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Categorical Characterizations of Some Results on Induced Mappings

Nitakshi Goyal[1], Navpreet Singh Noorie[2]

[1,2] Department of Mathematics, Punjabi University, Patiala-147002, Punjab, India.
[1,2] Email: goyal.nishu53@gmail.com, noorie@pbi.ac.in

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Abstract For a given morphism $f : X \to Y$ in a category $\mathcal{C}$ having pullbacks we study some properties of the adjoint string $f(-) \dashv f^{-1} \dashv f^{\sharp}$ and give new characterizations of monic and epic nature of the induced map $f(-)$.

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1. Introduction

It is elementary that associated to any map $f : X \to Y$ between sets, there exist induced maps $Imf : \wp(X) \to \wp(Y)$ known as the image map and $f^{-1} : \wp(Y) \to \wp(X)$ known as the inverse image map. In [3], Arkhangel'skii has also used another induced map defined with the help of fibers namely, $f^{\sharp} : \wp(X) \to \wp(Y)$, given by $f^{\sharp}(A) = \{y \in Y : f^{-1}(y) \subset A\}$. In [2], Al-shami discuss some properties of a somewhere dense set on topological spaces and in [1], Abbas et. al. discuss the concepts of fuzzy upper and fuzzy lower contra-continuous, contra-irresolute and contra semi-continuous multifunctions. In [9], the second author and Bala made extensive use of this induced map to give new characterizations of open, closed and continuous mappings. It is important to investigate in detail the relationships between these induced maps. Following MacLane [8] and his slogan that "Adjoint Functors Are Everywhere", Cicogna in [6] has given many interesting applications of functor adjunctions.

In this paper, we use categorical concepts and especially the concept of functor adjunctions to illuminate the nature and relationship of the above induced maps. In this paper, we (i) show that there is mapping between adjunctions involving the induced maps $f(-)$, f^{-1} and $f^{\sharp}$ (Theorem 2.1 below), (ii) prove the existence of functor $\sharp$ which results in the induced map $f^{\sharp}$ (Theorem 2.2 below), (iii) illuminate the categorical meaning of some results in [9] involving the induced map $f^{\sharp}$ (Remark 2.3 below), (iv) obtain triangle identity characterizations of the adjunctions $f(-) \dashv f^{-1} \dashv f^{\sharp}$, giving useful factorizations of these maps (Lemma 2.1 below), (v) give more comprehensive characterizations of the monicity of $f(-)$ map (Theorem 2.3 below) and epicity of $f(-)$ (Theorem 2.4 below). Further, we have defined saturated sets and have given the new characterization of Theorems 2.3 and 2.4 using these sets (Theorems 2.5 and 2.6 below). The importance of these results lie in the fact that they provide us with accurate and complete interrelationship between the induced maps and their interdependence. The aim of this paper is not just to find new results but also to bring out the clarity in the results with the application of categorical methods and to see how the properties of these induced maps are categorical in nature at the deeper level.

For the sake of completeness and readability we recall that a **category** $\mathcal{C}$ [4] has two collections $\mathcal{C}_0$ of objects and $\mathcal{C}_1$ of arrows such that (i) for all $f \in \mathcal{C}_1$, there exist two objects in $\mathcal{C}_0$, one is called $dom(f)$ and the other is called $cod(f)$, (ii) it is closed under composition of arrows and (iii) it has identity arrow for each object. Also associative law and unit law should be satisfied by it. A **Functor** [4] $F : \mathcal{C} \to \mathcal{D}$ is a mapping of objects to objects and arrows to arrows, preserving identity, domains and codomains and respecting composition.
Some other basic definitions can be found from Steve Awodey [4] viz. "Arrow category", "Pullback", "Natural Transformation" , "Full Functor" and "Faithful Functor" etc.

Definition 1.1. [4] An **adjunction** consists of functors, $G : \mathcal{C}' \rightleftarrows \mathcal{D}' : G'$ and a natural transformation $\eta' : 1_{\mathcal{C}'} \Rightarrow G' \circ G$ (written $G \dashv G'$) such that
For any $h : C' \to G'(D')$ in $\mathcal{C}'$ there exists unique $k : G(C') \to D'$ such that $h = G'(k) \circ \eta'_{C'}$ i.e.

$$\begin{array}{ccc} C' & & \\ {\scriptstyle \eta'_{C'}}\downarrow & \overset{h}{\searrow} & \\ G'G(C') & \xrightarrow[G'(k)]{} & G'(D') \end{array}$$

is commutative.
Here η' is known as the unit of adjunction.

