.

1.3. **Theorem.** For two continuous and positive functions g and h on $a \leq t \leq b$ where g is increasing on $[a, b]$ such that $g \leq h$ on $[a, b]$ and $\frac{g}{h}$ is decreasing, then for a convex function φ; $\varphi(0) = 0$, the relation

$$\frac{\mathfrak{J}^{\alpha}[g(t)]}{\mathfrak{J}^{\alpha}[h(t)]} \geq \frac{\mathfrak{J}^{\alpha}[\varphi(g(t))]}{\mathfrak{J}^{\alpha}[\varphi(h(t))]}, \tag{1. 6}$$

is valid.

1.4. **Theorem.** For three continuous and positive functions g_1, g_2 and h on $a \leq t \leq b$ where g_1 and g_2 are increasing on $[a, b]$ such that $g_1 \leq h$ on $[a, b]$ and $\frac{g_1}{h}$ is decreasing, then for a convex function φ; $\varphi(0) = 0$, the relation

$$\frac{\mathfrak{J}^{\alpha}[g_1(t)]}{\mathfrak{J}^{\alpha}[h(t)]} \geq \frac{\mathfrak{J}^{\alpha}[\varphi(g_1(t))g_2(t)]}{\mathfrak{J}^{\alpha}[\varphi(h(t))g_2(t)]}, \tag{1. 7}$$

holds. In [10], Chinchane has extended the above integral inequalities by using the Hadamard fractional integral operator.

1.5. **Theorem.** For two continuous and positive functions g and h on $a \leq t \leq b$ where g is increasing on $[a, b]$ such that $g \leq h$ on $[a, b]$ and $\frac{g}{h}$ is decreasing, then for a convex function φ; $\varphi(0) = 0$, the relation

$$\frac{\mathcal{H}^{\alpha}[g(t)]}{\mathcal{H}^{\alpha}[h(t)]} \geq \frac{\mathcal{H}^{\alpha}[\varphi(g(t))]}{\mathcal{H}^{\alpha}[\varphi(h(t))]}, \tag{1. 8}$$

is valid.

1.6. **Theorem.** Let g_1, g_2 and h be three continuous and positive functions on $a \leq t \leq b$ where g_1 and g_2 are increasing on $[a, b]$ such that $g_1 \leq h$ on $[a, b]$ and $\frac{g_1}{h}$ is decreasing, then for any convex function φ; $\varphi(0) = 0$, the relation

$$\frac{\mathcal{H}^{\alpha}[g_1(t)]}{\mathcal{H}^{\alpha}[h(t)]} \geq \frac{\mathcal{H}^{\alpha}[\varphi(g_1(t))g_2(t)]}{\mathcal{H}^{\alpha}[\varphi(h(t))g_2(t)]}, \tag{1. 9}$$

holds. Recently, a new generalized fractional integral operator known as generalized $k-$fractional conformable integral operator and the related integral inequalities are introduced by Habib et. al. [22].
Motivated by the above work, the objective of the presented manuscript is the generalization of some classical integral inequalities of [26] for convex functions by means of generalized $k-$fractional conformable integral operators. Theorem 1 and Theorem 2 can be concluded for our results as some particular cases.

2. Notations and Preliminaries

This section recalls some basic definitions of generalized $k-$fractional conformable derivative and integral as given in [22].

2.1. **Definition.** A function $f(z)$ is said to be in $L_p[a, b]$ if

$$\left(\int_a^b |f(z)|^p dz\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

2.2. **Definition.** A function $f(z)$ is said to be in $L_{p,s}[a, b]$ if

$$\left(\int_a^b |f(z)|^p z^s dz\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, s \geq 0.$$

2.3. **Definition.** If $f \in L_1[a,b]$. Then the left conformable fractional integral operator of order $s > 0$ defined by Abdeljawad [1] is given by

$$I_a^s f(x) = \int_a^x (t-a)^{s-1} f(t)dt, \quad 0 \le a < x < b \le \infty, 0 < s \le 1. \tag{2. 10}$$

2.4. **Definition.** If $f \in L_1[a,b]$, Then the right conformable fractional integral operator of order $s > 0$ defined by Abdeljawad [1] is given by

$$I_b^s f(x) = \int_x^b (b-t)^{s-1} f(t)dt, \quad 0 \le a < x < b \le \infty, 0 < s \le 1. \tag{2. 11}$$

2.5. **Definition.** If $f \in L_{1,s}[a,b]$, then the generalized left conformable fractional integral operator $\mathfrak{T}_a^{\alpha,s}$ of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ and $0 < s \le 1$, introduced by Jarad et al. [23] is defined by

$$\mathfrak{T}_a^{\alpha,s} f(t) = \frac{s^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left((t-a)^s - (x-a)^s\right)^{\alpha-1} (x-a)^{s-1} f(x)dx, \quad 0 \le a < t < b \le \infty, \tag{2. 12}$$

where Γ is the Euler gamma function.

