Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(6)(2019) pp. 1-17

New Hermite–Hadamard type Inequalities for Harmonically-Convex Functions

Muhammad Amer Latif
Department of Basic Sciences,
Deanship of Preparatory Year Program,
University of Hail, Hail 2440, Saudi Arabia.
Email: m_amer_latif@hotmail.com

Sabir Hussain
Department of Mathematics,
University of Engineering and Technology, Lahore, Pakistan.
Email: sabirhus@gmail.com

Received: 29 October, 2018 / Accepted: 18 January, 2019 / Published online: 20 March, 2018

Abstract. In this paper, some new integral inequalities are presented by using harmonic convexity of the mapping $|f'|^q$ for $q \in [1, \infty)$ over the interval $[a, b]$ of real numbers and mathematical analysis.

AMS (MOS) Subject Classification Codes: Primary 26D15; Secondary 26A51; 26A33
Key Words: convex functions, harmonically-convex functions , Hermite-Hadamard inequality.

1. Introduction

The past few decades have witnessed an explosion of research on inequalities, including a large number of papers and many fruitful applications. It is recognized that, in general, some specific inequalities provide a useful and essential gadget in the development of various branches of mathematics.

A function $z : I \subset \mathbb{R} \to \mathbb{R}$ is called convex function if

$$z(\nu x + (1-\nu) y) \leq \nu z(x) + (1-\nu) z(y),$$

where $x, y \in I$ and $\nu \in [0, 1]$.

The theory of convex functions is considered to play a fundamental part in the study of inequalities.

As is notable, the Hermite-Hadamard inequality is one of the most important mathematical inequalities for classical convex functions. It is stated as follows:

If $z : [\vartheta_1, \vartheta_2] \subset \mathbb{R} \to \mathbb{R}$ is a convex function, then

$$z\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} z(x)\, dx \leq \frac{z(\vartheta_1) + z(\vartheta_2)}{2} \tag{1. 1}$$

holds. The importance of the Hermite-Hadamard inequality is due to its role in different branches of modern mathematics such as numerical analysis, functional analysis and mathematical analysis. It was first observed by Hermite [8] and redeemed later by Hadamard [6]. It is considered as one of the most distinguished results on convex functions due to its strong geometrical significance and applications. Various refinements, generalizations and applications of Hermite-Hadamard inequality can be seen in [1]-[5], [7], [9]-[13] and [14]-[34].

Definition 1.1. [11] *Let $I \subset \mathbb{R}\setminus\{0\}$ be a real interval. A function $z : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$z\left(\frac{xy}{\nu x+(1-\nu)y}\right) \leq \nu z(y)+(1-\nu)z(x) \tag{1. 2}$$

for all $x, y \in I$ and $\nu \in [0,1]$. If the inequality in (1. 2) is reversed, then z is said to be harmonically concave.

A result which connects the usual convexity and the harmonic convexity can also be found in [11].

The following Hermite-Hadamard type inequalities were studies in [6].

Theorem 1.2. [11] *Let $z : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable mapping on I°, where ϑ_1, $\vartheta_2 \in I^\circ$ with $\vartheta_1 < \vartheta_2$. If $z' \in L_1[\vartheta_1, \vartheta_2]$, then*

$$z\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}\right) \leq \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx \leq \frac{z(\vartheta_1)+z(\vartheta_2)}{2}. \tag{1. 3}$$

The above inequalities are sharp.

The following error bounds for the difference between the middle and the rightmost terms in (1. 3) were also studied in [11].

Theorem 1.3. [11] *Let $z : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and ϑ_1, $\vartheta_2 \in I^\circ$ with $\vartheta_1 < \vartheta_2$. If $z' \in L_1[\vartheta_1, \vartheta_2]$ and $\left|z'\right|^\sigma$ is harmonically-convex on $[\vartheta_1, \vartheta_2]$ for $\sigma \geq 1$*

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \leq \frac{\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}{2}\lambda_1^{1-\frac{1}{\sigma}}\left(\lambda_2\left|z'(\vartheta_1)\right|^\sigma+\lambda_3\left|z'(\vartheta_2)\right|^\sigma\right)^{\frac{1}{\sigma}}, \tag{1. 4}$$

where

$$\lambda_1 = \frac{1}{\vartheta_1\vartheta_2} - \frac{2}{(\vartheta_1-\vartheta_2)^2}\ln\left(\frac{(\vartheta_1+\vartheta_2)^2}{4\vartheta_1\vartheta_2}\right),$$

$$\lambda_2 = -\frac{1}{\vartheta_2(\vartheta_1-\vartheta_2)} + \frac{(3\vartheta_1+\vartheta_2)}{(\vartheta_2-\vartheta_1)^3}\ln\left(\frac{(\vartheta_1+\vartheta_2)^2}{4\vartheta_1\vartheta_2}\right)$$

and

$$\lambda_3 = \frac{1}{\vartheta_1 (\vartheta_1 - \vartheta_2)} - \frac{(3\vartheta_2 + \vartheta_1)}{(\vartheta_2 - \vartheta_1)^3} \ln \left(\frac{(\vartheta_1 + \vartheta_2)^2}{4\vartheta_1 \vartheta_2} \right).$$

Theorem 1.4. [11] *Let $z : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° and ϑ_1, $\vartheta_2 \in I^\circ$ with $\vartheta_1 < \vartheta_2$. If $z' \in L_1 [\vartheta_1, \vartheta_2]$ and $\left| z' \right|^\sigma$ is harmonically-convex on $[\vartheta_1, \vartheta_2]$ for $\sigma > 1$, $\frac{1}{\xi} + \frac{1}{\sigma} = 1$, then*

$$\left| \frac{z(\vartheta_1) + z(\vartheta_2)}{2} - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} z(x)\, dx \right| \leq \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - \vartheta_1)}{2} \left(\frac{1}{\xi + 1} \right)^{\frac{1}{\xi}} \left(\mu_1 |z'(\vartheta_1)|^\sigma + \mu_2 |z'(\vartheta_2)|^\sigma \right)^{\frac{1}{\sigma}}, \quad (1.\ 5)$$

where

$$\mu_1 = \frac{\vartheta_1^{2-2\sigma} + \vartheta_2^{1-2\sigma} [(\vartheta_2 - \vartheta_1)(1 - 2\sigma) - \vartheta_1]}{2 (\vartheta_2 - \vartheta_1)^2 (1 - \sigma)(1 - 2\sigma)}$$

$$\mu_2 = \frac{\vartheta_2^{2-2\sigma} - \vartheta_1^{1-2\sigma} [(\vartheta_2 - \vartheta_1)(1 - 2\sigma) + \vartheta_2]}{2 (\vartheta_2 - \vartheta_1)^2 (1 - \sigma)(1 - 2\sigma)}.$$