Equivalently, we have

Definition 1.2. [4] An **adjunction** consists of functors, $G : \mathcal{C}' \rightleftarrows \mathcal{D}' : G'$ and a natural transformation $\epsilon' : G \circ G' \Rightarrow 1_{\mathcal{D}'}$ (written $G \dashv G'$) such that
For any $h : G(C') \to D'$ in $\mathcal{D}'$ there exists unique $k : C' \to G'(D')$ such that $\epsilon'_{D'} \circ G(k) = h$ i.e.

$$\begin{array}{ccc} G(C') & \xrightarrow{G(k)} & GG'(D') \\ & \underset{h}{\searrow} & \downarrow{\scriptstyle \epsilon'_{D'}} \\ & & D' \end{array}$$

is commutative.
Here ϵ' is known as the counit of adjunction.
In particular, if the counit of adjunction is identity i.e. $G \circ G' = 1_{\mathcal{D}'}$ then G is known as left-adjoint left inverse [8] of G'.

Also $G \dashv G'$ if and only if $Hom_{\mathcal{C}'}(C', G'D') \cong Hom_{\mathcal{D}'}(GC', D')$, where the isomorphism is natural in C' and D'.

Definition 1.3. [4] An arrow $f : X \to Y$ in a category $\mathcal{C}$ is said to be

(1) **epi**, if for any two arrows $g, h : Y \to Z$ whenever $g \circ f = h \circ f$ i.e. whenever the following is commutative:

$$X \xrightarrow{\;f\;} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

then g=h.

(2) **monic**, if for any two arrows $g, h : Z \to X$ whenever $f \circ g = f \circ h$ i.e. whenever the following is commutative:

$$Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{\;f\;} Y$$

then g=h.

For our work we shall need a better understanding of:

Subobjects: For a category $\mathcal{C}$ and for any object X in it, consider a category having as objects monic morphisms of $\mathcal{C}$ with codomain X and arrow between any two objects say $m_1 : M \to X$, $m_2 : N \to X$ is an arrow from M to N in $\mathcal{C}$ such that

$$\begin{array}{ccc} M & \longrightarrow & N \\ & \searrow^{m_1} & \downarrow m_2 \\ & & X \end{array}$$

is commutative.
We will denote this category by $Sub_{\mathcal{C}}(X)$. Here $Sub_{\mathcal{C}}(X)$ is a preorder category with relation $\prec$ given by $m_1 \prec m_2$ if and only if there is an arrow from m_1 to m_2 in $Sub_{\mathcal{C}}(X)$. Also by giving the equivalence relation R (where $m_1 R m_2$ if and only if there is an arrow from m_1 to m_2 and m_2 to m_1 in $Sub_{\mathcal{C}}(X)$) on it gives a partial order category. We will denote this category by $Sub_{\mathcal{C}}[X]$ having as objects the equivalence classes $[m_1]$ corresponding to an object $m_1 : M \to X$ of $Sub_{\mathcal{C}}(X)$ with relation say $\leq$ (where $[m_1] \leq [m_2]$ if and only if there is an arrow $m_1 \to m_2$ in $Sub_{\mathcal{C}}(X)$).

Remark 1.1. [4] In particular, for the category $\mathcal{C}$ = **Sets** and for any object X in **Sets** i.e. for any set X, $\phi : Sub_{Sets}[X] \cong \wp(X) : \psi$ where $\phi : Sub_{Sets}[X] \to \wp(X)$ is given by $\phi([m] : M \to X) = m(M)$ and $\psi : \wp(X) \to Sub_{Sets}[X]$ is given as $\psi(A) = [i_A]$ where $i_A : A \to X$ is the inclusion arrow. In further results for category **Sets** and any set X in it , we will denote $Sub_{Sets}[X]$ by $\wp(X)$.

