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The Marshall-Olkin Odd Lindley-G Family of Distributions: Theory and Applications

Farrukh Jamal
Department of Statistics,
Government S. A. Post-Graduate College
Dera Nawab Sahib, Pakistan.
Email: drfarrukh1982@gmail.com

Hesham Reyad
Department of Management Information Systems,
and Production Management,
Qassim University, Buraydah Saudi Arabia.
Email: hesham_reyad@yahoo.com

Christophe Chesneau
Department of Mathematics,
University of Caen-Normandie,
LMNO, France.
Email: christophe.chesneau@unicaen.fr

Muhammad Arslan Nasir
Department of Statistics,
Government S.E College,
Bahawalpur, Pakistan.
Email: arslannasir147@gmail.com

Soha Othman
Department of Applied Statistics,
Cairo University, Egypt.
Email: soha_othman@yahoo.com

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Abstract. In this paper, we propose a new family of distributions called the Marshall-Olkin odd Lindley-G family of distributions. It is constructed from the Marshall-Olkin transformation and the odd Lindley-G family of distributions introduced by Gomes-Silva *et al.* [10]. We study the fundamental mathematical properties of this new family and highlight its ability to provide appropriate statistical models for various kinds of data. A particular attention is paid on a special model with four parameters, using the Burr III distribution as baseline. We then estimate the model parameters by the maximum likelihood method. A simulation study is performed to check the asymptotic behavior of the maximum likelihood estimates. Four applications with practical data sets are considered to see the usefulness of the proposed family.

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Key Words: Odd Lindley distribution, Marshall-Olkin transformation, Hazard rate function, Maximum likelihood estimation.

1. INTRODUCTION

Among the existing one-parameter lifetime distributions, the Lindley distribution introduced by [16] is one of the most useful, with applications in various areas such as engineering, demography, reliability, medicine and biology. In some sense, it can provide a better statistical model than the standard exponential distribution (see [9] and the references therein). However, the deep analysis of certain real-life data by the standard Lindley model found some limitations, mainly due to the presence of only one tuning parameter. For this reason, numerous efforts have been made to provide suitable generalizations or extensions of this distribution. Among them, there are the two-parameter Lindley distribution by [21], the two-parameter weighted Lindley distribution by [9], the generalized Poisson Lindley distribution by [17], the exponentiated Lindley distribution by [5], the beta exponential Lindley distribution by [20], the transmuted Lindley geometric distribution by [19] and the complementary Lindley geometric distribution by [12]. For recent developments in this direction, we refer to the review by [23] and the references therein. On the other side, the Lindley distribution can be used to construct new general families of distributions, as those proposed by [6] and [10]. In particular, [10] introduced the odd Lindley-G (OL-G) family of distributions constructed from the T-X transformation developed by [4], the odd transformation and the Lindley distribution. It is characterized by the cumulative distribution function (cdf) given by

$$F_{OL}(x;\alpha,\phi) = 1 - \frac{\alpha + \bar{G}(x;\phi)}{(1+\alpha)\,\bar{G}(x;\phi)} \, \exp\left[-\alpha \frac{G(x;\phi)}{\bar{G}(x;\phi)}\right], \quad x \in \mathbb{R},$$

where $\alpha > 0$, $G(x;\phi)$ is a cdf of a baseline distribution with vector of parameters denoted by ϕ and $\bar{G}(x;\phi) = 1 - G(x;\phi)$. The associated probability density function (pdf) is given by

$$f_{OL}(x;\alpha,\phi) = \frac{\alpha^2}{1+\alpha} \, \frac{g(x;\phi)}{\bar{G}^3(x;\phi)} \, \exp\left[-\alpha \frac{G(x;\phi)}{\bar{G}(x;\phi)}\right], \quad x \in \mathbb{R}.$$

It is shown in [10] that the OL-G family of distributions has a strong physical interpretation and a great potential as statistical models to analyze data with different nature.

In this paper, we propose a natural extension of the OL-G family of distributions by using the Marshall-Olkin transformation introduced by [18]. This transformation is defined by

$$M(y;p) = \frac{y}{1 - p(1-y)}, \quad y \in (0,1), \tag{1. 1}$$

where $p \in (0,1)$ is an additional tuning parameter. The role of the Marshall-Olkin transformation is to bring more flexible shapes of a given model, which is welcome for the complete analysis of various types of practical data. Naturally, other transformations can be considered as, for instance, the beta transformation by [8], the McDonald transformation by [1], the Kumaraswamy Marshal-Olkin transformation by [2], the odd Burr transformation by [3], the beta Weibull transformation by [25], the odd Bur III transformation by [13] or the cosine-sine transformation by [7]. We however focus on the Marshall-Olkin transformation thanks to its simplicity and its great potential in terms of applicability. Hence we propose a new family of distributions called Marshall-Olkin Odd Lindley G family of distributions (MOOL-G for short) characterizing by the cdf given by

$$R(x;p,\alpha,\phi) = M(F_{OL}(x;\alpha,\phi);p) = \frac{F_{OL}(x;\alpha,\phi)}{1-p\bar{F}_{OL}(x;\alpha,\phi)}$$
$$= \frac{1-\frac{\alpha+\bar{G}(x;\phi)}{(1+\alpha)\,\bar{G}(x;\phi)}\,\exp\left[-\alpha\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right]}{1-p\frac{\alpha+\bar{G}(x;\phi)}{(1+\alpha)\,\bar{G}(x;\phi)}\,\exp\left[-\alpha\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right]}, \quad x\in\mathbb{R}. \tag{1. 2}$$

The main goal of this paper is to study and discuss the main features of the MOOL-G family of distributions. We also show that it is more reliable and gives better results compared to other existing models.