In Section 3, we will give a comparison of our results with those given in Theorem 1.3 and Theorem 1.4. For different generalizations and extensions of harmonic convex functions and related Hermite-Hadamard type inequalities, see for example [1, 2, 7], [10]-[13], [16], [23] and [32]-[34].

2. Hermite-Hadamard type inequalities for harmonically-convex functions

Lemma 2.1. *Let $z : I \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ be a differentiable mapping on I°, where ϑ_1, $\vartheta_2 \in I^\circ$ with $\vartheta_1 < \vartheta_2$. If $z' \in L_1 [\vartheta_1, \vartheta_2]$, then*

$$\frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} z(x)\, dx - z(\vartheta_1) = \vartheta_1 \vartheta_2 (\vartheta_2 - \vartheta_1) \left[\int_0^1 \frac{(1+\nu)}{(X_2(\vartheta_1, \vartheta_2; \nu))^2} z' \left(\frac{2\vartheta_1 \vartheta_2}{X_2(\vartheta_1, \vartheta_2; \nu)} \right) d\nu + \int_0^1 \frac{\nu}{(X_1(\vartheta_1, \vartheta_2; \nu))^2} z' \left(\frac{2\vartheta_1 \vartheta_2}{X_1(\vartheta_1, \vartheta_2; \nu)} \right) d\nu \right] \quad (2.\ 6)$$

and

$$z(\vartheta_2)-\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx$$
$$=\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left[\int_0^1 \frac{(1-\nu)}{(X_2(\vartheta_1,\vartheta_2;\nu))^2} z^{'}\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right)d\nu\right.$$
$$\left.+\int_0^1 \frac{(2-\nu)}{(X_1(\vartheta_1,\vartheta_2;\nu))^2} z^{'}\left(\frac{2\vartheta_1\vartheta_2}{X_1(\vartheta_1,\vartheta_2;\nu)}\right)d\nu\right], \quad (2.\ 7)$$

where $X_1(\vartheta_1,\vartheta_2;\nu)=\nu\vartheta_2+(2-\nu)\vartheta_1$ *and* $X_2(\vartheta_1,\vartheta_2;\nu)=(1+\nu)\vartheta_2+(1-\nu)\vartheta_1$.

Proof. By integration by parts, we have

$$I_1=\int_0^1 \frac{(1+\nu)}{(X_2(\vartheta_1,\vartheta_2;\nu))^2} z^{'}\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right)d\nu$$
$$=-\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}\int_0^1 (1+\nu)\,d\left[z\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right)\right]$$
$$=-\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}(1+\nu)\,z\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right)\Bigg|_0^1$$
$$+\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}\int_0^1 z\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right)d\nu$$
$$=-\frac{z(\vartheta_1)}{\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}+\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}z\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}\right)$$
$$+\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}\int_0^1 z\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right)d\nu$$
$$=-\frac{1}{\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}z(\vartheta_1)+\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}z\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}\right)$$
$$+\frac{1}{(\vartheta_2-\vartheta_1)^2}\int_{\vartheta_1}^{\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}} z(x)\,dx.$$

Similarly, we can get

$$I_2 = \int_0^1 \frac{\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^2} z'\left(\frac{2\vartheta_1\vartheta_2}{X_1(\vartheta_1,\vartheta_2;\nu)}\right) d\nu$$
$$= -\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)} z\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}\right) + \frac{1}{(\vartheta_2-\vartheta_1)^2}\int_{\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}}^{\vartheta_2} z(x)\,dx,$$

$$I_3 = \int_0^1 \frac{(1-\nu)}{(X_2(\vartheta_1,\vartheta_2;\nu))^2} z'\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right) d\nu$$
$$= \frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)} z\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}\right) - \frac{1}{(\vartheta_2-\vartheta_1)^2}\int_{\vartheta_1}^{\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}} z(x)\,dx$$

and

$$I_4 = \int_0^1 \frac{(2-\nu)}{(X_1(\vartheta_1,\vartheta_2;\nu))^2} z'\left(\frac{2\vartheta_1\vartheta_2}{X_1(\vartheta_1,\vartheta_2;\nu)}\right) d\nu$$
$$= -\frac{1}{2\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)} z\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}\right) + \frac{1}{\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)} z(\vartheta_2)$$
$$- \frac{1}{(\vartheta_2-\vartheta_1)^2}\int_{\frac{2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}}^{\vartheta_2} z(x)\,dx.$$

Adding I_1 and I_2 and multiplying the result by $\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$, we get (2. 6). Adding I_3 and I_4 and multiplying the result by $\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$, we get (2. 7). □

Corollary 2.2. *Let $z : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable mapping on I°, where ϑ_1, $\vartheta_2 \in I^\circ$ with $\vartheta_1 < \vartheta_2$. If $z' \in L_1[\vartheta_1,\vartheta_2]$, then*

$$\frac{z(\vartheta_1)+z(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx$$
$$= \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left[\int_0^1 \frac{(1-\nu)}{(X_1(\vartheta_1,\vartheta_2;\nu))^2} z'\left(\frac{2\vartheta_1\vartheta_2}{X_1(\vartheta_1,\vartheta_2;\nu)}\right) d\nu\right.$$
$$\left. - \int_0^1 \frac{\nu}{(X_2(\vartheta_1,\vartheta_2;\nu))^2} z'\left(\frac{2\vartheta_1\vartheta_2}{X_2(\vartheta_1,\vartheta_2;\nu)}\right) d\nu\right]. \quad (2.\ 8)$$