2.6. **Definition.** If $f \in L_{1,s}[a,b]$, then the generalized right conformable fractional integral operator $\mathfrak{T}_b^{\alpha,s}$ of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ and $0 < s \le 1$, introduced by Jarad et al. [23] is defined by

$$\mathfrak{T}_b^{\alpha,s} f(t) = \frac{s^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \left((b-x)^s - (b-t)^s\right)^{\alpha-1} (b-x)^{s-1} f(x)dx, \quad 0 \le a < t < b \le \infty, \tag{2. 13}$$

where Γ is the Euler gamma function.

2.7. **Definition.** If $f \in L_{1,s}[a,b]$, then the (k,s)-fractional conformable integrals (left and right) [22] of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of a continuous function $f(x)$ on $[0,\infty)$, are given as

$$\mathfrak{F}_{a^+,k}^{\alpha,s} f(t) = \frac{(s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1} (x-a)^{s-1} f(x)dx, \quad 0 \le a < t < b \le \infty, \tag{2. 14}$$

and

$$\mathfrak{F}_{b^-,k}^{\alpha,s} f(t) = \frac{(s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_t^b \left((b-x)^s - (b-t)^s\right)^{\frac{\alpha}{k}-1} (b-x)^{s-1} f(x)dx, \quad 0 \le a < t < b \le \infty, \tag{2. 15}$$

respectively, if integrals exist, where $k > 0, 0 < s \le 1$. The existence of the (k,s)-fractional conformable integrals (2. 14) and (2. 15) is proved in [22].

3. Fractional Integral Inequalities Involving Convex Functions

This section contains our main theorems.

3.1. **Theorem.** For two continuous and positive functions g, h on $a \leq t < \infty$ where g is increasing on $[a, \infty)$ such that $g \leq h$ on $[a, \infty)$ and $\frac{g}{h}$ is decreasing, then for a convex function φ; $\varphi(0) = 0$, for any $\alpha > 0, \beta > 0, t > a$, the relation

$$\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}(g(t))\,\mathfrak{F}^{\beta,s}_{a^+,k}(\varphi(h(t))) + \mathfrak{F}^{\beta,s}_{a^+,k}(g(t))\,\mathfrak{F}^{\alpha,s}_{a^+,k}(\varphi(h(t)))}{\mathfrak{F}^{\alpha,s}_{a^+,k}(h(t))\,\mathfrak{F}^{\beta,s}_{a^+,k}(\varphi(g(t))) + \mathfrak{F}^{\beta,s}_{a^+,k}(h(t))\,\mathfrak{F}^{\alpha,s}_{a^+,k}(\varphi(g(t)))} \geq 1, \tag{3. 16}$$

is valid.

Proof. Since the function φ is convex satisfying $\varphi(0) = 0$, so the function $\frac{\varphi(t)}{t}$ is increasing. The function g is increasing, then the function $\frac{\varphi(g(t))}{g(t)}$ is also increasing. Given that $\frac{g(t)}{h(t)}$ is decreasing, so for all $\tau, \rho \in [a, t), t > a$, we have

$$\left(\frac{\varphi(g(\tau))}{g(\tau)} - \frac{\varphi(g(\rho))}{g(\rho)}\right)\left(\frac{g(\rho)}{h(\rho)} - \frac{g(\tau)}{h(\tau)}\right) \geq 0, \tag{3. 17}$$

implies that

$$\frac{\varphi(g(\tau))}{g(\tau)}\frac{g(\rho)}{h(\rho)} + \frac{\varphi(g(\rho))}{g(\rho)}\frac{g(\tau)}{h(\tau)} - \frac{\varphi(g(\tau))}{g(\tau)}\frac{g(\tau)}{h(\tau)} - \frac{\varphi(g(\rho))}{g(\rho)}\frac{g(\rho)}{h(\rho)} \geq 0. \tag{3. 18}$$

Multiplying (3. 18) by $h(\tau)h(\rho)$, we have

$$\begin{aligned}&\frac{\varphi(g(\tau))}{g(\tau)}g(\rho)h(\tau) + \frac{\varphi(g(\rho))}{g(\rho)}g(\tau)h(\rho)\\ &-\frac{\varphi(g(\tau))}{g(\tau)}g(\tau)h(\rho) - \frac{\varphi(g(\rho))}{g(\rho)}g(\rho)h(\tau) \geq 0.\end{aligned} \tag{3. 19}$$

