Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(6)(2019) pp. 65-80

An Algorithm for Systems of Nonlinear Ordinary Differential Equations Based on Legendre Wavelets

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Received:05 October, 2018 / Accepted:05 March, 2019 / Published online:25 March, 2019

Abstract. In the present paper, we study Legendre Wavelet Method (LWM) and apply an algorithm based on this approach to solve systems of nonlinear differential equations. Differential equations of any degree can be expanded as a series in Legendre polynomials. So depending on the kind of problem, derivative or integral operational matrices appear, after simplification the nonlinear ordinary system changed to an algebraic system which includes the coefficients. In this step the suggested algorithm is approximated the system of coefficients by using an iterative method. For comparison, this equation is solved by Moving Least Squares Method (MLSM) and properties of LWM and MLSM approaches are expressed. These two approaches are applied to solve an equation that shows effect and transferring a kind of virus to a set of statistical society. Numerical results and figures of applying LWM and MLSM are shown finally.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Legendre wavelet method, Nonlinear differential equation, Moving least squares method.

1. Introduction

In recent years, there has been an increase usage among many scientists to apply wavelet technique to solve both linear and nonlinear problems. The main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations. The wavelet uses orthogonal bases in expansion and estimate solution of equations. With this description, the orthogonal functions include the following three classes, first: piecewise constant basis function, that it is locally constant in attached districts, second: orthogonal polynomials, and third : sets of sine cosine functions. The overview of this method can be found in [1, 6, 7, 14, 19, 22].
The essential properties of these three sets in use of multifold problems is converting them to a system of algebraic equations, thus the systems of differential equations are solved by

converting to the simpler system.
In this work we apply the Legendre wavelets method to solve a system of nonlinear differential equation whose unknown function depends on spatial and temporal variables. But the decomposition of this unknown function into Legendre wavelets basis will be done only on the spatial variable. Obviously, the coefcients of this decomposition will depend on the temporal variable. The shifted Legendre, among these orthogonal polynomials is defined on the distance $[0, \alpha]$. They are rather impressive because of in the shorter domain is defined and integral operational matrix is tridiagonal with fixed and unit weight [8].
The wavelet approach has many helpful properties as orthogonality, compact support, accurate exhibition and all of types of orthogonal constructor. Legendre polynomials are the simplest type of polynomials with having the fixed weight function and the other weight functions are the same with sine-cosine Fourier series or derivatives of polynomials and other factors which may change [15, 20].
This paper is organized as follows. Section 1 includes the introduction. Section 2 introduces formulation of LWM and MLSM with their features. Section 3 presents a model contains a kind of virus transfer process and application of LWM and MLSM on the model in their algorithm. In Section 4 numerical results are tabulated. Section 5 concludes.

2. Approaches

This section contains the preliminaries and some details of LWM and MLSM.

2.1. **Legendre Wavelets and its properties.** Structures of wavelets have arrived into many different segments of sciences and technology. A wave is usually defined as an oscillating function of time or space. Fourier analysis is wave analysis, it expands a signal or function in term of sin cosine. A wavelet is a small wave, which has its energy concentrated in time to give a tool for the analysis of transient, nonstationary, or time-varying phenomena. Legendre wavelet method is one of the powerful tools in accuracy and speed of execution in many fields of science to solve various differential equations [26]. Every wavelet needed bases and a collection of functions that created from the mother wavelet matching to kind of wavelet, which are made before [23].
Wavelet constitutes a family of functions constructed from dilation and transition of a single function called the mother wavelet. When the dilation parameter x and the transition parameter y vary continuously, we have the following family of continuous wavelets as [5].

$$\psi_{x,y}(t) = |x|^{-\frac{1}{2}}\psi(\frac{t-y}{x}), \quad x, y \in \mathbb{R}, \ x \neq 0 \qquad (2.\ 1)$$

If we restrict the parameters x and y to the discrete values as $x = x_0^{-k}$, $y = ny_0x_0^{-k}$ where $x_0 = 2$ and $y_0 = 1$ and use of positive integer of n and k discrete wavelets have made in $L^2(\mathbb{R})$ with orthogonal bases [13].

$$\psi_{k,n}(t) = |x_0|^{-\frac{k}{2}}\psi(x_0^k t - ny_0), \qquad (2.\ 2)$$

Legendre wavelet is adapted from Legendre polynomials which is made up of recursive relation

$$(1-x)^2 L''_n(x) - 2xL'_n(x) + n(n+1)L_n(x) = 0, \quad n = 0, 1, 2, ...$$

Where $L_n(x)$ is Legendre polynomials. Every two Legendre polynomials with different degrees and weight function of $w(t) = 1$, are orthogonal in interval $[-1, 1]$.
Arguments that used in Legendre wavelets are $k, m, \widehat{n}$ and t which $\widehat{n} = 2n - 1$, $m = 1, 2, \dots, 2^{k-1}$, and positive integer of k and time of t. Structure of a Legendre wavelet by dilation of $x = 2^{-k}$ and transiation of $y = \widehat{n}2^{-k}$ and combination of arguments on the interval $[0, 1)$ as

$$\psi_{m,n}(t) = \begin{cases} \sqrt{(m + \frac{1}{2})} 2^{\frac{k}{2}} L_m(2^k t - \widehat{n}) & \frac{\widehat{n}-1}{2^k} \leq t \leq \frac{\widehat{n}+1}{2^k} \\ 0 & o.w \end{cases} \tag{2. 3}$$