Condition A Let $z : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and ϑ_1, $\vartheta_2 \in I^{\circ}$ with $\vartheta_1 < \vartheta_2$. If $z^{'} \in L_1[\vartheta_1,\vartheta_2]$ and $\left|z^{'}\right|^{\sigma}$ is harmonically-convex on $[\vartheta_1,\vartheta_2]$ for $\sigma \geq 1$ or $\sigma > 1$, then

Theorem 2.3. *With reference to Condition A and $\sigma \geq 1$, then*

$$\begin{aligned}\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_1)\right| &\leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\\ &\times\left\{[\alpha_1(\vartheta_1,\vartheta_2)]^{1-\frac{1}{\sigma}}\left(\beta_1(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\beta_2(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right.\\ &\left.+[\alpha_2(\vartheta_1,\vartheta_2)]^{1-\frac{1}{\sigma}}\left(\beta_3(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\beta_4(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right\}, \quad (2.\ 9)\end{aligned}$$

where

$$\begin{aligned}\alpha_1(\vartheta_1,\vartheta_2) &= \frac{\vartheta_1}{\vartheta_2(\vartheta_1-\vartheta_2)}+\frac{1}{(\vartheta_1-\vartheta_2)^2}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right),\\ \alpha_2(\vartheta_1,\vartheta_2) &= \frac{1}{\vartheta_1-\vartheta_2}+\frac{1}{(\vartheta_1-\vartheta_2)^2}\ln\left(\frac{2\vartheta_1}{\vartheta_1+\vartheta_2}\right),\\ \beta_1(\vartheta_1,\vartheta_2) &= \frac{1}{\vartheta_1-\vartheta_2}-\frac{1}{(\vartheta_1-\vartheta_2)^2}\ln\left(\frac{2\vartheta_1}{\vartheta_1+\vartheta_2}\right),\\ \beta_2(\vartheta_1,\vartheta_2) &= \frac{2\vartheta_1-\vartheta_1\vartheta_2-\vartheta_2}{2\vartheta_2(\vartheta_1-\vartheta_2)^3(\vartheta_1+\vartheta_2)}\\ &\quad+\frac{2\vartheta_1}{(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right),\\ \beta_3(\vartheta_1,\vartheta_2) &= \frac{\vartheta_1+3\vartheta_2}{2(\vartheta_1-\vartheta_2)^2(\vartheta_1+\vartheta_2)}\\ &\quad-\frac{2\vartheta_1}{(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_1}{\vartheta_1+\vartheta_2}\right)\end{aligned}$$

and

$$\begin{aligned}\beta_4(\vartheta_1,\vartheta_2) &= -\frac{\vartheta_1+3\vartheta_2}{2(\vartheta_1-\vartheta_2)^2(\vartheta_1+\vartheta_2)}\\ &\quad-\frac{(\vartheta_1+\vartheta_2)}{(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_1}{\vartheta_1+\vartheta_2}\right).\end{aligned}$$

Proof. From (2. 6), using the power-mean inequality and the harmonic-convexity of $\left|z'\right|^{\sigma}$ on $[\vartheta_1, \vartheta_2]$ for $\sigma \geq 1$, we have

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_1)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left\{\left(\int_0^1 \frac{(1+\nu)\,d\nu}{(X_2(\vartheta_1,\vartheta_2;\nu))^2}\right)^{1-\frac{1}{\sigma}}\right.$$
$$\times\left(\int_0^1 \frac{(1+\nu)}{(X_2(\vartheta_1,\vartheta_2;\nu))^2}\left[\frac{1+\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{1-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu\right)^{\frac{1}{\sigma}}$$
$$\left.+\left(\int_0^1 \frac{\nu d\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^2}\right)^{1-\frac{1}{\sigma}}\left(\int_0^1 \frac{\nu\left[\frac{\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{2-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^2}\right)^{\frac{1}{\sigma}}\right\}. \quad (2.\ 10)$$

Calculations of the integrals in (2. 10) leads to the inequality (2. 9). □

Corollary 2.4. *If $\sigma = 1$ in Theorem 2.3, we have*

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_1)\right|$$
$$\leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left\{[\beta_1(\vartheta_1,\vartheta_2)+\beta_3(\vartheta_1,\vartheta_2)]\left|z'(\vartheta_1)\right|\right.$$
$$\left.+[\beta_2(\vartheta_1,\vartheta_2)+\beta_4(\vartheta_1,\vartheta_2)]\left|z'(\vartheta_2)\right|\right\}, \quad (2.\ 11)$$

where $\beta_1(\vartheta_1,\vartheta_2)$, $\beta_2(\vartheta_1,\vartheta_2)$, $\beta_3(\vartheta_1,\vartheta_2)$ and $\beta_4(\vartheta_1,\vartheta_2)$ are defined in Theorem 2.3.

Theorem 2.5. *With reference to Condition A and $\sigma > 1$, $\frac{1}{\xi}+\frac{1}{\sigma}=1$, then*

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_1)\right|$$
$$\leq \frac{\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}{(\xi+1)^{\frac{1}{\xi}}}\left\{\left(2^{\xi}-1\right)^{\frac{1}{\xi}}\left(\gamma_1(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\gamma_2(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\gamma_3(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\gamma_4(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right\}, \quad (2.\ 12)$$

where

$$\gamma_1(\vartheta_1,\vartheta_2)$$
$$= \frac{4^{1-\sigma}\vartheta_2^{1-2\sigma}(2\vartheta_1(\sigma-1)-\vartheta_2(2\sigma-1))-(\vartheta_1+\vartheta_2)^{1-2\sigma}(\vartheta_1(2\sigma-3)-\vartheta_2(2\sigma-1))}{4(\vartheta_1-\vartheta_2)^2(\sigma-1)(2\sigma-1)},$$

$$\gamma_2(\vartheta_1,\vartheta_2) = \frac{4^{1-\sigma}\vartheta_2^{2-2\sigma}-(\vartheta_1+\vartheta_2)^{1-2\sigma}(\vartheta_1(2\sigma-1)-\vartheta_2(2\sigma-3))}{4(\vartheta_1-\vartheta_2)^2(\sigma-1)(2\sigma-1)},$$