Multiplying (3. 19) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$, then integrating the resulting identity w.r.t τ from a to t, we get

$$\begin{aligned}&g(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right) + \frac{\varphi(g(\rho))}{g(\rho)}h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}(g(t))\\ &-h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right) - \frac{\varphi(g(\rho))}{g(\rho)}g(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}(h(t)) \geq 0.\end{aligned} \tag{3. 20}$$

Again, multiplying (3. 20) on both sides by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-s}}$, then integrating the resulting identity w.r.t ρ from a to t, we get

$$\begin{aligned}&\mathfrak{F}^{\beta,s}_{a^+,k}(g(t))\,\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right) + \mathfrak{F}^{\beta,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}(g(t))\\ &\geq \mathfrak{F}^{\alpha,s}_{a^+,k}(h(t))\,\mathfrak{F}^{\beta,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right) + \mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right)\mathfrak{F}^{\beta,s}_{a^+,k}(h(t)).\end{aligned} \tag{3. 21}$$

since $g \leq h$ on $[a, \infty)$ and function $\frac{\varphi(t)}{t}$ is increasing, so for $\tau, \rho \in [a, t)$, we have

$$\frac{\varphi(g(\tau))}{g(\tau)} \leq \frac{\varphi(h(\tau))}{h(\tau)}, \tag{3. 22}$$

Multiplying (3. 22) on both sides by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\tau-a)^{1-s}}h(\tau)$, then integrating the resulting identity w.r.t τ from a to t, we get

$$\mathfrak{F}^{\beta,s}_{a^+,k}\left(\frac{\varphi\left(g(t)\right)}{g(t)}h(t)\right)\leq\mathfrak{F}^{\beta,s}_{a^+,k}\left(\varphi\left(h(t)\right)\right) \tag{3. 23}$$

Hence, from (3. 33), (3. 21) and (3. 23), the required result (3. 16) is obtained. □

3.2. **Corollary.** For two continuous and positive continuous functions g, h on $a\leq t<\infty$ where g is increasing on $[a,\infty)$ such that $g\leq h$ on $[a,\infty)$ and $\frac{g}{h}$ is decreasing, then for a convex function φ; $\varphi(0)=0$, for any $\alpha>0, t>a$, the relation

$$\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)}\geq\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(g(t))\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(h(t))\right)}, \tag{3. 24}$$

is valid.

Proof. Since the function φ is convex satisfying $\varphi(0)=0$, so the function $\frac{\varphi(t)}{t}$ is increasing. The function g is increasing, then the function $\frac{\varphi(g(t))}{g(t)}$ is also increasing. Given that $\frac{g(t)}{h(t)}$ is decreasing, so for all $\tau,\rho\in[a,t), t>a$, we have

$$\left(\frac{\varphi(g(\tau))}{g(\tau)}-\frac{\varphi(g(\rho))}{g(\rho)}\right)\left(\frac{g(\rho)}{h(\rho)}-\frac{g(\tau)}{h(\tau)}\right)\geq 0, \tag{3. 25}$$

implies that

$$\frac{\varphi(g(\tau))}{g(\tau)}\frac{g(\rho)}{h(\rho)}+\frac{\varphi(g(\rho))}{g(\rho)}\frac{g(\tau)}{h(\tau)}-\frac{\varphi(g(\tau))}{g(\tau)}\frac{g(\tau)}{h(\tau)}-\frac{\varphi(g(\rho))}{g(\rho)}\frac{g(\rho)}{h(\rho)}\geq 0. \tag{3. 26}$$

Multiplying (3. 26) by $h(\tau)h(\rho)$, we have

$$\begin{aligned}&\frac{\varphi(g(\tau))}{g(\tau)}g(\rho)h(\tau)+\frac{\varphi(g(\rho))}{g(\rho)}g(\tau)h(\rho)\\ &-\frac{\varphi(g(\tau))}{g(\tau)}g(\tau)h(\rho)-\frac{\varphi(g(\rho))}{g(\rho)}g(\rho)h(\tau)\geq 0.\end{aligned} \tag{3. 27}$$

Multiplying (3. 27) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$, then integrating the resulting identity w.r.t τ from a to t, we get

$$\begin{aligned}&g(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right)+\frac{\varphi(g(\rho))}{g(\rho)}h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g(t)\right)\\ &-h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right)-\frac{\varphi(g(\rho))}{g(\rho)}g(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)\geq 0.\end{aligned} \tag{3. 28}$$

