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Comparative Analysis of Nonlinear Thirteenth Order Boundary Value Problems Utilizing OHAM and HPM

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Abstract. In this article, we employed Homotopy Perturbation Method (HPM) and Optimal Homotopy Asymptotic Method (OHAM) to investigate the semi analytical solution of thirteenth order boundary value problems. These analytical outcomes have been achieved in phrase of convergent series with simply computable constituents. Comparing the outcomes with exact solution, Variational Iterative Method (VIM) and Differential Transformative Method (DTM), excellent agreement has been found. It is observed that HPM and OHAM provide better solution than VIM and DTM which demonstrates the effectiveness, potentiality and validity of the proposed methods in reality and the acquired results are of top-level precision.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
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1. Introduction

The thirteenth-order boundary value problem is under consideration

$$\frac{d^{13}\check{\ell}}{dx^{13}} = f(t, \check{\ell}(t)),\ a \leq t \leq b$$
$$\check{\ell}^{(i)} = A_i,\ \check{\ell}^{(j)} = B_j,$$
$$i = 0, 1, 2, 3, 4, 5, 6.\ j = 0, 1, 2, 3, 4, 5. \tag{1. 1}$$

From last few decades, boundary value phenomena have great significance due to its wide-spread application in the field of engineering and physical sciences. Different types of boundary value flow problems occur in literature such as Marangoni boundary layer flow, natural convected boundary layer flow, asymmetric boundary layer flow, MHD boundary layer flow, atmospheric boundary layer flow through wind turbines and wind farms and power-law fluid based Falkner Skan boundary layer flow. Many researchers focus their observation to solve such types of problems by utilizing analytical, approximate and exact methods. To solve operator equations, approximate analytical techniques have been extensively used in last few years among these methodologies. To facilitate hard and long calculation some competent refinement has been made by Adomian Decomposition Method (ADM). The modified ADM illustrated promising mechanism for solution of linear and nonlinear based structure, Wazwaz and Al-Sayed [36] demonstrated ADM methodology for analytical and approximate solution of Riccati equations, ADM performs an excellent agreement while solving these types of equations [6] where ADM has been used to locate the series solution of multipoint boundary value problems. Aslam [2] used an iterative procedure which is based on decomposition process to find out the analytical result of fifth-order boundary value problem. Application of higher order boundary value problems are growing attention in many fields of physical sciences.

For solving different linear and nonlinear initial and boundary value problems, HPM [12, 13, 14], Perturbation Iterative Method (PIM) [15], ADM and Laplace transformation [3] have been extensively used in literature. Numerous mathematicians preferred to apply VIM [1], HPM, OHAM [29] and DTM [20] for solution of manipulative estimated system of linear and nonlinear differential equations [7, 32], rational conclusion for KdV, K(2,2), Burgers and cubic based Boussinesq equations [36]. To overcome shortcoming of perturbation methods, VIM and HPM both applying for simulation of heat transfer equations by porous media [10], though VIM need assessment study of Lagrange multiplier [36] if the solution exists it performs high accuracy and near to exact results. The convergence rate of VIM has been studied in [31] which illustrated that VIM is easy to interpret and based on few approximate terms for high precision. DTM [4] has been applied for analysis of revolving non-prismatic rays. Analysis of this method performs efficiently and reliably for lower and higher order boundary value problems.

To find out the solution of different physical problems, a lot of concentration has been devoted to solve different higher order boundary values problems by using approximate methods. In this regard, Sirajul Haq and Idress [11] reported the eighth order initial and boundary value problems using OHAM. They claimed that OHAM is useful for problems with large domains. Mohyud and Ahmet [8] solved the ninth and tenth order boundary value problems by taking different examples to verify the reliability of the HPM scheme. Further, Wazwaz [35] demonstrated that Modified Decomposition Method (MDM) simulated the ninth, tenth and twelfth order boundary value problems. Iftikhar and Rehman [21] represented the solution of thirteenth order boundary value problems by DTM. it has also been observed that HPM [16, 17, 18, 34] is working well for all other linear and non-linear boundary value problems. Moreover, some other numerical methods [5, 22, 23, 24, 25, 30, 33] have been used for the solution of different physical and boundary value problems.