2.2. **Function Approximation.** A function $f(t)$ defined over $[0, 1)$ can be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t), \tag{2. 4}$$

where $c = \langle f(t), \psi_{nm}(t) \ \rangle$ and $\langle . \rangle$ is the inner product. After infinite series (2. 4) is truncated it can be written as

$$f(t) = \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \tag{2. 5}$$

Where C and $\Psi(t)$ are two vectors, having $2^{k-1}M \times 1$ components [13],

$$C = \ [c_{10}, c_{11}, ..., c_{1M-1}, c_{20}, ..., c_{2M-1}, ..., c_{2^{k-1}0}, ..., c_{2^{k-1}M-1}]^T, \tag{2. 6}$$

$$\Psi(t) = \ [\Psi_{10}(t), \Psi_{11}(t), ..., \Psi_{1M-1}(t), \Psi_{20}(t), ..., \Psi_{2M-1}(t), ..., \Psi_{2^{k-1}0}(t), ... \tag{2. 7}$$
$$, \Psi_{2^{k-1}M-1}(t)]^T.$$

The integration of vector $\Psi(t)$ defined in $Eq.(5)$ is given by

$$\int_0^t \Psi(\alpha) d\alpha = P\Psi(t), \tag{2. 8}$$

Which P is a $(2^{k-1}M) \times (2^{k-1}M)$ operational matrix for integration and it is expressed as follows [24].

$$P = \frac{1}{2^k} \begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L \end{bmatrix}, \tag{2. 9}$$

Where F and L are $M \times M$ sub matrices as follows

$$F = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \end{bmatrix}, L = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Theorem 2.3. *A function $f(x)$ defind on $[0,1)$ is with bounded second derivative, say $|f''(x)| \leq M$, can be expanded as an infinite sum of the Legendre wavelets and the series converges uniformly to $f(x)$ that is*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{2. 10}$$

where $c_{nm} = \langle f(x), \psi_{nm}(x)\rangle$, and $\langle . \rangle$ is the inner product of $f(x)$ and $\psi_{nm}(x)$.

Proof: See [21].

Theorem 2.4. *Any function $f(x)$ defined on $[0,1)$ is with bounded first and second derivatives $|f'(x)| \leq M_1$ and $|f''(x)| \leq M_2$ can be expanded as an infinite sum of the extended Legendre wavelets and the series converges uniformly to $f(x)$ that is*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{2. 11}$$

Proof: Let $f(x)$ be a function defined on $[0,1)$ the bounds M_1 and M_2 for the first and second derivation respectively

$$c_{nm} = \int_0^1 f(x)\psi_{nm}(x)dx = \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(x)\sqrt{2m+1}\mu^{\frac{k}{2}} L_m(\mu^k x - 2n + 1)dx, \tag{2. 12}$$

Let $\widehat{n} = 2n-1$, then by the change of variable $t = 2\mu^k x - n$, we have $dx = \frac{dt}{2\mu^k}$ so

$$\begin{aligned} c_{nm} &= \frac{(2m+1)^{\frac{1}{2}}}{2\mu^{\frac{k}{2}}} \int_{-1}^{1} f(\frac{\widehat{n}+t}{2\mu^k}) L_{m+1}(t)dt, \qquad (2.\ 13) \\ &= \frac{1}{2\mu^{\frac{k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f(\frac{\widehat{n}+t}{2\mu^k}) d(L_{m+1}(t) - L_{m-1}(t)), \end{aligned}$$

Where the following property of the Legendre polynomial is used

$$(2m+1)L_m(t) = L'_{m+1}(t) - L'_{m-1}(t), \tag{2. 14}$$

Integrating in Eq.(2. 13), yields

$$\begin{aligned} c_{nm} &= \frac{1}{2\mu^{\frac{k}{2}}(2m+1)^{\frac{1}{2}}} \{ \frac{1}{2\mu^k} f'(\frac{\widehat{n}+t}{2\mu^k}) d(L_{m+1}(t) - L_{m-1}(t))|_{-1}^{1}, \qquad (2.\ 15) \\ & -\frac{1}{2\mu^k} \int_{-1}^{1} f'(\frac{\widehat{n}+t}{2\mu^k})(L_{m+1}(t) - L_{m-1}(t))dt\}, \end{aligned}$$

From Eq.(2. 15) we have

$$c_{nm} = \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f'(\frac{\widehat{n}+t}{2\mu^k})(L_{m+1}(t) - L_{m-1}(t))dt, \tag{2. 16}$$