$$\gamma_3(\vartheta_1,\vartheta_2)=\frac{4^{1-\sigma}\vartheta_1^{2-2\sigma}+(\vartheta_1+\vartheta_2)^{1-2\sigma}(\vartheta_1(2\sigma-3)-\vartheta_2(2\sigma-1))}{4(\vartheta_1-\vartheta_2)^2(\sigma-1)(2\sigma-1)}$$

and

$$\gamma_4(\vartheta_1,\vartheta_2)$$
$$=\frac{4^{1-\sigma}\vartheta_1^{1-2\sigma}(2\vartheta_2(\sigma-1)-\vartheta_1(2\sigma-1))+(\vartheta_1+\vartheta_2)^{1-2\sigma}(\vartheta_1(2\sigma-1)-\vartheta_2(2\sigma-3))}{4(\vartheta_1-\vartheta_2)^2(\sigma-1)(2\sigma-1)}.$$

Proof. By using the first equality of Lemma 2.1, application of the Hölder inequality and the usage of harmonic-convexity of $\left|z'\right|^{\sigma}$, $\sigma>1$, gives

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}z(x)\,dx-z(\vartheta_1)\right|\le\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{\left(\int_0^1(1+\nu)^{\xi}\,d\nu\right)^{\frac{1}{\xi}}\left(\int_0^1\frac{\left[\frac{1+\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{1-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu}{(X_2(\vartheta_1,\vartheta_2;\nu))^{2\sigma}}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_0^1\nu^{\xi}d\nu\right)^{\frac{1}{\xi}}\left(\int_0^1\frac{\left[\frac{\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{2-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^{2\sigma}}\right)^{\frac{1}{\sigma}}\right\}.$$

After evaluating the integrals in the above inequality, we get (2. 12). □

Theorem 2.6. *With reference to Condition A and $\sigma>1$, $\frac{1}{\xi}+\frac{1}{\sigma}=1$, then*

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}z(x)\,dx-z(\vartheta_1)\right|\le\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{[\gamma_5(\vartheta_1,\vartheta_2)]^{\frac{1}{\xi}}\left(\frac{\left(2^{\sigma+2}-1\right)(\sigma+1)\left|z'(\vartheta_1)\right|^{\sigma}+\left(2^{\sigma+2}-\sigma-3\right)\left|z'(\vartheta_2)\right|^{\sigma}}{2(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+[\gamma_6(\vartheta_1,\vartheta_2)]^{\frac{1}{\xi}}\left(\frac{(\sigma+2)\left|z'(\vartheta_1)\right|^{\sigma}+(\sigma+3)\left|z'(\vartheta_2)\right|^{\sigma}}{2(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}\right\},\quad(2.\ 13)$$

where

$$\gamma_5(\vartheta_1,\vartheta_2)=\frac{2^{1-2\xi}\vartheta_2-(\vartheta_1+\vartheta_2)^{1-2\xi}}{(2\xi-1)(\vartheta_1-\vartheta_2)}$$

and

$$\gamma_6(\vartheta_1,\vartheta_2)=-\frac{2^{1-2\xi}\vartheta_1-(\vartheta_1+\vartheta_2)^{1-2\xi}}{(2\xi-1)(\vartheta_1-\vartheta_2)}.$$

Proof. Usage of the first equality of Lemma 2.1, application of the Hölder inequality and applying the harmonic-convexity of $\left|z^{'}\right|^{\sigma}$, $\sigma > 1$, gives

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int\limits_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_1)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left\{\left(\int\limits_0^1 \frac{1}{(X_2(\vartheta_1,\vartheta_2;\nu))^{2\xi}}d\nu\right)^{\frac{1}{\xi}}\right.$$
$$\times\left(\int\limits_0^1 (1+\nu)^{\sigma}\left[\frac{1+\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{1-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu\right)^{\frac{1}{\sigma}}$$
$$\left.+\left(\int\limits_0^1 \frac{1}{(X_1(\vartheta_1,\vartheta_2;\nu))^{2\xi}}d\nu\right)^{\frac{1}{\xi}}\left(\int\limits_0^1 \nu^{\sigma}\left[\frac{\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{2-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu\right)^{\frac{1}{\sigma}}\right\}. \tag{2. 14}$$

By evaluating each integral in (2. 14), we get (2. 13). □

Theorem 2.7. *With reference to Condition A with $\sigma \geq 1$. Then*

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int\limits_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_2)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{[\theta_1(\vartheta_1,\vartheta_2)]^{1-\frac{1}{\sigma}}\left(\delta_1(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\delta_2(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+[\theta_2(\vartheta_1,\vartheta_2)]^{1-\frac{1}{\sigma}}\left(\delta_3(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\delta_4(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right\}, \tag{2. 15}$$

where

$$\theta_1(\vartheta_1,\vartheta_2) = \frac{1}{\vartheta_2-\vartheta_1} - \frac{1}{(\vartheta_1-\vartheta_2)^2}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right),$$
$$\theta_2(\vartheta_1,\vartheta_2) = \frac{\vartheta_2-\vartheta_1\vartheta_2}{\vartheta_1(\vartheta_1-\vartheta_2)^2(\vartheta_1+\vartheta_2)} + \frac{1}{(\vartheta_1-\vartheta_2)^2}\ln\left(\frac{2\vartheta_1}{\vartheta_1+\vartheta_2}\right),$$
$$\delta_1(\vartheta_1,\vartheta_2) = \frac{-3\vartheta_1+2\vartheta_1\vartheta_2-\vartheta_2}{2(\vartheta_1-\vartheta_2)^3(\vartheta_1+\vartheta_2)} + \frac{(\vartheta_1+\vartheta_2)}{(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right),$$
$$\delta_2(\vartheta_1,\vartheta_2) = \frac{\vartheta_1+2\vartheta_1\vartheta_2-3\vartheta_2}{2(\vartheta_1-\vartheta_2)^3(\vartheta_1+\vartheta_2)} - \frac{2\vartheta_2}{(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right),$$
$$\delta_3(\vartheta_1,\vartheta_2) = -\frac{\vartheta_1+2\vartheta_1\vartheta_2-3\vartheta_2}{2(\vartheta_1-\vartheta_2)^3(\vartheta_1+\vartheta_2)} - \frac{(\vartheta_1+\vartheta_2)}{(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_1}{\vartheta_1+\vartheta_2}\right)$$

and

$$\delta_4(\vartheta_1,\vartheta_2) = \frac{\vartheta_1+\vartheta_1\vartheta_2-2\vartheta_2}{2\vartheta_1(\vartheta_1-\vartheta_2)^3(\vartheta_1+\vartheta_2)} - \frac{2\vartheta_2}{(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_1}{\vartheta_1+\vartheta_2}\right).$$