From the literature survey, it is examined that no one yet attempts the thirteenth order boundary value problems by OHAM and HPM. Inspired by the above admirable and widely used semi analytical methods, we anticipated two methods, OHAM and HPM to solve thirteenth order boundary value problems, which yield encouraging solutions. The main purpose of applying both methods is to locate highly accurate analytical results for existing thirteenth order boundary value problems. OHAM and HPM are efficient and easier to interpret. The difference between these methods is of mathematical form of auxiliary function. OHAM is uses auxiliary constants, while HPM does not. However, both methods show excellent performance over the solution of proposed boundary value problems; observing the minimum numerical error with exact solution. In this paper, OHAM and HPM are used in order to investigate the thirteenth order boundary value problems. For instance, the papers of Esmaeilpour and Ganji [9], Mabood et al. [26, 27], Herisanu and Marinca [19] have established the efficiency and generalizability of OHAM.

2. Basic idea of HPM

The mathematical structure of HPM is given by considering the nonlinear differential equation with boundary value conditions

$$C(\check{\ell}) = f(n); n \in \Omega,\ K\left(\check{\ell}, \frac{\partial \check{\ell}}{\partial s}\right), n \in I \tag{2. 2}$$

where $C(\check{\ell})$ represents differential operator, K is a boundary operator, $f(n)$ is an analytical function, I stands for the boundary of the domain Ω. Suppose Equation (2.2) may be rephrased as:

$$H(\check{\ell}) + N(\check{\ell}) = f(n). \tag{2. 3}$$

Applying a homotopy

$$\alpha(n, h) : \Omega \times [0, 1] \longrightarrow M, \tag{2. 4}$$

which satisfies

$$K(\alpha, h) = (1 - h)[(H(v) - H(w_0)] + h[C(\alpha) - f(n)] = 0, \tag{2. 5}$$

where $h \in$[0,1] is an inserting parameter, consider as a small parameter, and the primary estimation of equation (2.2) is w_0. By applying Homotopy technique on equation (2.5), we can get the series solution of equation (2.5), as follows:

$$v = v_0 + hv_1 + h^2 v_2 + h^3 v_3 + ..., \tag{2. 6}$$

if $h \longrightarrow$1 equation (2.6) will converges to the approximate solution of equation (2.5), i.e.

$$\check{\ell}(x) = \lim_{h \longrightarrow 1} v = v_0 + v_1 + v_2 + v_3 + ... \tag{2. 7}$$

3. Fundamental Theme of OHAM

The essential ideology of OHAM follow as:
(a) The subsequent differential equation with boundary constraint is under consideration:

$$L(z(x)) + g(x) + N(z(x)) = 0\ , K\left(z, \frac{dz}{dx}\right), \tag{3. 8}$$

where L and N denotes the linear and non-linear operators correspondingly, independent variable shows by x, $z(x)$ is unspecified function, $g(x)$ is defined function and K is a boundary operator.
(b) An equation suggested as bend equation is created

$$(1-r)[L(\check{\phi}(x,r))+g(x)] = S(r)[L(\check{\phi}(x,r))+g(x)+N(\check{\phi}(x,r))]\ K\left(\check{\phi}(x,r), \frac{\partial \check{\phi}(x,r)}{\partial x}\right) = 0, \tag{3. 9}$$

where $r \in [0,1]$is an inserting variable, the convergence of the general solution is dependent on $S(r)$, which is stand for an auxiliary function.
(c) The Taylor series about r is an expanded approximate solution for $\check{\phi}(x,r)$.

$$\check{\phi}(x,r,L_i) = z_0(x) + z_1(x,L_1)r^1 + z_2(x,L_2)r^2.... \tag{3. 10}$$

The rate of convergence of the series (3.10) has been observed by previous researchers which is dependent upon the auxiliary constants L_i.... If it is converges then we obtain

$$z(x,L_1,\tilde{L_2},...,L_m) = z_0(x) + \sum_{i=1}^{\infty} z_i(x,L_1,L_2,...,L_m). \tag{3. 11}$$

(d) Substituting (3.10) into (3.8) we have mention residual:

$$R(x,L_1,L_2,\ldots,L_m) = L(\bar{z}(x,L_1,L_2,\ldots,L_m)) + N(\bar{z}(x,L_1,L_2,\ldots,L_m)). \tag{3. 12}$$

If $R(x,L_1,L_2,...,L_m)$=0, then the exact solution will be $\bar{z}$. Generally, this case will not be for non-linear problems. For considering L_1, L_2... the method of least square can be used.
(e) Finally put these constants in (3.11) and can get the approximate solution.