Considering Eq.(2. 14)

$$c_{nm} = \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f'(\frac{\widehat{n}+t}{2\mu^k})(\frac{L_{m+2}(t) - L_m(t)}{2m+3} - \frac{L_m(t) - L_{m-2}(t)}{2m-1})dt, \tag{2. 17}$$

Solving the equation similar to the previous step yields

$$c_{nm} = \frac{1}{8\mu^{\frac{5k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f'(\frac{\widehat{n}+t}{2\mu^k})(\frac{L_{m+2}(t)-L_m(t)}{2m+3} - \frac{L_m(t)-L_{m-2}(t)}{2m-1})dt, \quad (2.\ 18)$$

Now with

$$\tau_m(t) = (2m-1)L_{m+2}(2) - 2(2m+1)L_m(t) + (2m+3)L_{m-2}(t),$$

$$c_{nm} = \frac{1}{8\mu^{\frac{5k}{2}}(2m+1)^{\frac{1}{2}}} \frac{1}{(2m-1)(2m+3)} \int_{-1}^{1} f''(\frac{\widehat{n}+t}{2\mu^k})\tau_m(t)dt, \quad (2.\ 19)$$

And also

$$|c_{nm}| \leq \Omega(\mu,k,m) \int_{-1}^{1} |f''(\frac{\widehat{n}+t}{2\mu^k})||\tau_m(t)|dt, \quad (2.\ 20)$$

Where

$$\Omega(\mu,k,m) = \frac{1}{8\mu^{\frac{5k}{2}}(2m+1)^{\frac{1}{2}}} \frac{1}{(2m-1)(2m+3)},$$

It is showing that [15],

$$\int_{-1}^{1} |\tau_m(t)dt \leq \sqrt{24}\frac{2m+3}{\sqrt{2m-3}}, \quad (2.\ 21)$$

Therefore since $n \leq \mu^k$ for $m > 1$ we get

$$|c_{nm}| \leq \frac{\sqrt{6}M_2}{2n^{\frac{5}{2}}(2m-3)^2}, \quad (2.\ 22)$$

Moreover for $m = 1$,

$$|c_{n1}| \leq \frac{M_1}{\sqrt{3}n^{\frac{3}{2}}}. \quad (2.\ 23)$$

Hence the series

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}$$

Is absolutely convergent. For $m = 0$, sequence $\{\psi_{n0}(x)\}_{n=1}^{\infty}$ transforms into an orthogonal system by the Haar function and thus $\sum_{n=1}^{\infty} c_{nm}\psi_{n0}(x)$ is convergent.Consequently, it follows that the series

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{n0}(x)$$

Converges to the function $f(x)$ uniformly.

2.5. **Moving Least Squares Approximation.** In real world many problems are modeled by ordinary differential equations and partial or integral differential equations. Generally such issues cannot be solved in analytical form. Hence development of numerical methods has attached the attention of scientists. One of them is Meshless method due to flexibility of method for solving various models. This numerical method dont need the whole domain of problem and just can be used by select a few points of related domain [16, 27].
In applying of the MLSM let $X = \{x_1, x_2, ..., x_n\} \in \overline{\Omega}$ and at nods X is used $u = \{u_j\}_{j=1}^{N}$ so that $u_j = u(x_j), j = 1, 2, ..., n$, and set of $u^h \in C^s(R^d)$ as the approximate values of u in a weighted square sense, in fact the main objective of least square approximation in a weighted square in domain of problem presents a method which just points nearby nodes X have an effect in approximate function of u. For this purpose weight function is applied to approximate function of u therefore the weighted function of moving least square is considered with the weight function of w.
The space of polynomials of degree m and $d$variable is shown by P_m^d and $q = dim(P_m^d)$ Let $\{p_0, p_1, ..., p_m\}$ be each basis of P_m^d then the prime approximation of u can be defined as

$$u^h(x) = \sum_{i=1}^{m} \phi_i(x)p_i(x) = P^T(x)\Phi(x), \tag{2. 24}$$

Where $P(x)$ is a matrix with components of $\{p_0, p_1, ..., p_q\}$ and $\Phi(x) = \{\phi_1, \phi_2, ..., \phi_m\}$ are shapes functions and they are made by applying nods of X and minimizing in space of Ω as follows

$$min_x[\sum_{i=1}^{n}(u^h(x) - u_j)^2 w_j(x)], \tag{2. 25}$$

That is $w_j(x) = w(x, x_j)$ weight function and $w_j(x) \ > 0, u^h(x_j) \neq u_j$, the matrix of W is diagonal with diameter components of $w_j(x), j = 1, ..., n$ and the matrix of P as

$$P = \begin{bmatrix} p_0(x_1) & \cdots & p_m(x_1) \\ \cdots & \ddots & \cdots \\ p_0(x_1) & \cdots & p_m(x_1) \end{bmatrix}^T, \tag{2. 26}$$

By derivative from the side of (25), and equal to zero, as following

$$[\sum_{j=1}^{n} w_j(x)P(x_j)P^T(x_j)]\Phi(x) = [w_1(x)P(x_1), w_2(x)P(x_2), ..., w_n(x)P(x_n)]u, \tag{2. 27}$$