Proof. Taking the absolute value on both sides of (2. 7), using the power-mean inequality and the harmonic-convexity of $\left|z^{'}\right|^{\sigma}$ on $[\vartheta_1, \vartheta_2]$ for $\sigma \geq 1$, we have

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int\limits_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_2)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left\{\left(\int\limits_0^1 \frac{(1-\nu)}{(X_2(\vartheta_1,\vartheta_2;\nu))^2}d\nu\right)^{1-\frac{1}{\sigma}}\right.$$
$$\times\left(\int\limits_0^1 \frac{(1-\nu)}{(X_2(\vartheta_1,\vartheta_2;\nu))^2}\left[\frac{1+\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{1-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu\right)^{\frac{1}{\sigma}}$$
$$+\left(\int\limits_0^1 \frac{(2-\nu)}{(X_1(\vartheta_1,\vartheta_2;\nu))^2}d\nu\right)^{1-\frac{1}{\sigma}}$$
$$\left.\times\left(\int\limits_0^1 \frac{(2-\nu)}{(X_1(\vartheta_1,\vartheta_2;\nu))^2}\left[\frac{\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{2-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu\right)^{\frac{1}{\sigma}}\right\}. \quad (2.\ 16)$$

Calculations of the integrals in (2. 16) give the inequality (2. 15). □

Corollary 2.8. *If $\sigma = 1$ in Theorem 2.3, we have*

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int\limits_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_2)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{\left[\delta_1(\vartheta_1,\vartheta_2)+\delta_3(\vartheta_1,\vartheta_2)\right]\left|z'(\vartheta_1)\right|+\left[\delta_2(\vartheta_1,\vartheta_2)+\delta_4(\vartheta_1,\vartheta_2)\right]\left|z'(\vartheta_2)\right|\right\}, \quad (2.\ 17)$$

where $\delta_1(\vartheta_1,\vartheta_2)$, $\delta_2(\vartheta_1,\vartheta_2)$, $\delta_3(\vartheta_1,\vartheta_2)$ and $\delta_4(\vartheta_1,\vartheta_2)$ are defined in Theorem 2.7.

Theorem 2.9. *With reference to Condition A and $\sigma > 1$, $\frac{1}{\xi}+\frac{1}{\sigma}=1$, then*

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int\limits_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_2)\right|$$
$$\leq \frac{\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}{(\xi+1)^{\frac{1}{\xi}}}\left\{\left(\gamma_1(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\gamma_2(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(2^{\xi+1}-1\right)^{\frac{1}{\xi}}\left(\gamma_3(\vartheta_1,\vartheta_2)\left|z'(\vartheta_1)\right|^{\sigma}+\gamma_4(\vartheta_1,\vartheta_2)\left|z'(\vartheta_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}\right\}, \quad (2.\ 18)$$

where $\gamma_1(\vartheta_1,\vartheta_2)$, $\gamma_2(\vartheta_1,\vartheta_2)$, $\gamma_3(\vartheta_1,\vartheta_2)$ and $\gamma_4(\vartheta_1,\vartheta_2)$ are defined in Theorem 2.5.

Proof. By using the second equality of Lemma 2.1, application of the Hölder inequality and the usage of harmonic-convexity of $\left|z^{'}\right|^{\sigma}$ on $[\vartheta_1, \vartheta_2]$ for $\sigma > 1$, gives

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_2)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{\left(\int_0^1 (1-\nu)^{\xi}\,d\nu\right)^{\frac{1}{\xi}}\left(\int_0^1 \frac{\left[\frac{1+\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{1-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu}{\left(X_2(\vartheta_1,\vartheta_2;\nu)\right)^{2\sigma}}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_0^1 (2-\nu)^{\xi}\,d\nu\right)^{\frac{1}{\xi}}\left(\int_0^1 \frac{\left[\frac{\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{2-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu}{\left(X_1(\vartheta_1,\vartheta_2;\nu)\right)^{2\sigma}}\right)^{\frac{1}{\sigma}}\right\}.$$

After evaluating the integrals in the above inequality, we get (2. 18). □

Theorem 2.10. *With reference to Condition A and $\sigma > 1$, $\frac{1}{\xi}+\frac{1}{\sigma}=1$, then*

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_2)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{\left[\gamma_5(\vartheta_1,\vartheta_2)\right]^{\frac{1}{\xi}}\left(\frac{(\sigma+3)\left|z'(\vartheta_1)\right|^{\sigma}+(\sigma+1)\left|z'(\vartheta_2)\right|^{\sigma}}{2(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}\right.$$
$$+\left[\gamma_6(\vartheta_1,\vartheta_2)\right]^{\frac{1}{\xi}}$$
$$\left.\times\left(\frac{\left(2^{\sigma+2}-\sigma-3\right)\left|z'(\vartheta_1)\right|^{\sigma}+(\sigma+1)\left(2^{\sigma+2}-1\right)\left|z'(\vartheta_2)\right|^{\sigma}}{2(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}\right\}, \quad (2.\ 19)$$

where $\gamma_5(\vartheta_1,\vartheta_2)$ and $\gamma_6(\vartheta_1,\vartheta_2)$ are defined in Theorem 2.6.