Definition 1.4. [7]

(1) **Inverse image of an object**: Let $\mathcal{C}$ be a category with pullbacks. Then pullbacks preserve monics implies there exists functor $f^{-1} : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[X]$, which is the restriction of the pullback functor to $Sub_{\mathcal{C}}[Y]$ corresponding to morphism $f : X \to Y$ of $\mathcal{C}$. Therefore, for an object $[n_1]$ of $Sub_{\mathcal{C}}[Y]$; f^{-1} is given by the

following pullback square:

$$\begin{array}{ccc} f^{-1}(N) & \longrightarrow & N \\ {\scriptstyle f^{-1}[n_1]}\downarrow & & \downarrow{\scriptstyle [n_1]} \\ X & \xrightarrow[f]{} & Y \end{array}$$

So for any object $[n_1] : N \to Y$ in $Sub_{\mathcal{C}}[Y]$, **the inverse image of** $[n_1]$ is $f^{-1}[n_1]$ given by the above pullback square. Also $f^{-1}[n_1] = [f^{-1}(n_1)]$.

(2) **Image of an object**: The image functor denoted by $f(-)$ is given as the left adjoint of f^{-1} **if** the left adjoint exists. So

$$f(-) : Sub_{\mathcal{C}}[X] \to Sub_{\mathcal{C}}[Y]$$

and for an object $[m_1] : M \to X$ in $Sub_{\mathcal{C}}[X]$, **image** $f(-)[m_1]$ is given by the property $[m_1] \leq f^{-1}[n_1]$ if and only if $f(-)[m_1] \leq [n_1]$. Also $f(-)[m_1] = [f(m_1)]$.

Remark 1.2. [4] Since the category **Sets** has all pullbacks and for morphism $f : X \to Y$ in **Sets**, inverse image-functor $f^{-1} : \wp(Y) \to \wp(X)$ has both left adjoint $f(-)$ which (will be denoted by Imf in **Sets**) as well as right adjoint which here will be denoted by $f^{\sharp} : \wp(X) \to \wp(Y)$. Therefore, $Imf \dashv f^{-1} \dashv f^{\sharp}$ holds in **Sets**.

Theorem 1.1. [5] *Let $G : \mathcal{X} \rightleftarrows \mathcal{D} : K$ with $G \dashv K$ and $\alpha : 1_{\mathcal{X}} \Rightarrow K \circ G$, $\beta : G \circ K \Rightarrow 1_{\mathcal{D}}$ be corresponding unit, counit of adjunction respectively. Then following are equivalent:*

(a) *$K(G)$ is full and faithful.*
(b) *The counit(unit) of adjunction $\beta(\alpha)$ is iso.*

Theorem 1.2. [5] *If $G : \mathcal{X} \to \mathcal{D}$ has both left adjoint K and right adjoint H. Then K is full and faithful if and only if H is full and faithful.*

Theorem 1.3. [4] *If $G : \mathcal{X} \rightleftarrows \mathcal{D} : H$, $\gamma : 1_{\mathcal{X}} \Rightarrow H \circ G$ and $\delta : G \circ H \Rightarrow 1_{\mathcal{D}}$*
Then $G \dashv H$ with unit γ and counit δ if and only if following diagrams are commutative:

$$\begin{array}{ccc} H & & \\ {\scriptstyle \gamma_H}\downarrow & \searrow^{1_H} & \\ H \circ G \circ H & \xrightarrow[H(\delta)]{} & H \end{array} \qquad \begin{array}{ccc} G & & \\ {\scriptstyle G_{\gamma}}\downarrow & \searrow^{1_G} & \\ G \circ H \circ G & \xrightarrow[\delta_G]{} & G \end{array}$$

These diagrams are known as triangle identities.
i.e. for any object $X \in \mathcal{X}$ and $D \in \mathcal{D}$, we have $H(\delta_D) \circ \gamma_{HD} = 1_{HD}$ and $\delta_{GX} \circ G(\gamma_X) = 1_{GX}$.