So that by multiplying the sides (27), in the coefficients inversion Φ yields

$$\Phi(x) = [\sum_{j=1}^{n} w_j(x)P(x_j)P^T(x_j)]^T[w_1(x)P(x_1), w_2(x)P(x_2), ..., w_n(x)P(x_n)]u, \tag{2. 28}$$

The approximate of u^h from u as follows

$$u^h(x) = \Phi^T(x)u. \tag{2. 29}$$

In this paper the Gaussian weight function as follows [3, 11, 18, 29],

$$w_j(x) = \begin{cases} \frac{exp[-(r_j/\alpha)^2 - exp[(-h_j/\alpha)^2]}{1-exp[(-h_j/\alpha)^2} & 0 \leq r_j \leq h_j, \\ 0 & r_j \geq h_j \end{cases} \tag{2. 30}$$

Where $r_j = \|x - x_j\|$ (the Euclidean space between x and x_j) and α is a fixed shape of the weight function w_j and h_j is the measure of the support domain.

3. THE MODEL AND METHODOLOGY

In this section, structure, variables and parameters of the model are defined and methodology of LWM and MLSM will investigate on the model.

3.1. **Description of the Model.** A theoretical model has been introduced based in nonlinear ordinary differential equation to describe the dynamic of the population incidence of the infected pregnant that may present fetal microcephaly induced by the ZIKA virus, the variables and parameters of the model are as follows

- u_1 :Average number of susceptible people
- u_2 : Average number of ZIKV infected pregnant women that may induce fetal microcephaly
- u_3 :Average number of persons infected by ZIKV
- v_1 : Average number of non-carrier mosquitoes
- v_2 : Average number of virus-carrier mosquitoes

Parameters applied in the simulations are,
δ : constant flux of susceptible people, μ : the natural death rate, β : the virus transmission probability from the virus-carrier mosquitoes to the susceptible people, σ : the virus transmission probability from the infected pregnant women to the non-carrier mosquitoes, γ : virus transmission probability from infected people to the non-carrier mosquitoes, ε : the full mosquitoes death rate, θ : the recovery rate of the infected pregnant women, α : the infected people recovery rate, f : the fraction of infected people, $1 - f$: is the fraction of pregnant women infected by ZIKV [4, 9, 12].
Let

$$U(t) = (u_1(t), u_2(t), u_3(t), v_1(t), v_2(t)),$$

And

$$S_1(v_1(t), v_2(t)) = \frac{v_1(t)}{v_1(t) + v_2(t)},$$

$$S_2(u_1(t), u_2(t), u_3(t), v_1(t)) = \frac{v_1(t)}{u_1(t) + u_2(t) + u_3(t)},$$

System of nonlinear differential equation is obtained

$$DU^T = QU^T + B^T, \tag{3. 31}$$

Which D is first order linear differential operator and matrix of Q and the vector of B are as follows

$$Q = \begin{bmatrix} -\mu - \beta S_1 & 0 & 0 & 0 & 0 \\ (1-f)\beta & -(\theta+\mu) & 0 & 0 & 0 \\ -f\beta & 0 & -(\alpha+\mu) & 0 & 0 \\ -\delta S_1 \gamma S_2 & 0 & 0 & -\varepsilon & 0 \\ 0 & \delta S_2 & -\gamma S_2 & 0 & -\varepsilon \end{bmatrix}, B = \begin{bmatrix} \delta \\ 0 \\ 0 \\ \rho \\ 0 \end{bmatrix}$$

In the other words

$$\begin{aligned} \frac{\partial u_1(t)}{\partial t} &= \delta - \beta \frac{v_2(t)}{V(t)} u_1(t) - \mu u_1(t), \qquad (3.\ 32) \\ \frac{\partial u_2(t)}{\partial t} &= (1-f)\beta \frac{v_2(t)}{V(t)} u_1(t) - (\theta+\mu) u_2(t), \\ \frac{\partial u_3(t)}{\partial t} &= f\beta \frac{v_2(t)}{V(t)} u_1(t) - (\alpha+\mu) u_3(t), \\ \frac{\partial v_1(t)}{\partial t} &= \rho - \sigma \frac{u_2(t)}{U(t)} v_1(t) - \gamma \frac{u_3(t)}{U(t)} v_1(t) - \epsilon v_1(t), \\ \frac{\partial v_2(t)}{\partial t} &= \sigma \frac{u_2(t)}{U(t)} v_1(t) + \gamma \frac{u_3(t)}{U(t)} v_1(t) - \epsilon v_2(t), \end{aligned}$$

$U(t)$, is the vector of unknown functions and

$$U_0(t) = (u_{10}(t), u_{20}(t), u_{30}(t), v_{10}(t), v_{20}(t)),$$

is the vector of initial conditions, and the parameters $(\delta, \alpha, \mu, \theta, \rho)$ are positive and $(\beta, \gamma, f, \sigma) \in (0, 1)$.