The rest of the paper is organized as follows. In Section 2, important properties of the MOOL-G family of distributions are given, including the associated pdf, the hazard rate function and quantile function, a study of the shapes of the pdf and the hrf, moments, skewness and kurtosis. A special model using the Burr III distribution as baseline is presented in Section 3, with plots of the related pdfs and hrfs, and numerical results on moments, skewness and kurtosis. Section 4 is devoted to the estimation of the parameters for this special model, with simulation. Applications to four data sets are presented to illustrate the potentially of the MOOL-G family of distributions. A conclusion is given in Section 5.

2. Properties of the MOOL-G distribution

2.1. **Crucial functions.** By using the cdf $R(x;p,\alpha,\phi)$ given by (1. 2), the pdf and hazard rate function (hrf) of the of the MOOL-G family of distributions are respectively given by

$$r(x;p,\alpha,\phi) = \frac{(1-p)\,\frac{\alpha^2}{1+\alpha}\,\frac{g(x;\phi)}{\bar{G}^3(x;\phi)}\,\exp\left[-\alpha\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right]}{\left[1-p\frac{\alpha+\bar{G}(x;\phi)}{(1+\alpha)\,\bar{G}(x;\phi)}\,\exp\left[-\alpha\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right]\right]^2} \tag{2. 3}$$

and

$$h(x;p,\alpha,\phi) = \frac{\frac{\alpha^2}{1+\alpha}\,\frac{g(x;\phi)}{\bar{G}^3(x;\phi)}\,\left[\frac{\alpha+\bar{G}(x;\phi)}{(1+\alpha)\,\bar{G}(x;\phi)}\right]^{-1}}{1-p\frac{\alpha+\bar{G}(x;\phi)}{(1+\alpha)\,\bar{G}(x;\phi)}\,\exp\left[-\alpha\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right]}. \tag{2. 4}$$

By solving the equation $R(Q(u;p,\alpha,\phi);p,\alpha,\phi)=u$, $u\in(0,1)$ and using [10, Equation (14)], the quantile function of the MOOL-G family of distributions is given by

$$Q(u;p,\alpha,\phi) = G^{-1}\left[1+\alpha\left\{1+W_{-1}\left(\frac{(1+\alpha)(u-1)}{1-u\,p}\,e^{-(1+\alpha)}\right)\right\}^{-1};\phi\right],$$

where $G^{-1}(x;\phi)$ denotes the inverse function (or quantile function) of $G(x;\phi)$, $W_{-1}(x)$ denotes the lower branch of the so-called Lambert function $W(x)$ defined the solution of the equation: $W(x)e^{W(x)} = x$.

2.2. **Asymptotes and shapes.** Let us now present the asymptotes of the crucial functions of the MOOL-G family of distributions. When $G(x;\phi) \to 0$, we have

$$R(x;p,\alpha,\phi) \sim \frac{\alpha}{1-p} G(x;\phi), \quad r(x;p,\alpha,\phi) \sim \frac{\alpha^2\, g(x;\phi)}{(1+\alpha)(1-p)}$$

and

$$h(x;p,\alpha,\phi) \sim \frac{\alpha^2\, g(x;\phi)}{(1+\alpha)(1-p)}.$$

When $G(x;\phi) \to 1$, we have

$$R(x;p,\alpha,\phi) \sim \frac{1 - \frac{\alpha}{(1+\alpha)\bar{G}(x;\phi)} \exp\left[-\frac{\alpha}{\bar{G}(x;\phi)}\right]}{1 - p\,\frac{\alpha}{(1+\alpha)\bar{G}(x;\phi)} \exp\left[-\frac{\alpha}{\bar{G}(x;\phi)}\right]},$$

$$r(x;p,\alpha,\phi) \sim \frac{(1-p)\,\frac{\alpha^2}{1+\alpha}\,\frac{g(x;\phi)}{\bar{G}^3(x;\phi)} \exp\left[-\frac{\alpha}{\bar{G}(x;\phi)}\right]}{\left[1 - p\frac{\alpha}{(1+\alpha)\,\bar{G}(x;\phi)} \exp\left[-\frac{\alpha}{\bar{G}(x;\phi)}\right]\right]^2}$$

and

$$h(x;p,\alpha,\phi) \sim \frac{\frac{\alpha^2}{1+\alpha}\,\frac{g(x;\phi)}{\bar{G}^3(x;\phi)} \left[\frac{\alpha}{(1+\alpha)\,\bar{G}(x;\phi)}\right]^{-1}}{1 - p\frac{\alpha}{(1+\alpha)\,\bar{G}(x;\phi)} \exp\left[-\frac{\alpha}{\bar{G}(x;\phi)}\right]}.$$