Again, multiplying (3. 28) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\rho-a)^{1-s}}$, then integrating the resulting identity w.r.t ρ from a to t, we get

$$\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right)+\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g(t)\right)$$

$$\geq \mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right)+\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right). \quad (3.\ 29)$$

which follows that

$$\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right)\geq \mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right) \quad (3.\ 30)$$

$$\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}g(t)\right)}\geq\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g(t))}{g(t)}h(t)\right)} \quad (3.\ 31)$$

since $g\leq h$ on $[a,\infty)$ and function $\frac{\varphi(t)}{t}$ is increasing, so for $\tau,\rho\in[a,t)$, we have

$$\frac{\varphi\left(g(\tau)\right)}{g(\tau)}\leq\frac{\varphi\left(h(\tau)\right)}{h(\tau)}, \quad (3.\ 32)$$

Multiplying (3. 32) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}h(\tau)$, then integrating the resulting identity w.r.t τ from a to t, we get

$$\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi\left(g(t)\right)}{g(t)}h(t)\right)\leq\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi\left(h(t)\right)\right) \quad (3.\ 33)$$

Hence, from (3. 31) and (3. 33), the required result (3. 24) is obtained. □

3.3. **Remark.** Clearly, Theorem (1.1) would follow as a special case of Corollary (3.2) when $k=1,\alpha=1,s=1$ and $t=b$.

3.4. **Theorem.** For three continuous and positive functions g_1,g_2 and h on $a\leq t<\infty$ where g_1 and g_2 are increasing on $[a,\infty)$ such that $g_1\leq h$ on $[a,\infty)$ and $\frac{g_1}{h}$ is decreasing, then for a convex function φ; $\varphi(0)=0$, for any $\alpha>0,\beta>0,t>a$, the relation

$$\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(f_1(t)\right)\mathfrak{F}^{\beta,s}_{a^+,k}\left(\varphi(h(t))g_2(t)\right)+\mathfrak{F}^{\beta,s}_{a^+,k}\left(f_1(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(h(t))g_2(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)\mathfrak{F}^{\beta,s}_{a^+,k}\left(\varphi(g_1(t))g_2(t)\right)+\mathfrak{F}^{\beta,s}_{a^+,k}\left(h(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(g_1(t))g_2(t)\right)}\geq 1, \quad (3.\ 34)$$

is valid.

Proof. since the function φ is convex satisfying $\varphi(0)=0$, so the function $\frac{\varphi(t)}{t}$ is increasing. The function g_1 is increasing, then the function $\frac{\varphi(g_1(t))}{g_1(t)}$ is also increasing. Given that $\frac{g_1(t)}{h(t)}$ is decreasing, so for all $\tau,\rho\in[a,t),t>a$, we have

$$\left(\frac{\varphi(g_1(\tau))}{g_1(\tau)}g_2(\tau)-\frac{\varphi(g_1(\rho))}{g_1(\rho)}g_2(\rho)\right)\left(g_1(\rho)h(\tau)-g_1(\tau)h(\rho)\right)\geq 0, \quad (3.\ 35)$$

implies that

$$\frac{\varphi(g_1(\tau))g_2(\tau)}{g_1(\tau)}g_1(\rho)h(\tau)+\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\tau)h(\rho)$$
$$-\frac{\varphi(g_1(\tau))g_2(\tau)}{g_1(\tau)}g_1(\tau)h(\rho)-\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)h(\tau)\geq 0. \quad (3.\ 36)$$

Multiplying (3. 36) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$, then integrating the resulting identity w.r.t τ from a to t, we get

$$g_1(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g_1(t))g_2(t)}{g_1(t)}h(t)\right)+\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g_1(t)\right)$$

$$-h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g_1(t))g_2(t)}{g_1(t)}g_1(t)\right)-\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)\geq 0. \quad (3.\ 37)$$