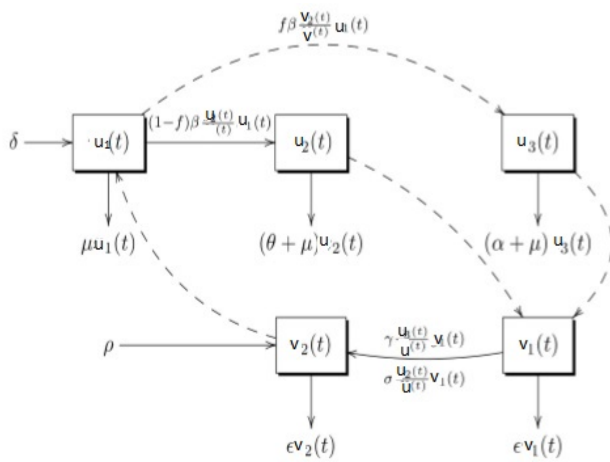

FIGURE 1. The relationship between variables in terms of virus transmission

3.2. **LWM and MLSM methodology in model.** In this section methodology of LWM and MLS is explained in model

LWM: In this model suppose the rate variable $X(t)$ can be expressed as

$$X(t) = C^T\Psi(t) \tag{3. 33}$$

Where C^T is transpose of C vector. Using Eq.(2. 8), $X(t)$ can be represented as

$$X(t) = \int_0^t X(\alpha)d\alpha + X(0) = C^T P\Psi(t), \tag{3. 34}$$

Which $t \in [0, 1]$ and also

$$U^2(t) = C^T P\Psi(t)\Psi(t)^T C, \tag{3. 35}$$

Eq.(2. 11), can be simplified by using the following property of the product of two Legendre wavelet function vectors

$$C^T P\Psi(t)\Psi(t)^T \approx \Psi(t)^T(\widehat{C}), \tag{3. 36}$$

By applying relation $Eq.(33)$ to $Eq(36)$, in Eq.(3. 31), the following

$$\begin{aligned}
\frac{du_1(t)}{dt} &= C_1^T\Psi(t), \quad u_1(t) = C_1^T P\Psi(t) + u_1(0), \qquad (3.\ 37)\\
\frac{du_2(t)}{dt} &= C_2^T\Psi(t), \quad u_2(t) = C_2^T P\Psi(t) + u_2(0),\\
\frac{du_3(t)}{dt} &= C_3^T\Psi(t), \quad u_3(t) = C_3^T P\Psi(t) + u_3(0),\\
\frac{dv_1(t)}{dt} &= C_4^T\Psi(t), \quad v_1(t) = C_4^T P\Psi(t) + v_1(0),\\
\frac{dv_2(t)}{dt} &= C_5^T\Psi(t), \quad v_2(t) = C_5^T P\Psi(t) + v_2(0),
\end{aligned}$$

In continue by using Eq.(33), in Eq.(3. 31), we get the following

$$\begin{aligned}
f_1 &: \quad C_1^T - \delta + (\beta g_1 + \mu)C_1^T P + u_1(0) = 0, \qquad (3.\ 38)\\
f_2 &: \quad C_2^T - (1-f)\beta g_1(C_1^T P + u_1(0)) + (\theta + \mu)(C_2^T P + u_2(0)) = 0,\\
f_3 &: \quad C_3^T - f\beta g_1(C_1^T P + u_1(0)) + (\alpha + \mu)(C_3^T P + u_3(0)) = 0,\\
f_4 &: \quad C_4^T - \rho + (\sigma g_2 + \gamma g_3 + \varepsilon)(C_4^T P + v_1(0)) = 0,\\
f_5 &: \quad C_5^T - (\sigma g_2 + \gamma g_3)(C_4^T P + v_1(0)) + \varepsilon(C_5^T P + v_2(0)) = 0,
\end{aligned}$$

And note that

$$\begin{cases} 1 = d^T\Psi(x) \\ x = e^T\Psi(x) \end{cases} \tag{3. 39}$$

Where g_1, g_2 and g_3 given as

$$\begin{aligned} g_1 &= \frac{(C_5^T P + v_2(0))}{(C_4 + C_5)^T P + v_1(0) + v_2(0)}, \\ g_2 &= \frac{(C_2^T P + u_2(0))}{(C_1 + C_2 + C_3)^T P + u_1(0) + u_2(0) + u_3(0)}, \\ g_3 &= \frac{(C_3^T P + v_2(0))}{(C_1 + C_2 + C_3)^T P + u_1(0) + u_2(0) + u_3(0)}. \end{aligned} \tag{3. 40}$$

On the other hand by derivative Eq.(3. 41), to all of coefficients C_i the Jacobian of coefficients matrix have calculated as following

$$|Z^{(i)}| = \begin{bmatrix} f^{(i)}_{1C_1} & f^{(i)}_{1C_2} & \cdots & f^{(i)}_{1C_5} \\ f^{(i)}_{2C_1} & f^{(i)}_{2C_2} & \cdots & f^{(i)}_{2C_5} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(i)}_{5C_1} & f^{(i)}_{5C_2} & \cdots & f^{(i)}_{5C_5} \end{bmatrix}, \tag{3. 41}$$