We thus now see the role of the parameters p and α in the asymptotes. The shapes of the pdf and the hrf the MOOL-G family of distributions can be described analytically. Let us set $w = \frac{\alpha+\bar{G}(x;\phi)}{(1+\alpha)\bar{G}(x;\phi)}$ and $z = \exp\left[-\alpha\,\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right]$. The critical points of the pdf are the roots of the following equation:

$$\frac{\partial_x g(x;\phi)}{g(x;\phi)} + 3\,\frac{g(x;\phi)}{\bar{G}(x;\phi)} - \frac{\alpha\, g(x;\phi)}{\bar{G}^2(x;\phi)} + 2p\frac{\partial_x(w)\,z + \partial_x(z)\,w}{1-p\,w\,z} = 0$$

and the critical points of the hrf are the roots of the following equation:

$$\frac{\partial_x g(x;\phi)}{g(x;\phi)} + 3\,\frac{g(x;\phi)}{\bar{G}(x;\phi)} - \frac{\alpha\, g(x;\phi)}{\bar{G}^2(x;\phi)} + 2p\frac{\partial_x(w)\,z + \partial_x(z)\,w}{1-p\,w\,z} + h(x;p,\alpha,\phi) = 0,$$

with $\partial_x(w) = \frac{\alpha g(x;\phi)}{(1+\alpha)\bar{G}^2(x;\phi)}$ and $\partial_x(z) = -\alpha\frac{g(x;\phi)}{\bar{G}^2(x;\phi)} \exp\left[-\alpha\,\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right]$. Note that the above equations may have more than one roots.

2.3. **Moments, skewness and kurtosis.** Let X be a random variable having the MOOL-G pdf. Then the s-th moment of X can be obtained as

$$\mu'_s = \mathbb{E}(X^s) = \int_{-\infty}^{+\infty} x^s r(x; p, \alpha, \phi) dx = \int_0^1 Q^s(u; p, \alpha, \phi) du, \tag{2. 5}$$

which can be computed numerically for a given cdf $G(x;\phi)$. The mean of X is given by $\mathbb{E}(X) = \mu'_1$. The variance of X is given by $\mathbb{V}(X) = \mu'_2 - (\mu'_1)^2$. The s-th central moment of X is given by

$$\mu_s = \mathbb{E}\left[(X - \mu'_1)^s\right] = \sum_{k=0}^{s} \binom{s}{k} (-1)^k (\mu'_1)^k \mu'_{s-k}.$$

The s-th cumulants of X can be obtained by the following equation:

$$\kappa_s = \mu'_s - \sum_{k=1}^{s-1} \binom{s-1}{k-1} \kappa_k \mu'_{s-k},$$

with $\kappa_1 = \mu'_1$. The skewness and the kurtosis of X are respectively given by

$$\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}, \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2}.$$

Again, for a given $G(x;\phi)$, all these quantities can be evaluated, as it is done for a special case in Section 3.

3. Marshall-Olkin Odd Lindley Burr III (MOOL-BIII) distribution

In this section, we present a special case of the MOOL-G family of distribution. We chose the Burr III distribution as baseline distribution, with cdf and pdf respectively given by

$$G(x;\theta,\beta) = \left(1 + x^{-\theta}\right)^{-\beta}, \quad g(x;\theta,\beta) = \theta\,\beta\,x^{-\theta-1}\left(1 + x^{-\theta}\right)^{-\beta-1}, \quad x > 0,$$

with $\theta > 0$ and $\beta > 0$. We thus define the Marshall-Olkin Odd Lindley Burr III distribution (MOOL-BIII for short) by putting $G(x;\theta,\beta)$ in the definition of the cdf given by (1. 2). Thus the associated cdf of the MOOL-BIII distribution is given by

$$R(x;p,\alpha,\theta,\beta) = \frac{1 - \left[\frac{\alpha+1-\left(1+x^{-\theta}\right)^{-\beta}}{(1+\alpha)\{1-(1+x^{-\theta})^{-\beta}\}}\right] \exp\left[-\frac{\alpha}{(1+x^{-\theta})^{\beta}-1}\right]}{1 - p\left[\frac{\alpha+1-(1+x^{-\theta})^{-\beta}}{(1+\alpha)\{1-(1+x^{-\theta})^{-\beta}\}}\right] \exp\left[-\frac{\alpha}{(1+x^{-\theta})^{\beta}-1}\right]}, \quad x > 0. \tag{3. 6}$$

The associated pdf and the hrf are respectively given by

$$r(x;p,\alpha,\theta,\beta) = \frac{(1-p)\frac{\alpha^2}{1+\alpha}\left(\frac{\theta\,\beta\,x^{-\theta-1}}{(1+x^{-\theta})^{\beta+1}\left[1-(1+x^{-\theta})^{-\beta}\right]^3}\right) \exp\left[-\frac{\alpha}{(1+x^{-\theta})^{\beta}-1}\right]}{\left[1 - p\left[\frac{\alpha+1-(1+x^{-\theta})^{-\beta}}{(1+\alpha)\{1-(1+x^{-\theta})^{-\beta}\}}\right] \exp\left[-\frac{\alpha}{(1+x^{-\theta})^{\beta}-1}\right]\right]^2}, \quad x > 0, \tag{3. 7}$$

and

$$h(x;p,\alpha,\theta,\beta)=\frac{\frac{\alpha^2}{1+\alpha}\left(\frac{\theta\,\beta\,x^{-\theta-1}}{(1+x^{-\theta})^{\beta+1}\left[1-(1+x^{-\theta})^{-\beta}\right]^3}\right)\left[\frac{\alpha+1-\left(1+x^{-\theta}\right)^{-\beta}}{(1+\alpha)\{1-(1+x^{-\theta})^{-\beta}\}}\right]^{-1}}{1-p\left[\frac{\alpha+1-(1+x^{-\theta})^{-\beta}}{(1+\alpha)\{1-(1+x^{-\theta})^{-\beta}\}}\right]\exp\left[-\frac{\alpha}{(1+x^{-\theta})^{\beta}-1}\right]},\quad x>0. \tag{3. 8}$$

Plots for pdfs $r(x;p,\alpha,\theta,\beta)$ given by (3. 7) and hrfs $h(x;p,\alpha,\theta,\beta)$ given by (3. 8), with selected values for the parameters p, α, θ and β are presented in Figure 1.