4. Example 1

4.1. **HPM Solution.**

Consider the thirteenth order linear boundary value problem

$$\check{\ell}^{(13)}(x) = -\sin(x) + \cos(x)\ , x \in [0,1], \tag{4. 13}$$

with boundary constraints:
$\check{\ell}(0) = 1, \check{\ell}^{(1)}(0) = 1, \check{\ell}^{(2)}(0) = 1, \check{\ell}^{(3)}(0) = -1, \check{\ell}^{(4)}(0) = 1, \check{\ell}^{(5)}(0) = 1, \check{\ell}^{(6)}(0) = -1$
$\check{\ell}(1) = \sin(1)+\cos(1), \check{\ell}^{(1)}(1) = -\sin(1)+\cos(1), \check{\ell}^{(2)}(1) = -\sin(1)-\cos(1), \check{\ell}^{(3)}(1) = \sin(1)-\cos(1), \check{\ell}^{(4)}(1) = \sin(1)+\cos(1), \check{\ell}^{(5)}(1) = -\sin(1)+\cos(1)$. where $\check{\ell}(x) = \sin(x)+\cos(x)$ shows the exact out come of the problem. By applying the HPM scheme on (4.13) we attained

- **Zeroth order problem**

$$\check{\ell}_0^{(13)}(x) = 0, \tag{4. 14}$$

with boundary constraints:
$\check{\ell}_0(0) = 1, \check{\ell}_0^{(1)}(0) = 1, \check{\ell}_0^{(2)}(0) = 1, \check{\ell}_0^{(3)}(0) = -1, \check{\ell}_0^{(4)}(0) = 1, \check{\ell}_0^{(5)}(0) = 1, \check{\ell}_0^{(6)}(0) = -1,$
$\check{\ell}_0(1) = \sin(1)+\cos(1), \check{\ell}_0^{(1)}(1) = -\sin(1)+\cos(1), \check{\ell}_0^{(2)}(1) = -\sin(1)-\cos(1), \check{\ell}_0^{(3)}(1) = \sin(1)-\cos(1), \check{\ell}_0^{(4)}(1) = \sin(1)+\cos(1), \check{\ell}_0^{(5)}(1) = \cos(1)-\sin(1),$
it's solution is

$$\check{\ell}_0(x) = 1 + 1x - 0.5x^2 - \ldots + 2.774138104741695 \times 10^{-9}x^{12}. \tag{4. 15}$$

- **First order problem**

$$\check{\ell}_1^{(13)}(x) = [\cos(x) - \sin(x)], \tag{4. 16}$$

with boundary constraints:
$\check{\ell}_1(0) = 0, \check{\ell}_1^{(1)}(0) = 0, \check{\ell}_1^{(2)}(0) = 0, \check{\ell}_1^{(3)}(0) = 0, \check{\ell}_1^{(4)}(0) = 0, \check{\ell}_1^{(5)}(0) = 0, \check{\ell}_1^{(6)}(0) = 0,$

$\check{\ell}_1(1) = 0, \check{\ell}_1^{(1)}(1) = 0, \check{\ell}_1^{(2)}(1) = 0, \check{\ell}_1^{(3)}(1) = 0, \check{\ell}_1^{(4)}(1) = 0, \check{\ell}_1^{(5)}(1) = 0.$
It's solution is

$$\check{\ell}_1(x) = \frac{1}{720}\left(-720 + 720x + 360x^2 + 120x^3 + ... + 450456x^{12}\sin(1) + 720\sin(x)\right). \quad (4.\ 17)$$