Proof. Usage of the second equality of Lemma 2.1, application of the Hölder inequality and applying the harmonic-convexity of $\left|z^{'}\right|^{\sigma}$ on $[\vartheta_1, \vartheta_2]$ for $\sigma > 1$, we get

$$\left|\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx - z(\vartheta_2)\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left\{\left(\int_0^1 \frac{d\nu}{\left(X_2(\vartheta_1,\vartheta_2;\nu)\right)^{2\xi}}\right)^{\frac{1}{\xi}}\right.$$
$$\times\left(\int_0^1 (1-\nu)^{\sigma}\left[\frac{1+\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{1-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu\right)^{\frac{1}{\sigma}}+\left(\int_0^1 \frac{d\nu}{\left(X_1(\vartheta_1,\vartheta_2;\nu)\right)^{2\xi}}\right)^{\frac{1}{\xi}}$$
$$\left.\times\left(\int_0^1 (2-\nu)^{\sigma}\left[\frac{\nu}{2}\left|z'(\vartheta_1)\right|^{\sigma}+\frac{2-\nu}{2}\left|z'(\vartheta_2)\right|^{\sigma}\right]d\nu\right)^{\frac{1}{\sigma}}\right\}. \quad (2.\ 20)$$

By evaluating each integral in (2. 20), we get (2. 19). □

Now we present some trapezoidal type inequalities for harmonically-convex functions.

Theorem 2.11. *With reference to Condition A and $\sigma \geq 1$, then*

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2}-\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{[\eta_1(\vartheta_1,\vartheta_2)]^{1-\frac{1}{\sigma}}\left(\mu_1(\vartheta_1,\vartheta_2)|z'(\vartheta_1)|^{\sigma}+\mu_2(\vartheta_1,\vartheta_2)|z'(\vartheta_2)|^{\sigma}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+[\eta_1(\vartheta_2,\vartheta_1)]^{1-\frac{1}{\sigma}}\left(\mu_2(\vartheta_2,\vartheta_1)|z'(\vartheta_1)|^{\sigma}+\mu_1(\vartheta_2,\vartheta_1)|z'(\vartheta_2)|^{\sigma}\right)^{\frac{1}{\sigma}}\right\}, \quad (2.\ 21)$$

where

$$\eta_1(\vartheta_1,\vartheta_2)=\frac{1}{2(\vartheta_1-\vartheta_2)\vartheta_2}+\frac{1}{(\vartheta_1-\vartheta_2)^2}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right),$$
$$\mu_1(\vartheta_1,\vartheta_2)=\frac{\vartheta_1+\vartheta_2}{2(\vartheta_1-\vartheta_2)^2\vartheta_2}+\frac{(3\vartheta_1+\vartheta_2)}{2(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right)$$

and

$$\mu_2(\vartheta_1,\vartheta_2)=-\frac{1}{(\vartheta_1-\vartheta_2)^2}-\frac{(3\vartheta_2+\vartheta_1)}{2(\vartheta_1-\vartheta_2)^3}\ln\left(\frac{2\vartheta_2}{\vartheta_1+\vartheta_2}\right).$$

Proof. Taking the absolute value on both sides of (2. 7), using the power-mean inequality and the harmonic-convexity of $\left|z'\right|^{\sigma}$ on $[\vartheta_1,\vartheta_2]$ for $\sigma \geq 1$, we have

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2}-\frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)$$
$$\times\left\{\left(\int_0^1\frac{\nu d\nu}{(X_2(\vartheta_1,\vartheta_2;\nu))^2}\right)^{1-\frac{1}{\sigma}}\left(\int_0^1\frac{\nu\left[\frac{1+\nu}{2}|z'(\vartheta_1)|^{\sigma}+\frac{1-\nu}{2}|z'(\vartheta_2)|^{\sigma}\right]d\nu}{(X_2(\vartheta_1,\vartheta_2;\nu))^2}\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_0^1\frac{(1-\nu)\,d\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^2}\right)^{1-\frac{1}{\sigma}}\left(\int_0^1\frac{(1-\nu)\left[\frac{\nu}{2}|z'(\vartheta_1)|^{\sigma}+\frac{2-\nu}{2}|z'(\vartheta_2)|^{\sigma}\right]d\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^2}\right)^{\frac{1}{\sigma}}\right\}. \quad (2.\ 22)$$

Calculations of the integrals in (2. 22) give the inequality (2. 21). □

Corollary 2.12. *According to the assumptions of Theorem 2.11 and for* $\sigma = 1$, *we have*

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \le \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1) \times \{[\mu_1(\vartheta_1,\vartheta_2)+\mu_2(\vartheta_2,\vartheta_1)]\,|z'(\vartheta_1)| + [\mu_2(\vartheta_1,\vartheta_2+\mu_1(\vartheta_2,\vartheta_1))]\,|z'(\vartheta_2)|\}, \quad (2.\ 23)$$

where $\mu_1(\vartheta_1,\vartheta_2)$, $\mu_2(\vartheta_2,\vartheta_1)$, $\mu_2(\vartheta_1,\vartheta_2$ *and* $\mu_1(\vartheta_2,\vartheta_1))$ *are defined in Theorem 2.11.*

Theorem 2.13. *With reference to Condition A and* $\sigma > 1$, $\frac{1}{\xi}+\frac{1}{\sigma}=1$, *then*

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \le \frac{\vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)}{(\xi+1)^{\frac{1}{\xi}}}\left\{\left(\gamma_1(\vartheta_1,\vartheta_2)\,|z'(\vartheta_1)|^{\sigma}+\gamma_2(\vartheta_1,\vartheta_2)\,|z'(\vartheta_2)|^{\sigma}\right)^{\frac{1}{\sigma}} + \left(\gamma_3(\vartheta_1,\vartheta_2)\,|z'(\vartheta_1)|^{\sigma}+\gamma_4(\vartheta_1,\vartheta_2)\,|z'(\vartheta_2)|^{\sigma}\right)^{\frac{1}{\sigma}}\right\}, \quad (2.\ 24)$$

where $\gamma_1(\vartheta_1,\vartheta_2)$, $\gamma_2(\vartheta_1,\vartheta_2)$, $\gamma_3(\vartheta_1,\vartheta_2)$ *and* $\gamma_4(\vartheta_1,\vartheta_2)$ *are defined in Theorem 2.5.*