(a) (b)

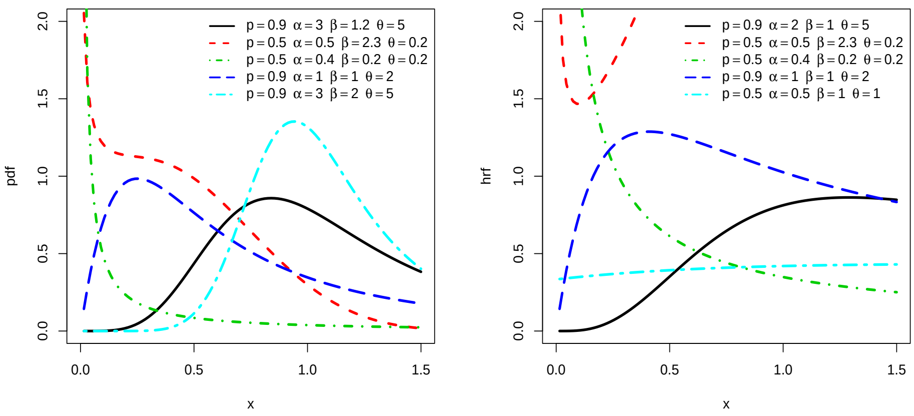

FIGURE 1. Plots for (a) pdfs (left skewed, right skewed and reverse J shape) and (b) hrfs (increasing, decreasing, constant, bathtub and upside down bathtub) of the MOOL-BIII distribution.

Let X be a random variable following the MOOL-BIII distribution, i.e. with cdf given by (3. 6). Table 1 presents the numerical values of some moments (of order 1, 2, 3 and 4), the skewness γ_1 and the kurtosis γ_2 of X for selected values of the parameters.

To conclude this section, let us mention that other special cases of this family can be studied with other choices for $G(x;\phi)$.

4. ESTIMATION OF THE PARAMETERS, SIMULATION AND APPLICATIONS

4.1. **Estimation of the parameters.** In this section, we will consider the maximum likelihood estimates (MLEs) of the model parameters of the MOOL-G family of distributions. Let $x_1, x_2, \ldots, x_n$ be a independent and identically distributed random sample having the cdf given in (3. 6). The log-likelihood function for the vector of parameters $\Theta=(p,\alpha,\phi)$

TABLE 1. Some moments, skewness and kurtosis of X for MOOL-BIII distribution for the following selected parameters values in order (p,α,θ,β); (i): $(0.5,0.5,1.5,2)$, (ii): $(0.5,1.5,0.5,2)$, (iii): $(0.5,1.5,5,0.2)$ and (iv): $(0.1,2,1,0.2)$.

	(i)	(ii)	(iii)	(iv)
$\mathbb{E}(X)$	2.7767	1.7478	0.3209	0.0254
$\mathbb{E}(X^2)$	10.4402	8.3125	0.1478	0.0037
$\mathbb{E}(X^3)$	48.5986	52.9391	0.08039	0.0009
$\mathbb{E}(X^4)$	264.5586	385.2934	0.0483	0.0003
$\mathbb{V}(X)$	2.7297	5.2576	0.0447	0.0031
γ_1	1.4406	2.2089	1.4148	4.3180
γ_2	2.4271	5.5760	2.2151	26.6793

is given by

$$\ell(\Theta)=n\log(1-p)+2\,n\,\log(\alpha)-n\,\log(1+\alpha)+\sum_{i=1}^{n}\log[g(x_i;\phi)]$$
$$-\,3\sum_{i=1}^{n}\log\left[\bar{G}(x_i;\phi)\right]-\alpha\sum_{i=1}^{n}\frac{G(x_i;\phi)}{\bar{G}(x_i;\phi)}-2\sum_{i=1}^{n}\log\left(1-p\,w_i\,z_i\right),$$

where $w_i=\frac{\alpha+\bar{G}(x_i;\phi)}{(1+\alpha)\,\bar{G}(x_i;\phi)}$ and $z_i=\exp\left[-\alpha\frac{G(x_i;\phi)}{\bar{G}(x_i;\phi)}\right]$.
The components of the score vector $U=(U_p,U_\alpha,U_\phi)$ are given by

$$U_p=-\frac{n}{1-p}+2\sum_{i=1}^{n}\frac{w_i\,z_i}{1-p\,w_i\,z_i},$$

$$U_\alpha=\frac{2\,n}{\alpha}-\frac{n}{1+\alpha}-\sum_{i=1}^{n}\frac{G(x_i;\phi)}{\bar{G}(x_i;\phi)}+2\,p\sum_{i=1}^{n}\left[\frac{\partial_\alpha(w_i)\,z_i+\partial_\alpha(z_i)\,w_i}{1-p\,w_i\,z_i}\right],$$