Using (4.15) and (4.17) we get the solution

$$\check{\ell}(x) = \check{\ell}_0(x) + \check{\ell}_1(x) \quad (4.\ 18)$$

4.2. **OHAM Solution.** Consider the thirteenth order linear boundary value problem

$$\check{\ell}^{(13)}(x) = -\sin(x) + \cos(x),\ x \in [0, 1] \quad (4.\ 19)$$

with boundary constraints:
$\check{\ell}(0) = 1, \check{\ell}^{(1)}(0) = 1, \check{\ell}^{(2)}(0) = 1, \check{\ell}^{(3)}(0) = -1, \check{\ell}^{(4)}(0) = 1, \check{\ell}^{(5)}(0) = 1, \check{\ell}^{(6)}(0) = -1,$
$\check{\ell}(1) = \sin(1)+\cos(1), \check{\ell}^{(1)}(1) = -\sin(1)+\cos(1), \check{\ell}^{(2)}(1) = -\sin(1)-\cos(1), \check{\ell}^{(3)}(1) = \sin(1) - \cos(1)+, \check{\ell}^{(4)}(1) = \sin(1) + \cos(1), \check{\ell}^{(5)}(1) = -\sin(1) + \cos(1).$
where $\check{\ell}(x) = \sin(x) + \cos(x)$ presented the exact solution of the problem designed
By applying the OHAM to (4.19) we acquire

- **Zeroth order problem**

$$\check{\ell}_0^{(13)}(x) = 0, \quad (4.\ 20)$$

with boundary constraints:
$\check{\ell}_0(0) = 1, \check{\ell}_0^{(1)}(0) = 1, \check{\ell}_0^{(2)}(0) = 1, \check{\ell}_0^{(3)}(0) = -1, \check{\ell}_0^{(4)}(0) = 1, \check{\ell}_0^{(5)}(0) = 1, \check{\ell}_0^{(6)}(0) = -1,$
$\check{\ell}_0(1) = \sin(1)+\cos(1), \check{\ell}_0^{(1)}(1) = -\sin(1)+\cos(1), \check{\ell}_0^{(2)}(1) = -\sin(1)-\cos(1), \check{\ell}_0^{(3)}(1) = \sin(1) - \cos(1), \check{\ell}_0^{(4)}(1) = \sin(1) + \cos(1), \check{\ell}_0^{(5)}(1) = -\sin(1) + \cos(1).$
It's solution is

$$\check{\ell}_0(x) = 1+1x-0.5x^2-0.16666666666666666x^3+\ldots+2.774138104741695\times10^{-9}x^{12}. \quad (4.\ 21)$$

- **First order problem**

$$\check{\ell}_1^{(13)}(x, L_1) = (1 + L_1)\check{\ell}_0^{(13)}(x) + L_1[-\sin(x) + \cos(x)], \quad (4.\ 22)$$

with boundary constraints:
$\check{\ell}_1(0) = 0, \check{\ell}_1^{(1)}(0) = 0, \check{\ell}_1^{(2)}(0) = 0, \check{\ell}_1^{(3)}(0) = 0, \check{\ell}_1^{(4)}(0) = 0, \check{\ell}_1^{(5)}(0) = 0, \check{\ell}_1^{(6)}(0) = 0,$
$\check{\ell}_1(1) = 0, \check{\ell}_1^{(1)}(1) = 0, \check{\ell}_1^{(2)}(1) = 0, \check{\ell}_1^{(3)}(1) = 0, \check{\ell}_1^{(4)}(1) = 0, \check{\ell}_1^{(5)}(1) = 0.$
It's solution is

$$\check{\ell}_1(x) = \frac{1}{720}\left(720L_1 + 720xL_1 - 360x^2L_1 +\ \ldots - 450456x^{12}\sin(1)L_1 - 720\sin(x)L_1\right). \quad (4.\ 23)$$

- **Second order problem**

$$\check{\ell}_2^{(13)}(x, L_1, L_2) = (1 + L_1)\check{\ell}_1^{(13)}(x) - L_2[\cos(x) - \sin(x)] + L_2\check{\ell}_0^{(13)}(x), \quad (4.\ 24)$$

with boundary constraints:
$\check{\ell}_2(0) = 0, \check{\ell}_2^{(1)}(0) = 0, \check{\ell}_2^{(2)}(0) = 0, \check{\ell}_2^{(3)}(0) = 0, \check{\ell}_2^{(4)}(0) = 0, \check{\ell}_2^{(5)}(0) = 0, \check{\ell}_2^{(6)}(0) = 0,$