Proof. By using the second equality of Lemma 2.1, application of the Hölder inequality and the usage of harmonic-convexity of $\left|z^{'}\right|^{\sigma}$ on $[\vartheta_1,\vartheta_2]$ for $\sigma > 1$, gives

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \le \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1) \times \left\{\left(\int_0^1 \nu^{\xi}d\nu\right)^{\frac{1}{\xi}}\left(\int_0^1 \frac{\left[\frac{1+\nu}{2}\,|z'(\vartheta_1)|^{\sigma}+\frac{1-\nu}{2}\,|z'(\vartheta_2)|^{\sigma}\right]d\nu}{(X_2(\vartheta_1,\vartheta_2;\nu))^{2\sigma}}\right)^{\frac{1}{\sigma}} + \left(\int_0^1 (1-\nu)^{\xi}\,d\nu\right)^{\frac{1}{\xi}}\left(\int_0^1 \frac{\left[\frac{\nu}{2}\,|z'(\vartheta_1)|^{\sigma}+\frac{2-\nu}{2}\,|z'(\vartheta_2)|^{\sigma}\right]d\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^{2\sigma}}\right)^{\frac{1}{\sigma}}\right\}.$$

After evaluating the integrals in the above inequality, we get (2. 24). □

Theorem 2.14. *With reference to Condition A and $\sigma > 1$, $\frac{1}{\xi} + \frac{1}{\sigma} = 1$, then*

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1) \times \left\{[\gamma_5(\vartheta_1,\vartheta_2)]^{\frac{1}{\xi}}\left(\frac{(2\sigma+3)|z'(\vartheta_1)|^\sigma + |z'(\vartheta_2)|^\sigma}{2(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}} + [\gamma_6(\vartheta_1,\vartheta_2)]^{\frac{1}{\xi}}\left(\frac{|z'(\vartheta_1)|^\sigma + (2\sigma+3)|z'(\vartheta_2)|^\sigma}{2(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}\right\}, \quad (2.\ 25)$$

where $\gamma_5(\vartheta_1,\vartheta_2)$ and $\gamma_6(\vartheta_1,\vartheta_2)$ are defined in Theorem 2.6.

Proof. Usage of the second equality of Lemma 2.1, application of the Hölder inequality and applying the harmonic-convexity of $\left|z'\right|^\sigma$ on $[\vartheta_1,\vartheta_2]$ for $\sigma > 1$, we get

$$\left|\frac{z(\vartheta_1)+z(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} z(x)\,dx\right| \leq \vartheta_1\vartheta_2(\vartheta_2-\vartheta_1)\left\{\left(\int_0^1 \frac{1}{(X_2(\vartheta_1,\vartheta_2;\nu))^{2\xi}}d\nu\right)^{\frac{1}{\xi}} \times \left(\int_0^1 \nu^\sigma\left[\frac{1+\nu}{2}|z'(\vartheta_1)|^\sigma + \frac{1-\nu}{2}|z'(\vartheta_2)|^\sigma\right]d\nu\right)^{\frac{1}{\sigma}} + \left(\int_0^1 \frac{d\nu}{(X_1(\vartheta_1,\vartheta_2;\nu))^{2\xi}}\right)^{\frac{1}{\xi}} \times \left(\int_0^1 (1-\nu)^\sigma\left[\frac{\nu}{2}|z'(\vartheta_1)|^\sigma + \frac{2-\nu}{2}|z'(\vartheta_2)|^\sigma\right]\right)^{\frac{1}{\sigma}} d\nu\right\}. \quad (2.\ 26)$$

By evaluating each integral in (2. 20), we get (2. 19). □

3. Comparison of the Results

In this section we give a numerical comparison of our results given in Theorem 2.11, Theorem 2.13 and Theorem 2.14 with those given in Theorem 1.3 and Theorem 1.4. Let E_1, E_2, E_3, E_4 and E_5 be the error bounds in Theorem 1.3, Theorem 1.4, Theorem 2.11, Theorem 2.13 and Theorem 2.14 respectively, then the following tables have been prepared to compare the results by using the software Mathematica From Table 1 and Table 2, it is obvious that the error bounds in Theorem 2.11, Theorem 2.13 and Theorem 2.14 are better than those given in Theorem 1.3, Theorem 1.4.

TABLE 1

	E_1	E_3
$\vartheta_1 = 5, \vartheta_2 = 10, \sigma = \frac{5}{4}, z'(x) = x$	11.1168	11.0697
$\vartheta_1 = 0.5, \vartheta_2 = 0.75, \sigma = \frac{5}{4}, z'(x) = x$	0.0416266	0.0415459
$\vartheta_1 = 5, \vartheta_2 = 10, \sigma = 2, z'(x) = x$	11.4336	11.2709
$\vartheta_1 = 0.5, \vartheta_2 = 0.75, \sigma = 2, z'(x) = x$	0.0421681	0.041863

TABLE 2

	E_2	E_4	E_5
$\vartheta_1 = 5, \vartheta_2 = 10, \sigma = \frac{5}{4}, \xi = 5, z'(x) = x$	14.7838	14.3739	11.6059
$\vartheta_1 = 0.5, \vartheta_2 = 0.75, \sigma = \frac{5}{4}, \xi = 5, z'(x) = x$	0.0570742	0.0564602	0.0432951
$\vartheta_1 = 5, \vartheta_2 = 10, \sigma = 2, \xi = 2, z'(x) = x$	13.8193	12.5324	12.2711
$\vartheta_1 = 0.5, \vartheta_2 = 0.75, \sigma = 2, \xi = 2, z'(x) = x$	0.0495933	0.041863	0.0476111

4. ACKNOWLEDGEMENT

The authors are thankful to the anonymous reviewers for their very useful and constructive comments which have been incorporated in the revised version of the manuscript.

REFERENCES

[1] F. Chen and S. Wu, *Hermite-Hadamard type inequalities for harmonically s-convex functions*, Sci. World **7**, (2014) Article ID 279158.