with $\partial_\alpha(w_i)=\frac{G(x_i;\phi)}{(1+\alpha)^2\bar{G}(x_i;\phi)}$ and $\partial_\alpha(z_i)=-\frac{G(x_i;\phi)}{\bar{G}(x_i;\phi)}\exp\left[-\alpha\frac{G(x_i;\phi)}{\bar{G}(x_i;\phi)}\right]$ and

$$U_\phi=\sum_{i=1}^{n}\frac{\partial_\phi g(x_i;\phi)}{g(x_i;\phi)}+3\sum_{i=1}^{n}\frac{\partial_\phi G(x_i;\phi)}{\bar{G}(x_i;\phi)}-\alpha\sum_{i=1}^{n}\frac{\partial_\phi G(x_i;\phi)}{\bar{G}^2(x_i;\phi)}$$
$$+\,2\,p\sum_{i=1}^{n}\left[\frac{\partial_\phi(w_i)\,z_i+\partial_\phi(z_i)\,w_i}{1-p\,w_i\,z_i}\right],$$

with $\partial_\phi(w_i)=\frac{\alpha\partial_\phi G(x_i;\phi)}{(1+\alpha)\bar{G}^2(x_i;\phi)}$ and $\partial_\phi(z_i)=-\alpha\frac{\partial_\phi G(x_i;\phi)}{\bar{G}^2(x_i;\phi)}\exp\left[-\alpha\frac{G(x_i;\phi)}{\bar{G}(x_i;\phi)}\right]$. The MLEs of p, α and ϕ, say $\hat{p}$, $\hat{\alpha}$ and $\hat{\phi}$, are the simultaneous solutions of the equations $U_{\hat{p}}=0$, $U_{\hat{\alpha}}=0$ and $U_{\hat{\phi}}=0$. Since they are non linear, these equations can not be solved analytically. However, numerical alternatives exist by using softwares supporting the iterative techniques like the Newton-Raphson algorithm. Using the well-known normal approximation of the MLEs for n large, confidence intervals and parametric tests can be constructed for the parameters. Finally, note that other estimation methods can be investigated, as the least square method or the Bayesian method (see for instance [22]).

4.2. **Simulation study.** In this section, we study the performance of the maximum likelihood method for the MOOL-BIII distribution by conducting various simulations with different sizes for n ($n = 50, 100, 200, 300$ and 500). The software R is used. We simulate 2000 samples for the true parameters values for six different sets of values : Set I: $p = 0.5$, $\alpha = 0.5$, $\theta = 1.5$, $\beta = 2.5$, Set II: $p = 0.2$, $\alpha = 2.5$, $\theta = 0.5$, $\beta = 1.5$, Set III: $p = 0.6$, $\alpha = 3.5$, $\theta = 1.5$, $\beta = 4$, Set IV: $p = 0.8$, $\alpha = 2.5$, $\theta = 0.5$, $\beta = 2$, Set V: $p = 0.5$, $\alpha = 2$, $\theta = 2.5$, $\beta = 2$ and Set VI: $p = 0.6$, $\alpha = 2$, $\theta = 1$, $\beta = 2$, in order to obtain the mean estimates and the mean square errors (MSEs) of the parameters. The numerical results are listed in Table 2. We observe that MSEs decrease as the sample size increases. The results indicate that the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.

4.3. **Applications to real data sets.** This section provides four applications to show how the MOOL-BIII distribution can be applied in practice. For the comparisons, we consider the Generalized Gamma Burr III distribution (GG-BIII), the Kumaraswamy BIII distribution (Kw-BIII), the beta Burr III distribution (B-BIII) and the BIII distribution. The MLEs are computed using Quasi-Newton Code for Bound Constrained Optimization and the log-likelihood function evaluated. The computed goodness-of-fit measures are the Anderson-Darling (A*), the Cramer-von Mises (W*), the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC) and the log-likelihood $(\hat{\ell})$. The lower the values of these criteria, the better the fit. The value for the Kolmogorov Smirnov (KS) statistic and its p-value are also provided. The plots of the fitted pdfs and cdfs of some distributions are displayed for visual comparison. The required computations are carried out using the R software. The fours considered data sets are presented below.

Data set 1 The first data set consists of 63 observations of the gauge lengths of 10 mm from [14]. The data set is given as follows: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Data set 2 The second data set is taken from [11]. It represents there lief times of 20 patients receiving analgesic. The data set is given as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2,1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6,2.0.

Data set 3 The third data refers breaking strength (in Gpa) of carbon fibers (https://www.biz.uiowa.edu/faculty/jledolter/StatisticalQualityControl/Data/ch12ex10.txt) The data set is given as follows: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35,

TABLE 2. Empirical means and the MSEs (in parenthesis) of the MOOL-BIII distribution for six sets of selected parameters values in order (p, α, θ, β); Set I: $(0.5, 0.5, 1.5, 2.5)$, Set II: $(0.2, 2.5, 0.5, 1.5)$, Set III: $(0.6, 3.5, 1.5, 4)$, Set IV: $(0.8, 2.5, 0.5, 2)$, Set V: $(0.5, 2, 2.5, 2)$ and Set VI: $(0.6, 2, 1, 2)$.