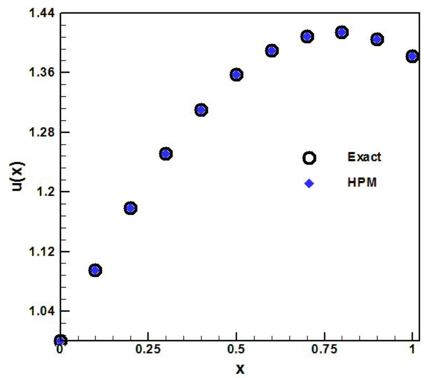

FIGURE 1. Comparison of exact solution with HPM

TABLE 1. Comparison of absolute error of HPM and OHAM with VIM [1] and DTM [20].

x	Exact solution	Exact-HPM	Exact-OHAM	Exact-VIM	Exact-DTM
0	1	0	0	0	0
0.1	1.094	2.2204×10^{-16}	0	3.885×10^{-15}	3.885×10^{-15}
0.2	1.178	0	0	1.462×10^{-13}	1.462×10^{-13}
0.3	1.250	2.2204×10^{-16}	2.2204×10^{-16}	8.805×10^{-13}	8.805×10^{-13}
0.4	1.310	8.8817×10^{-16}	8.8817×10^{-16}	2.358×10^{-12}	2.358×10^{-12}
0.5	1.357	2.2204×10^{-16}	4.4408×10^{-16}	3.801×10^{-12}	3.801×10^{-12}
0.6	1.389	6.8833×10^{-15}	2.0650×10^{-14}	5.147×10^{-11}	5.147×10^{-11}
0.7	1.409	3.8857×10^{-14}	1.9317×10^{-14}	1.562×10^{-11}	1.562×10^{-11}
0.8	1.414	3.1397×10^{-13}	2.8377×10^{-13}	8.994×10^{-11}	8.994×10^{-11}
0.9	1.404	5.8619×10^{-13}	1.2145×10^{-13}	4.700×10^{-10}	4.700×10^{-10}
1	1.381	1.6187×10^{-13}	8.0602×10^{-14}	2.063×10^{-9}	2.063×10^{-9}

$\check{\ell}_2(1)=0, \check{\ell}_2^{(1)}(1)=0, \check{\ell}_2^{(2)}(1)=0, \check{\ell}_2^{(3)}(1)=0, \check{\ell}_2^{(4)}(1)=0, \check{\ell}_2^{(5)}(1)=0.$
It's solution is too long, here we take few terms

$$\check{\ell}_2(x)=\frac{1}{720}\left(720L_1+720xL_1-360x^2L_1-\ldots-450456x^{12}\sin(1)L_2-720\sin(x)L_2\right), \tag{4. 25}$$

using Eq's (4.21), (4.23) and (4.25) the second order approximate solution by OHAM for $r=1$

$$\tilde{\check{\ell}}(x,L_1,L_2)=\check{\ell}_0(x)+\check{\ell}_1(x,L_1)+\check{\ell}_2(x,L_1,L_2), \tag{4. 26}$$

we use the least square procedure to acquire the unknown convergent constants in $\tilde{\check{\ell}}$. Switching the values of L_1 and L_2 in (4.26) we attain the second order approximation utilizing OHAM.

5. Example 2

5.1. **HPM Solution.** The thirteenth order non-linear boundary value problem esteemed as:

$$\check{\ell}^{(13)}(x) = e^{-x}\check{\ell}^2(x)\; x \in [0,1], \tag{5. 27}$$

$\check{\ell}(0)=1, \check{\ell}^{(1)}(0)=1, \check{\ell}^{(2)}(0)=1, \check{\ell}^{(3)}(0)=1, \check{\ell}^{(4)}(0)=1, \check{\ell}^{(5)}(0)=1, \check{\ell}^{(6)}(0)=1,$
$\check{\ell}(1)=e, \check{\ell}^{(1)}(1)=e, \check{\ell}^{(2)}(1)=e, \check{\ell}^{(3)}(1)=e, \check{\ell}^{(4)}(1)=e, \check{\ell}^{(5)}(1)=e,$
where $\check{\ell}(x)=e^x$ shows the exact solution of problem.
By applying HPM to (5.27) we conclude
Zeroth order problem