Again, multiplying (3. 37) on both sides by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-s}}$, then integrating the resulting identity w.r.t ρ from a to t, we get

$$\mathfrak{F}^{\beta,s}_{a^+,k}\left(g_1(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g_1(t))g_2(t)}{g_1(t)}h(t)\right)+\mathfrak{F}^{\beta,s}_{a^+,k}\left(\frac{\varphi(g_1(t))g_2(t)}{g_1(t)}h(t)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g_1(t)\right)\geq$$

$$\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)\mathfrak{F}^{\beta,s}_{a^+,k}\left(\varphi(g_1(t))g_2(t)\right)+\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(g_1(t))g_2(t)\right)\mathfrak{F}^{\beta,s}_{a^+,k}\left(h(t)\right). \quad (3.\ 38)$$

since $g_1\leq h$ on $[a,\infty)$ and the function $\frac{\varphi(t)g_2(t)}{t}$ is increasing, we obtain

$$\mathfrak{F}^{p,s}_{a^+,k}\left(\frac{\varphi(g_1(t))g_2(t)}{g_1(t)}h(t)\right)\leq\mathfrak{F}^{p,s}_{a^+,k}\left(\varphi(h(t))g_2(t)\right),p=\alpha,\beta. \quad (3.\ 39)$$

Hence from (3. 38) and (3. 39), we obtain (3. 34). □

3.5. **Corollary.** For three continuous and positive functions g_1,g_2 and h on $a\leq t<\infty$ where g_1 and g_2 are increasing on $[a,\infty)$ such that $g_1\leq h$ on $[a,\infty)$ and $\frac{g_1}{h}$ is decreasing, then for a convex function φ; $\varphi(0)=0$, for any $\alpha>0,t>a$, the relation

$$\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g_1(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)}\geq\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(g_1(t))g_2(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(h(t))g_2(t)\right)}, \quad (3.\ 40)$$

is valid.

Proof. since $g_1\leq h$ on $[a,\infty)$ and function $\frac{\varphi(t)}{t}$ is increasing, so for $\tau,\rho\in[a,t)$, we have

$$\frac{\varphi\left(g_1(\tau)\right)}{g_1(\tau)}\leq\frac{\varphi\left(h(\tau)\right)}{h(\tau)}, \quad (3.\ 41)$$

Multiplying (3. 41) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}h(\tau)g_2(\tau)$, then integrating the resulting identity w.r.t τ from a to t, we get

$$\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi\left(g_1(t)\right)}{g_1(t)}h(t)g_2(t)\right)\leq\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi\left(h(t)g_2(t)\right)\right) \quad (3.\ 42)$$

Also, since the function φ is convex satisfying $\varphi(0)=0$, so the function $\frac{\varphi(t)}{t}$ is increasing. The function g_1 is increasing, then the function $\frac{\varphi(g_1(t))}{g_1(t)}$ is also increasing. Given that $\frac{g_1(t)}{h(t)}$ is decreasing, so for all $\tau,\rho\in[a,t),t>a$, we have

$$\left(\frac{\varphi(g_1(\tau))}{g_1(\tau)}g_2(\tau)-\frac{\varphi(g_1(\rho))}{g_1(\rho)}g_2(\rho)\right)\left(g_1(\rho)h(\tau)-g_1(\tau)h(\rho)\right)\geq 0, \quad (3.\ 43)$$

implies that

$$\frac{\varphi(g_1(\tau))g_2(\tau)}{g_1(\tau)}g_1(\rho)h(\tau)+\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\tau)h(\rho)$$

$$-\frac{\varphi(g_1(\tau))g_2(\tau)}{g_1(\tau)}g_1(\tau)h(\rho)-\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)h(\tau)\geq 0. \tag{3. 44}$$

Multiplying (3. 44) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$, then integrating the resulting identity w.r.t τ from a to t, we get

$$g_1(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g_1(t))g_2(t)}{g_1(t)}h(t)\right)+\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g_1(t)\right)$$

$$-h(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g_1(t))g_2(t)}{g_1(t)}g_1(t)\right)-\frac{\varphi(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)\geq 0. \tag{3. 45}$$

Again, multiplying (3. 45) on both sides by $\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\rho-a)^{1-s}}$, then integrating the resulting identity w.r.t ρ from a to t, we get

$$\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(g_1(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(h(t)\right)}\geq\frac{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(g_1(t))g_2(t)\right)}{\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\frac{\varphi(g_1(t))}{g_1(t)}h(t)g_2(t)\right)}, \tag{3. 46}$$

Hence, from (3. 42) and (3. 46), we obtain (3. 40). □

3.6. **Remark.** Clearly, Theorem (1.2) would follow as a special case of Corollary (3.5) when $k=1,\alpha=1,s=1$ and $t=b$.

4. CONCLUSION

The inequalities presented in this manuscript contribute the fractional calculus theory and related integral inequalities and are projected to direct to applications for developing uniqueness of solutions in fractional differential and integral equations.

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6. CONFLICT OF INTERESTS

The author(s) declare(s) that there is no conflict of interests regarding the publication of this article.

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