[2] F. Chen and S. Wu, *Fejér and Hermite-Hadamard type inequalities for harmonically convex functions*, J. Appl. Math. (2014), Article ID 386806, 6 pages.

[3] H. Darwish and A. M. Lashin, *S. Soileh, Fekete-Szego type coefficient inequalities for certain subclasses of analytic functions involving salagean pperator*, Punjab Univ. j. math. **48**, No. 2 (2016) 65-80.

[4] S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11**, No. 5 (1998) 91-95.

[5] S. S. Dragomir, *Generalization, refinement and reverses of the right Fejer inequality for convex functions*, Punjab Univ. j. math. **49**, No. 2 (2017)1-13.

[6] J. Hadamard, *Étude sur les Propriétés des Fonctions Entières en Particulier d'une Fonction Considérée par Riemann*, Journal de Mathématique Pures et Appliquées **58**, 171-215.

[7] C. -Y. He, Y. Wang, B. -Y. Xi and F. Qi, *Hermite–Hadamard type inequalities for (α, m)-HA and strongly (α, m)-HA convex functions*, J. Nonlinear Sci. Appl. **10**, (2017) 205–214.

[8] Ch. Hermite, *Sur deux limites d'une integrale define*, Mathesis **3**, (1883) 82.

[9] S. Hussain and S. Qaisar, *New integral inequalities of the type of Hermite-Hadamard through quasi convexity*, Punjab Univ. j. math. **45**, (2013) 33-38.

[10] İ. İşcan and M. Kunt, *Fejér and Hermite-Hadamard-Fejér type inequalities for harmonically s-convex functions via Fractional Integrals*, The Australian Journal of Mathematical Analysis and Applications, **12**, No. 1 (2015) 1-6.

[11] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics **43**, No. 6 (2014) 935-942.

[12] İ. İşcan and S. Wu, *Hermite-Hadamard type inequalities for harmonically-convex functions via fractional integrals*, Applied Mathematics and Computation **238**, (2014) 237–244.

[13] İ. İşcan and M. Kunt, *Hermite-Hadamard-Fejer type inequalities for harmonically convex functions via fractional integrals*, RGMIA Research Report Collection **18**, (2015) Article 107, pp. 1-16.

[14] M. A. Khan, Y. Khurshid, S. S. Dragomir and R. Ullah, *New Hermite-Hadamard type inequalities with applications*, Punjab Univ. j. math. **50**, No. 3 (2018) 1-12.

[15] M. A. Khan, Y. Khurshid, T. Ali and N. Rehman, *Inequalities for three times differentiable functions*, Punjab Univ. j. math. **48**, No. 2 (2016) 35-48.

[16] M. A. Latif, S. S. Dragomir and E. Momoniat, *Fejér type inequalities for harmonically-convex functions with applications*, Journal of Applied Analysis and Computation **7**, No. 3 (2017) 795-813.

[17] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some Fejér type inequalities for harmonically-convex functions with applications to special means*, International Journal of Analysis and Applications **13**, No. 1 (2017) 1-14.

[18] M. A. Latif, *New Hermite Hadamard type integral inequalities for GA-convex functions with applications*, Analysis **34**, No. 4 (2014) 379–389.

[19] M. A. Latif, S. S. Dragomir and E. Momoniat, *Fejér type integral inequalities related with geometrically-arithmetically-convex functions with applications* (Submitted).

[20] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some Fejér type integral inequalities related with geometrically-arithmetically-convex functions with applications* (Accepted).

[21] M. A. Latif, *New Fejer and Hermite-Hadamard type inequalities for differentiable p-convex mappings*, Punjab Univ. j. math. **51**, No. 2 (2019) 39-59.

[22] M. A. Latif and W. Irshad, *Some Fejer and Hermite-Hadamard type inequalities considering ϵ-convex and (σ, ϵ)-convex functions*, Punjab Univ. j. math. **50**, No. 3 (2018) 13-24.

[23] M. A. Latif, *Estimates of Hermite-Hadamard inequality for twice differentiable harmonically-convex functions with applications*, Punjab Univ. j. math. **50**, No. 1 (2018) 1-13.

[24] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some ϕ-analogues of Hermite-Hadamard inequality for s-convex functions in the second sense and related estimates*, Punjab Univ. j. math. **48**, No. 2 (2016) 147-166.

[25] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some weighted Hermite-Hadamard-Noor type inequalities for differentiable preinvex and quasi preinvex functions*, Punjab Univ. j. math. **47**, No. 1 (2015) 57-72.

[26] B. Meftah, *Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of derivatives are co-ordinated log-preinvex*, Punjab Univ. j. math. **51**, No. 2 (2019) 21-37.

[27] M. Muddassar and A. Ali, *New integral inequalities through generalized convex functions*, Punjab Univ. j. math. **46**, No. 2 (2014) 47-51.

[28] M. Muddassar and M. I. Bhatti, *Some generalizations of Hermite-Hadamard type integral inequalities and their applications*, Punjab Univ. j. math. **46**, No. 1 (2014) 9-18.

[29] M. A. Noor, K. I. Noor and S. Iftikhar, *Nonconvex functions and integral inequalities*, Punjab Univ. j. math. **47**, No. 2 (2015) 19-27.

[30] C. P. Niculescu, *Convexity according to means,* Math. Inequal, Appl. **6**, No. 4 (2003) 571–579.

[31] C. P. Niculescu, *Convexity according to geometric mean*, Math. Inequal. Appl. **3**, No. 2 (2000) 155-167.

[32] W. Wang, İ. İscan and H. Zhou, *Fractional integral inequalities of Hermite-Hadamard type for m-HH convex functions with applications*, Advanced Studies in Contemporary Mathematics (Kyungshang) **26**, No. 3 (2016) 501-512.

[33] W. Wang and J. Qi, *Some new estimates of Hermite-Hadamard inequalities for harmonically convex functions with applications*, International Journal of Analysis and Applications **13**, No. 1 (2017) 15-21.

[34] T. -Y. Zhang and Feng Qi, *Integral inequalities of Hermite-Hadamard type for m-AH convex functions*, Turkish Journal of Analysis and Number Theory **3**, No. 2 (2014) 60-64.