$$\check{\ell}_0^{(13)}(x) = 0, \tag{5. 28}$$

with boundary constraints:
$\check{\ell}_0(0)=1, \check{\ell}_0^{(1)}(0)=1, \check{\ell}_0^{(2)}(0)=1, \check{\ell}_0^{(3)}(0)=1, \check{\ell}_0^{(4)}(0)=1, \check{\ell}_0^{(5)}(0)=1, \check{\ell}_0^{(6)}(0)=1,$
$\check{\ell}_0(1)=e, \check{\ell}_0^{(1)}(1)=e, \check{\ell}_0^{(2)}(1)=e, \check{\ell}_0^{(3)}(1)=e, \check{\ell}_0^{(4)}(1)=e, \check{\ell}_0^{(5)}(1)=e.$
It's solution is

$$\check{\ell}_0(x) = \frac{1}{720}(720+720x+360x^2+120x^3+30x^4+6x^5+...+566827x^{12}). \tag{5. 29}$$

Using (5.29) we get the solution,

$$\check{\ell}(x) = \check{\ell}_0(x). \tag{5. 30}$$

5.2. **OHAM Solution.** The nonlinear thirteenth order boundary value problem is follow as:

$$\check{\ell}^{(13)}(x) = e^{-x}\check{\ell}^2(x),\; x \in [0,1] \tag{5. 31}$$

with boundary constraints
$\check{\ell}(0)=1, \check{\ell}^{(1)}(0)=1, \check{\ell}^{(2)}(0)=1, \check{\ell}^{(3)}(0)=1, \check{\ell}^{(4)}(0)=1, \check{\ell}^{(5)}(0)=1, \check{\ell}^{(6)}(0)=1,$
$\check{\ell}(1)=e, \check{\ell}^{(1)}(1)=e, \check{\ell}^{(2)}(1)=e, \check{\ell}^{(3)}(1)=e, \ell^{(4)}(1)=e, \check{\ell}^{(5)}(1)=e,$
where $\check{\ell}(x)=e^x$ presented the exact solution of problem.
By applying OHAM to (5.31) we obtain

- **Zeroth order problem**

$$\check{\ell}_0^{(13)}(x) = 0, \tag{5. 32}$$

with boundary constraints:
$\check{\ell}_0(0)=1, \check{\ell}_0^{(1)}(0)=1, \check{\ell}_0^{(2)}(0)=1, \check{\ell}_0^{(3)}(0)=1, \check{\ell}_0^{(4)}(0)=1, \check{\ell}_0^{(5)}(0)=1, \check{\ell}_0^{(6)}(0)=1,$
$\check{\ell}_0(1)=e, \check{\ell}_0^{(1)}(1)=e, \check{\ell}_0^{(2)}(1)=e, \check{\ell}_0^{(3)}(1)=e, \check{\ell}_0^{(4)}(1)=e, \check{\ell}_0^{(5)}(1)=e.$
It's solution is

$$\check{\ell}_0(x) = \frac{1}{720}(720+720x+360x^2+120x^3+\ldots-208524ex^{12}). \tag{5. 33}$$

- **First order problem**

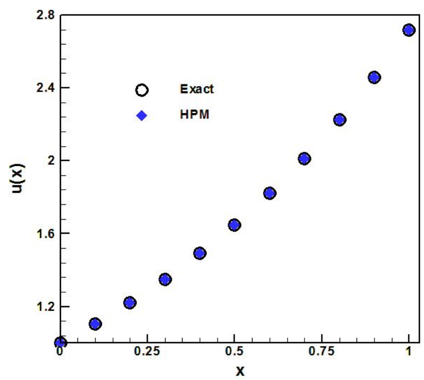

FIGURE 2. Comparison of exact solution with HPM

$$\breve{\ell}_1^{(13)}(x) = (1 + L_1)u_0^{(13)}(x) + e^{-x}L_1u_0^2(x), \tag{5. 34}$$

with boundary constraints:
$\breve{\ell}_1(0) = 0, \breve{\ell}_1^{(1)}(0) = 0, \breve{\ell}_1^{(2)}(0) = 0, \breve{\ell}_1^{(3)}(0) = 0, \breve{\ell}_1^{(4)}(0) = 0, \breve{\ell}_1^{(5)}(0) = 0, \breve{\ell}_1^{(6)}(0) = 0,$
$\breve{\ell}_1(1) = 0, \breve{\ell}_1^{(1)}(1) = 0, \breve{\ell}_1^{(2)}(1) = 0, \breve{\ell}_1^{(3)}(1) = 0, \breve{\ell}_1^{(4)}(1) = 0, \breve{\ell}_1^{(5)}(1) = 0.$
It's solution is too long so we take few terms