	Set I				Set II			
n	$\hat{p}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
50	0.776	0.635	1.918	3.928	0.523	3.181	0.499	1.982
	(2.114)	(3.135)	(1.050)	(5.321)	(0.991)	(2.177)	(0.098)	(0.992)
100	0.643	0.559	1.881	3.192	0.419	2.915	0.485	1.772
	(1.171)	(2.041)	(0.956)	(4.012)	(0.851)	(1.169)	(0.051)	(0.812)
200	0.438	0.449	1.597	2.982	0.598	1.985	0.478	1.722
	(1.007)	(2.039)	(0.817)	(3.112)	(0.627)	(0.959)	(0.049)	(0.712)
300	0.405	0.495	1.602	2.121	0.322	2.623	0.474	1.461
	(0.939)	(1.094)	(0.727)	(2.921)	(0.519)	(0.808)	(0.032)	(0.582)
500	0.510	0.498	1.501	1.671	0.258	2.490	0.493	1.489
	(0.186)	(0.910)	(0.493)	(2.012)	(0.121)	(0.093)	(0.020)	(0.444)
	Set III				Set IV			
n	$\hat{p}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
50	0.995	3.545	1.654	4.913	1.368	2.727	0.445	2.308
	(0.137)	(0.033)	(0.115)	(0.025)	(0.105)	(0.061)	(0.032)	(0.070)
100	0.670	3.518	1.649	4.701	1.037	2.541	0.504	2.084
	(0.035)	(0.054)	(0.090)	(0.013)	(0.052)	(0.021)	(0.022)	(0.028)
200	0.633	3.451	1.607	4.511	0.895	2.509	0.515	2.031
	(0.017)	(0.010)	(0.036)	(0.010)	(0.027)	(0.011)	(0.009)	(0.014)
300	0.604	3.504	1.539	4.169	0.908	2.524	0.492	2.050
	(0.008)	(0.008)	(0.025)	(0.006)	(0.018)	(0.009)	(0.007)	(0.009)
500	0.558	3.493	1.502	3.998	0.722	2.484	0.554	1.993
	(0.006)	(0.005)	(0.014)	(0.004)	(0.004)	(0.004)	(0.002)	(0.004)
	Set V				Set VI			
n	$\hat{p}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
50	0.596	2.105	2.636	2.155	0.933	2.141	1.438	2.351
	(0.038)	(0.088)	(0.151)	(0.055)	(0.195)	(0.056)	(0.109)	(0.097)
100	0.679	2.101	2.370	2.131	0.656	2.129	1.032	2.146
	(0.033)	(0.048)	(0.074)	(0.028)	(0.046)	(0.030)	(0.045)	(0.027)
200	0.621	2.108	2.566	2.162	0.498	2.000	1.079	1.988
	(0.033)	(0.018)	(0.050)	(0.008)	(0.009)	(0.011)	(0.022)	(0.005)
300	0.505	2.195	2.582	2.121	0.554	2.169	1.051	2.082
	(0.012)	(0.014)	(0.027)	(0.005)	(0.009)	(0.007)	(0.017)	(0.007)
500	0.499	2.103	2.515	2.101	0.601	2.068	0.957	2.045
	(0.003)	(0.008)	(0.014)	(0.001)	(0.003)	(0.010)	(0.009)	(0.001)

2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

Data set 4 The fourth data represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (see [15]). The data set is given as follows: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

TABLE 3. MLEs and their standard errors (in parentheses) for Data set 1.

Distribution	p	α	θ	β	σ
MOOL-BIII	0.6329	1.2801	3.5801	62.6197	-
	(0.6104)	(3.1734)	(1.9651)	(23.2331)	-
GG-BIII	12.8369	1.4154	0.3953	13.3746	19.4312
	(111.7384)	(1.9364)	(2.2296)	(91.5497)	(50.9298)
B-BIII	-	1.4042	2.8480	22.2076	16.5927
	-	(1.8052)	(11.0929)	(110.8130)	(48.2007)
Kw-BIII	-	2.5824	2.6979	14.2263	6.0050
	-	(0.9680)	(53.6599)	(282.9576)	(6.5035)
BIII	-	293.4286	-	96.3766	-
	-	(373.9396)	-	(122.6920)	-

TABLE 4. The $\hat{\ell}$, AIC, BIC, *W**, *A**, *KS*, p-value values for Data set 1.

Dist	$\hat{\ell}$	AIC	BIC	*W**	*A**	*KS*	p-value
MOOL-BIII	55.9894	119.7788	128.3513	0.0484	0.2682	0.0724	0.8954
GG-BIII	56.2993	122.5987	133.3143	0.0610	0.3250	0.0800	0.8148
B-BIII	56.3183	120.6367	129.2093	0.0604	0.3249	0.0802	0.8114
Kw-BIII	56.3886	120.7773	129.3498	0.0626	0.3354	0.0813	0.7991
BIII	133.7594	271.5179	275.8042	0.0587	0.3611	0.4864	0.0000

Tables 3, 5, 7 and 9 compare the MOOL-BIII model with the GG-BIII, B-BIII, Kw-BIII and BIII models by using the goodness-of-fits measures presented above. We note that the MOOL-BIII model gives the lowest values for the AIC, BIC, KS, A* and W* statistics and maximum KS p-values among all fitted models. With these criteria, the MOOL-BIII model could be considered as the best model. The histograms of the data and estimated pdfs and cdfs for the fitted models are displayed in Figures 2, 3, 4 and 5. It is clear from Tables 3, 5, 7 and 9 and Figures 2, 3, 4 and 5 that the MOOL-BIII distribution provides a better fit to the histogram and therefore could be chosen as the best model for the considered data sets. So MOOL-BIII model is superior to the other competitive models.