By applying the Jacobian iterative method, we solve following equation,

$$Z^{(i)} C^{(i+1)} = -F^{(i)} + Z^{(i)} C^{(i)}, \tag{3. 42}$$

Where

$$\begin{aligned} C^{(i)} &= (C_1^{(i)}, C_2^{(i)}, C_3^{(i)}, C_4^{(i)}, C_5^{(i)}), \\ C^{(i+1)} &= (C_1^{(i+1)}, C_2^{(i+1)}, C_3^{(i+1)}, C_4^{(i+1)}, C_5^{(i+1)}), \\ F^{(i)} &= (F_1^{(i)}, F_2^{(i)}, F_3^{(i)}, F_4^{(i)}, F_5^{(i)}). \end{aligned} \tag{3. 43}$$

MLSM: The proposed method is applied to solve an ODE system. Consider the system of Eq.(3. 31), to apply the method assumed that $\Phi = \{\phi_1, \phi_2, ..., \phi_n\}$ are the MLS shape functions. So in order to solve the system of ZIKV, n nodal points t_i are selected on the domain Ω which $\{t_j\}_{j=1}^n$ are q-unisolver them, then instead of u_1, u_2, u_3, v_1 and v_2, replaced approximation value as the following [17, 30]

$$\begin{gathered} u_1^{(h)}(t) = \sum_{j=1}^n \phi_j(t) u_1(t_j), \quad u_2^{(h)}(t) = \sum_{j=1}^n \phi_j(t) u_2(t_j), \quad u_3^{(h)}(t) = \sum_{j=1}^n \phi_j(t) u_3(t_j), \\ v_1^{(h)}(t) = \sum_{j=1}^n \phi_j(t) v_1(t_i), \quad v_2^{(h)}(t) = \sum_{j=1}^n \phi_j(t) v_2(t_j), \end{gathered} \tag{3. 44}$$

If $\phi_j, j = 1, 2, ..., n$, are sufficiently smooth, derivatives of u_1, u_2, u_3 and v_1, v_2 are usually approximated by derivative of $u^{(h)}$ and $v^{(h)}$

$$Du_i = Du_i^{(h)} = \sum_{j=1}^n D\phi_j(t) u_i(t_j), \qquad i = 1, 2, 3, \;\; t \in \Omega. \tag{3. 45}$$

$$Dv_k = Dv_k^{(h)} = \sum_{j=1}^{n} D\phi_j(t)v_k(t_j), \qquad k = 1, 2$$

So the system of Eq.(3. 31), for $i = 1, 2, 3$, $k = 1, 2$ becomes as fallows

$$\begin{aligned}
F_1(u_i^{(h)}, v_k^{(h)}, t) \quad &: \quad Du_1^{(h)}(t) + \beta\frac{v_2^h(t)}{V^h(t)}u_1^h(t) + \mu u_1^h(t) = \delta + r_1(t), \qquad (3.\ 46)\\
F_2(u_i^{(h)}, v_k^{(h)}, t) \quad &: \quad Du_2^{(h)}(t) - (1-f)\beta\frac{v_2^h(t)}{V^h(t)}u_1^h(t) + (\theta+\mu)u_2^h(t) = r_2(t),\\
F_3(u_i^{(h)}, v_k^{(h)}, t) \quad &: \quad Du_3^{(h)}(t) - f\beta\frac{v_2^h(t)}{V^h(t)}x_1^h(t) + (\alpha+\mu)u_3^h(t) = r_3(t),\\
F_4(u_i^{(h)}, v_k^{(h)}, t) \quad &: \quad Dv_1^{(h)}(t) + \sigma\frac{u_1^h(t)}{U^h(t)}v_1^h(t) + \gamma\frac{u_1^h(t)}{U^h(t)}v_1^h(t) + \varepsilon v_1^h(t) = \rho + r_4(t),\\
F_5(u_i^{(h)}, v_k^{(h)}, t) \quad &: \quad Dv_2^{(h)}(t) - \sigma\frac{u_2^h(t)}{U^h(t)}v_1^h(t) + \gamma\frac{u_3^h(t)}{U^h(t)}v_1^h(t) + \varepsilon v_2^h(t) = r_5(t).
\end{aligned}$$

Where $r_m(t), m = 1, 2, ..., 5$, is residual error of function which vanishes to zero in collocation points, thus by using these points

$$t_r = 0, \ \ r = 1, ..., n,$$

So

$$F_i(u_1^{(h)}, u_2^{(h)}, u_3^{(h)}, v_1^{(h)}, v_2^{(h)}, r_m) = 0$$

hence by imposing the initial conditions at $t = 0$, and solving the nonlinear system of Eq.(3. 31) leads to finding quantities $u_1^{(h)}, u_2^{(h)}, u_3^{(h)}, v_1^{(h)}$, and $v_2^{(h)}$. Then the values of $u_1^{(h)}, u_2^{(h)}, u_3^{(h)}, v_1^{(h)}$, and $v_2^{(h)}$, at any point of $t \in \Omega$, can be approximated by following equations.

$$\begin{aligned}
u_i(t) \quad &= \quad \sum_{j=1}^{n} \phi_j(t)u_i^h(t_j), i = 1, 2, 3 \qquad (3.\ 47)\\
v_k(t) \quad &= \quad \sum_{j=1}^{n} \phi_j(t)v_k^h(t_j), k = 1, 2
\end{aligned}$$

3.3. **Algorithm.** In this subsection an efficient algorithm proposed for solving the nonlinear system of Eq.(3. 31). As we know in LWM the goal is to find the coefficients c_{nm} which introduced in $Eq.(4)$. In order to find the coefficients one of the suitable approaches using an iterative method as follows [28].