$$\breve{\ell}_1(x, L_1) = \frac{1}{518400}e^{(-1-x)}(183427375828478504057 60eL_1 \ldots + 43482258576e^3x^{24}L_1 \tag{5. 35}$$

using (5.33) and (5.35), for $r = 1$ OHAM conclude the first order approximate solution,

$$\tilde{\breve{\ell}}(x, L_1) = \breve{\ell}_0(x) + \breve{\ell}_1(x, L_1), \tag{5. 36}$$

we use the least square scheme to achieve the unknown convergent constants in $\tilde{\breve{\ell}}$. Inserting the values of L_1 in (5.36), OHAM conclude the first order approximation.

6. RESULTS AND DISCUSSION

In order to achieve clear depiction of accuracy and proficiency of proposed methods, tabular and graphical analysis is performed. Table 1 illustrates the comparison of absolute error of HPM and OHAM with VIM and DTM for linear thirteenth order boundary value problem by keeping different values of x. The exact solutions of proposed examples have already been existed. The error difference between exact and approximate solutions via HPM, OHAM, VIM and DTM are demonstrated in Table 1 and Table 2 for the values of $x = 0.0, 0.1, 0.2, ..., 1.0$. In Table 1, the performance of proposed methods reveals that HPM and OHAM much better than VIM and DTM for proposed problem as the error difference between exact and approximate solution of HPM and OHAM is very smaller comparative to VIM and DTM. Table 2 exhibits a comparison of absolute error of HPM and OHAM with VIM and DTM for non-linear thirteenth order boundary value problem. The results indicate that the error difference of approximate and exact solutions of HPM

TABLE 2. Comparison of absolute error of HPM and OHAM with VIM [1] and DTM [20].

x	Exact solution	Exact- HPM	Exact- OHAM	Exact-VIM	Exact- DTM
0	1	0	0	0	0
0.1	1.105	0	0	4.174×10^{-14}	4.171×10^{-14}
0.2	1.221	8.8817×10^{-16}	8.8817×10^{-16}	2.641×10^{-12}	2.641×10^{-12}
0.3	1.349	6.6613×10^{-15}	6.6613×10^{-15}	2.993×10^{-11}	2.993×10^{-11}
0.4	1.491	1.9984×10^{-14}	1.9984×10^{-14}	1.671×10^{-10}	1.671×10^{-10}
0.5	1.648	2.7089×10^{-14}	2.7089×10^{-14}	6.309×10^{-10}	6.309×10^{-10}
0.6	1.822	1.1102×10^{-15}	1.1102×10^{-15}	1.847×10^{-9}	1.847×10^{-9}
0.7	2.013	1.1279×10^{-13}	1.1279×10^{-13}	4.478×10^{-9}	4.478×10^{-9}
0.8	2,225	3.4639×10^{-13}	3.4639×10^{-13}	9.215×10^{-9}	9.215×10^{-9}
0.9	2.459	7.9092×10^{-13}	7.9092×10^{-13}	1.589×10^{-8}	1.589×10^{-8}
1	2.718	8.8284×10^{-13}	8.8284×10^{-13}	2.090×10^{-8}	2.090×10^{-8}

and OHAM smaller than VIM and DTM for all values of x. The comparison of these four methods with each other establishes that OHAM and HPM are superior to VIM and DTM due to approximate solution. A comparison of exact solution with HPM solution is also presented in Figure 1 and 2. These figures show low difference between exact and HPM solutions for both boundary value problems. The solution curves are very smooth and exhibit excellent agreement with exact solution, which conform the validity of HPM.

7. CONCLUSION

In this work, we studied approximate analytical solution of linear and nonlinear thirteenth order boundary value problems via HPM and OHAM. The obtained results propose that the HPM and OHAM could be a valuable and efficient technique in solving linear and nonlinear differential equations. The techniques have advantages over other existing analytical approximation techniques. Interestingly a highly accurate results are obtained with only one or two p-terms of the series using HPM or OHAM. The low error and convergence of both methods are remarkable.

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