5. CONCLUSION

We propose a new family of distributions, called the Marshall-Olkin odd Lindely-G family (MOOL-G), constructed from the Marshall-Olkin transformation and the odd Lindley-G

(a) (b)

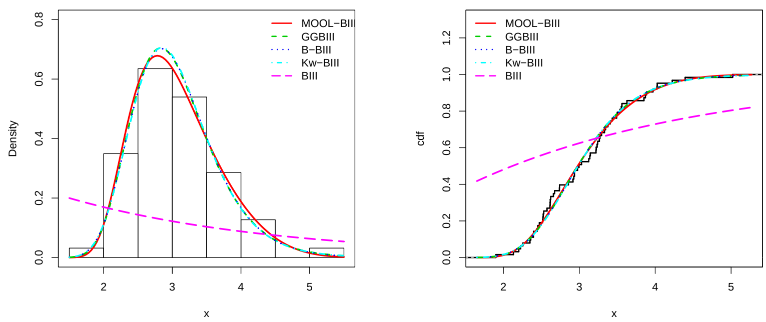

FIGURE 2. Plots of (a) estimated pdfs and (b) estimated cdfs of the MOOL-BIII distribution for Data set 1.

TABLE 5. MLEs and their standard errors (in parentheses) for Data set 2.

Distribution	p	α	θ	β	σ
MOOL-BIII	0.9602	0.9257	2.3687	10.3997	-
	(0.0968)	(2.6460)	(1.4048)	(4.2907)	-
GG-BIII	27.5220	13.3554	0.7115	4.8176	0.5292
	(0.3020)	(3.5140)	(0.2448)	(2.4372)	(0.4183)
B-BIII	-	3.0410	0.1304	67.1304	1.8025
	-	(2.5845)	(0.1481)	(76.3337)	(2.8647)
Kw-BIII	-	4.6281	0.8628	8.9592	0.8815
	-	(2.3871)	(2.6857)	(1.3827)	(0.8064)
BIII	-	277.1693	-	146.3734	-
	-	(798.5144)	-	(421.1494)	-

TABLE 6. The $\hat{\ell}$, AIC, BIC, *W**, *A**, *KS*, p-value values for Data set 2.

Dist	$\hat{\ell}$	AIC	BIC	*W**	*A**	*KS*	p-value
MOOL-BIII	15.2256	38.4512	42.5341	0.0248	0.1447	0.0871	0.9981
GG-BIII	15.3726	40.7452	45.7239	0.0321	0.1805	0.0981	0.9805
B-BIII	15.6360	39.2720	43.2549	0.0331	0.1908	0.1090	0.9714
Kw-BIII	15.4202	38.8413	42.8243	0.0267	0.1526	0.0959	0.9808
BIII	32.8962	69.7929	71.7844	0.1047	0.6208	0.4399	0.0008

(a) (b)

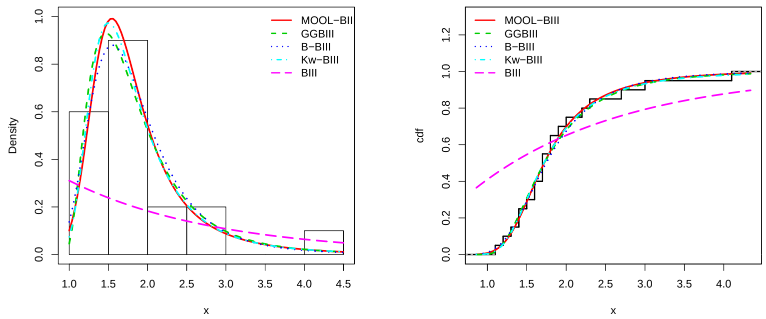

FIGURE 3. Plots of (a) estimated pdfs and (b) estimated cdfs of the MOOL-BIII distribution for Data set 2.

TABLE 7. MLEs and their standard errors (in parentheses) for Data set 3.

Distribution	p	α	θ	β	σ
MOOL-BIII	0.0001	0.3452	2.1076	1.7075	-
	(1.5479)	(0.7217)	(0.5232)	(1.9270)	-
GG-BIII	8.1977	10.1111	0.1692	23.8495	0.9639
	(0.0548)	(0.0074)	(0.0037)	(3.4005)	(0.1377)
B-BIII	-	0.9614	19.1986	0.3903	101.0804
	-	(0.3316)	(7.7909)	(0.2656)	(122.7663)
Kw-BIII	-	0.5842	3.5937	3.6343	281.5802
	-	(0.1194)	(3.9788)	(1.9689)	(64.2313)
BIII	-	294.8660	-	112.9853	-
	-	(342.5489)	-	(130.9986)	-

TABLE 8. The $\hat{\ell}$, AIC,BIC, $W*$, $A*$, KS, p-value values for Data set 3.