Algorithm of iterative nonlinear system of Eq.(3. 31)

Algorithm 1

Require: (input; P, $u_1(0)$, $u_2(0)$, $u_2(0)$, $u_3(0)$, $v_1(0)$, $v_2(0)$)

Set:

$$
\begin{aligned}
g_1 &\leftarrow \frac{(C_5^T P + v_2(0))}{(C_4 + C_5)^T P + v_1(0) + v_2(0)} \\
g_2 &\leftarrow \frac{(C_2^T P + u_2(0))}{(C_1 + C_2 + C_3)^T P + u_1(0) + u_2(0) + u_3(0)} \\
g_3 &\leftarrow \frac{(C_3^T P + u_3(0))}{(C_1 + C_2 + C_3)^T P + u_1(0) + u_2(0) + u_3(0)} \\
f_1 &: \quad C_1^T - \delta + (\beta g_1 + \mu) C_1^T P + u_1(0) = 0 \\
f_2 &: \quad C_2^T - (1-f)\beta g_1 (C_1^T P + u_1(0)) + (\theta + \mu)(C_2^T P + u_2(0)) = 0 \\
f_3 &: \quad C_3^T + f\beta g_1 (C_1^T P + u_1(0)) + (\alpha + \mu)(C_3^T P + u_3(0)) = 0 \\
f_4 &: \quad C_4^T - \rho + (\sigma g_2 + \gamma g_3 + \varepsilon) C_4^T P + v_1(0) = 0 \\
f_5 &: \quad C_5^T - (\delta g_1 + \gamma g_3)(C_4^T P + v_1(0)) + \varepsilon (C_5^T P + v_2(0)) = 0
\end{aligned}
$$

procedure CALCULATING THE MATRIXOF Z AND SOLVE THE NON-LINEAR SYSTEM OF EQUATIONS

$Z \leftarrow [jacobian\, of\; equations]$

$inv(Z) \leftarrow Inv(Z)$ (Compute the inverse of the matrix jacobian)

$C_0 \leftarrow [C_{11}(0), C_{12}(0), ..., C_{15}(0), C_{21}(0), ..., C_{25}(0), ..., C_{n1}(0), ...C_{n5}(0)]$(Those are obtained by X_0)

for $i = 0 : n$ (n is the number of numerical itarative) **do**

$C[i+1] = inv(Z[i])(-f[i] + Z[i]C[i])$

end

4. NUMERICAL RESULTS

In this section the results of implementation of the proposed methods are presented. At first we obtained the coefficients c_{nm}, $n = 5$ and $m = 2$, for $f = 0.6$, and values of parameters Table 1, by applying LWM. The results are shown in Table 2.

Then from Eq.(10), by using Table 2 the equations of solution $u_i, i = 1, 2, 3$, can be

TABLE 1. The values of parameters

Parameter	γ	β	σ	ϵ	α	μ	θ	ρ	η	f
Value	0.773	0.7913	0.6	0.0352	0.14	0.0003	0.05	30	20	$0.3, 0.6$

TABLE 2. Corresponding coefficients by LWM for f=0.6.

c_1	c_2	c_3	c_4	c_5
0.1845	0.1590	0.1948	0.4815	0.3801
0.0824	0.0562	0.0829	0.6811	0.6050
0.1111	0.0456	0.0678	0.4554	0.7279
0.1336	0.0368	0.0549	0.3391	0.7353
0.1685	0.0229	0.0342	0.3393	0.7206
0.1865	0.0157	0.0235	0.3393	0.5953
0.2100	0.0063	0.0294	0.3393	0.4803
0.2165	0.0038	0.0056	0.3394	0.1935
0.2389	−0.0052	−0.0078	0.3394	0.1313
0.2290	−0.0012	−0.0018	0.3391	−0.3763

written as

$$\begin{aligned} u_1 &= 1408.513x^9 - 6316.638x^8 + 11844.788x^7 - 12039.632x^6 + \\ &\quad 7188.203x^5 - 2552.357x^4 + 519.925x^3 - 55.235x^2 + 2.618x - 0.0263 \\ u_2 &= -30.633x^9 - 145.903x^8 - 286.522x^7 + 301.136x^6 - 83.441x^5 + \\ &\quad 65.437x^4 - 13.128x^3 - 1.347x^2 + 0.059x + 0.00014 \\ u_3 &= -46.067x^9 - 219.263x^8 - 430.319x^7 + 452.005x^6 - 275.187x^5 + \\ &\quad 8.107x^4 - 19.670x^3 + 2.017x^2 + 0.047x - 0.0002 \end{aligned}$$