Dist	$\hat{\ell}$	AIC	BIC	$W*$	$A*$	KS	p-value
MOOL-BIII	141.3292	290.6585	301.0792	0.0650	0.3979	0.0611	0.8479
GG-BIII	159.0499	328.0998	341.1256	0.4902	2.7634	0.1406	0.0382
B-BIII	141.7565	291.5129	301.9336	0.0851	0.4728	0.0746	0.6328
Kw-BIII	142.1229	292.2458	302.6665	0.1087	0.5626	0.0831	0.4945
BIII	196.7476	397.4952	402.7055	0.1509	0.7727	0.3211	0.0000

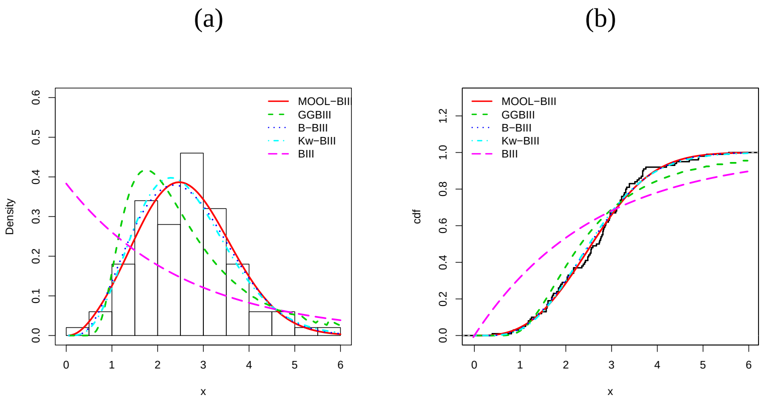

FIGURE 4. Plots of (a) estimated pdfs and (b) estimated cdfs of the MOOL-BIII distribution for Data set 3.

TABLE 9. MLEs and their standard errors (in parentheses) for Data set 4.

Distribution	p	α	θ	β	σ
MOOL-BIII	0.0001	0.0687	0.9087	1.1297	-
	(0.6126)	(0.0763)	(0.0945)	(1.2058)	-
GG-BIII	10.8764	0.7522	18.3425	0.2765	12.1608
	(0.0061)	(0.0066)	(0.0061)	(0.0279)	(2.4645)
B-BIII	-	0.3596	27.4173	0.4455	152.2599
	-	(0.0723)	(9.9050)	(0.2204)	(99.9625)
Kw-BIII	-	0.2374	6.1533	2.8660	379.3748
	-	(0.0299)	(44.2252)	(20.6027)	(320.0348)
BIII	-	386.6204	-	9.0637	-
	-	(163.0798)	-	(3.6128)	-

TABLE 10. The $\hat{\ell}$, AIC,BIC, *W**, *A**, *KS*, p-value values for Data set 4.

Dist	$\hat{\ell}$	AIC	BIC	*W**	*A**	*KS*	p-value
MOOL-BIII	579.1797	1166.3590	1177.5431	0.0556	0.3823	0.0706	0.5811
GG-BIII	583.0440	1176.0880	1190.0670	0.0941	0.6429	0.1104	0.1044
B-BIII	582.6887	1173.3770	1184.5610	0.0822	0.5968	0.0932	0.2439
Kw-BIII	582.7962	1173.5920	1184.7760	0.0946	0.6398	0.0937	0.2377
BIII	588.1276	1180.2550	1185.847	0.0720	0.4876	0.1334	0.0268

(a) (b)

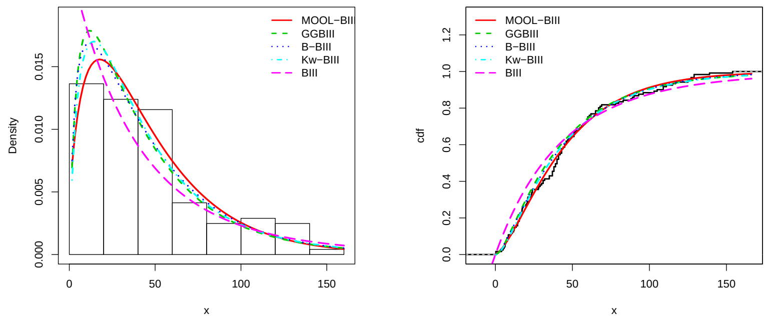

FIGURE 5. Plots of (a) estimated pdfs and (b) estimated cdfs of the MOOL-BIII distribution for Data set 4.

family of distributions. We investigate some statistical properties of the MOOL-G family of distributions such as the crucial functions (pdf, hrf and quantile function), shapes of the pdf and hrf, moments, generating functions, skewness and kurtosis. A focus is done on a special model with four parameters using the Burr III distribution as baseline. The method of maximum likelihood is applied to estimate the model parameters. Four real data sets are used to show that some models corresponding to the MOOL-G family can give better fit than similar models generated by well-known competitors. Naturally, for a given data set, other baseline distributions can be examined with possible better results in terms of goodness of fit. A possible extension of this work is to consider successful generalization of the Marshall-Olkin transformation instead of (1. 1), as the one considered in [24], i.e.

$$M(y; \upsilon, \delta) = \frac{1 - (1 - y)^{\delta}}{1 - (1 - \upsilon)(1 - y)^{\delta}},$$

with $\upsilon, \delta > 0$. The presence of these new parameters opens the door to new models of interest. This generalization needs further investigations that we leave for a future work.

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