Also the equations of solution $v_j, j = 1, 2,$ can be approximated by

$$\begin{aligned} v_1 &= 2000.88x^9 - 8940.693x^8 + 16715.827x^7 - 16952.516x^6 + 10106.969x^5 - \\ &\quad 3587.307x^4 + 731.959x^3 - 78.0099x^2 + 3.424x - 0.0389 \\ v_2 &= 777.244x^9 - 2607.122x^8 + 3224.353x^7 - 1514.351x^6 - 217.151x^5 + \\ &\quad 514.801x^4 - 207.352x^3 - 35.342x^2 - 2.3820x - 0.04317 \end{aligned}$$

By computing values of coefficients in $f = 0.3$ and $f = 0.85$, the other equations of solution are determined too.

In figures $(2-4)$, conduct of the infected pregnant women,u_2 and infected people,u_3 are shown in the ZIKV. Also, it is noted that if f pick up the pregnant women people arrives a major epidemic point by comparing the LWM by $m = 10$, and MLS at $t \in [0, 1]$ by various f and difference $\triangle x = 0.1$, we see that figures of them are almost similar in the same interval $[0, 1]$. In MLSM precision of solution depend on number of points and parameters of weight function. In LWM as a spectral method a closed form of the solutions are obtained and the accuracy of the method is related to degree of Legendre polynomials. In other hands MLSM is a method without mesh and approximation of unknown functions is based on points where they can be regularly or unregularly [2]. As we know according to approximation theory, each continues function could be approximate with a sequence of polynomial that is base of space. Therefore, in comparison between bases of two methods

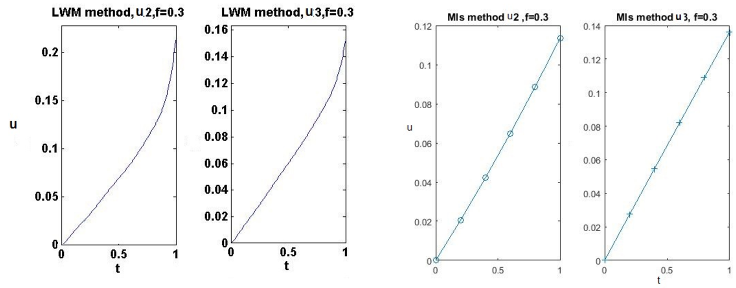

FIGURE 2. Results of applying LWM and MLS in $f = 0.3$ and $t \in [0, 1]$.

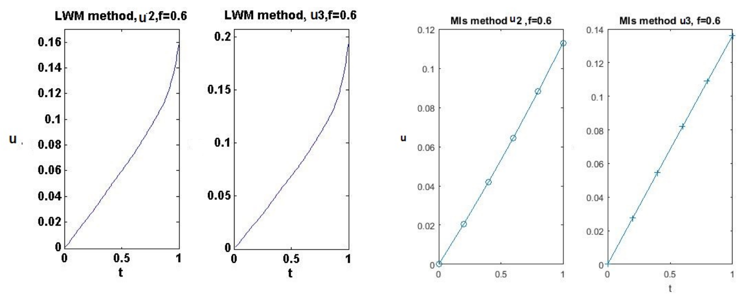

FIGURE 3. Results of applying LWM and MLS in $f = 0.6$ and $t \in [0, 1]$.

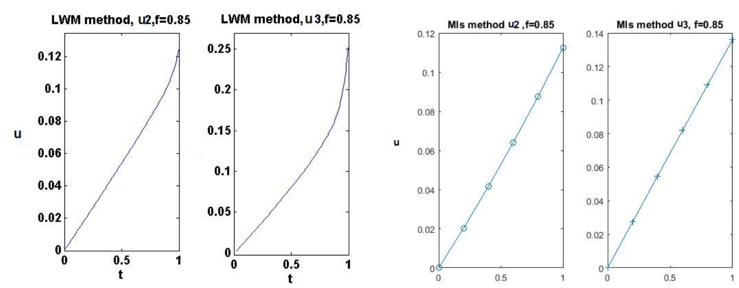

FIGURE 4. Results of applying LWM and MLS in $f = 0.85$ and $t \in [0, 1]$.

in LWM the bases are unknown and fixed but in MLSM the shape functions as the bases are produced in each trial points [10, 25].

5. Conclusion

Two methods of LWM and MLSM on the system of nonlinear differential equations are implemented. To find the equation of the solution the coefficients of wavelet must be obtained. Thus in this step we applied Jacobian iterative method in this algorithm. In theorem 2.2 we found an upper bond for coefficients and for the fraction infected people (f=0.6), the coefficients are tabulated in Table 2. In this study while we didnt have exact solution for Zika model and as it can see in the figures 2 to 4 the approximate solutions are similar in these two approaches.

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