Theorem 1.4. [5] *Let $G : \mathcal{X} \rightleftarrows \mathcal{D} : H$ with $\mathcal{X}$ and $\mathcal{D}$ partial order categories and $G \dashv H$ then $G \circ H \circ G = G$ and $H \circ G \circ H = H$.*

In our further results, for the category $\mathcal{C}$, we will assume that like Sets, $\mathcal{C}$ has all pullbacks and for a morphism $f : X \to Y$ of $\mathcal{C}$, there exists string of adjoints $f(-) \dashv f^{-1} \dashv f^{\sharp}$ where $f^{\sharp} : Sub_{\mathcal{C}}[X] \to Sub_{\mathcal{C}}[Y]$ is the right adjoint of f^{-1}. Also $\eta : 1_{Sub_{\mathcal{C}}}[X] \Rightarrow f^{-1} \circ f(-)$, $\epsilon : f(-) \circ f^{-1} \Rightarrow 1_{Sub_{\mathcal{C}}[Y]}$ will denote the unit and counit of adjunction $f(-) \dashv f^{-1}$ respectively and $\alpha : 1_{Sub_{\mathcal{C}}[Y]} \Rightarrow f^{\sharp} \circ f^{-1}$, $\beta : f^{-1} \circ f^{\sharp} \Rightarrow 1_{Sub_{\mathcal{C}}[X]}$ will denote the unit and counit of adjunction $f^{-1} \dashv f^{\sharp}$ respectively.

2. Results

As our first result we prove that there is a mapping between adjunctions $f(-)\dashv f^{-1}$ and $f^{-1}\dashv f^{\sharp}$. Before this we will give some useful factorizations of f^{-1}, $f(-)$ and $f^{\sharp}$.

Lemma 2.1. *(Triangle identities for $f(-) \dashv f^{-1} \dashv f^{\sharp}$): For a morphism $f : X \to Y$ of $\mathcal{C}$, following holds:*

(i) $f^{-1} \circ f(-) \circ f^{-1} = f^{-1}$, $f(-) \circ f^{-1} \circ f(-) = f(-)$.
(ii) $f^{-1} \circ f^{\sharp} \circ f^{-1} = f^{-1}$, $f^{\sharp} \circ f^{-1} \circ f^{\sharp} = f^{\sharp}$.

Proof. Proof follows from the fact that $Sub_{\mathcal{C}}[X]$ and $Sub_{\mathcal{C}}[Y]$ are partial order categories and by Theorem 1.4. □

In particular, for **Sets** category, we have $f^{-1} \circ Imf \circ f^{-1} = f^{-1}$ and $Imf \circ f^{-1} \circ Imf = Imf$.

Theorem 2.1. *Let $f : X \to Y$ be a morphism in the category $\mathcal{C}$. For the adjunctions $f(-)\dashv f^{-1}$ with η, ϵ as the unit and counit of adjunction resp. and $f^{-1}\dashv f^{\sharp}$ with α, β as the unit and counit of adjunction resp., there is a functor $L : Sub_{\mathcal{C}}[X]\to Sub_{\mathcal{C}}[Y]$ such that following squares become commutative.*

$$\begin{array}{ccccc} Sub_{\mathcal{C}}[X] & \xrightarrow{f(-)} & Sub_{\mathcal{C}}[Y] & \xrightarrow{f^{-1}} & Sub_{\mathcal{C}}[X] \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle L} \\ Sub_{\mathcal{C}}[Y] & \xrightarrow[f^{-1}]{} & Sub_{\mathcal{C}}[X] & \xrightarrow[f^{\sharp}]{} & Sub_{\mathcal{C}}[Y] \end{array}$$

and $L(\eta_{[m]}) = \alpha_{L([m])}$, $\beta_{f^{-1}([n])} = f^{-1}(\epsilon_{[n]})$ for objects $[m]$ of $Sub_{\mathcal{C}}[X]$ and $[n]$ of $Sub_{\mathcal{C}}[Y]$.

Proof. Firstly, we define $L : Sub_{\mathcal{C}}[X]\to Sub_{\mathcal{C}}[Y]$ as $L = f^{\sharp} \circ f^{-1} \circ f(-)$. Since composition of functors is also a functor, so L is a functor. Further $f^{-1}\circ L = f^{-1}\circ f^{\sharp}\circ f^{-1}\circ f(-) = f^{-1}\circ f(-)$ using triangle identity of Lemma 2.1(ii) and $L\circ f^{-1} = f^{\sharp}\circ f^{-1}\circ f(-)\circ f^{-1} = f^{\sharp} \circ f^{-1}$ using triangle identity of Lemma 2.1(i). Therefore, $f^{-1} \circ L = f^{-1} \circ f(-)$ and $L \circ f^{-1} = f^{\sharp} \circ f^{-1}$, proving that both the squares are commutative. Also for any object $[m]$ of $Sub_{\mathcal{C}}[X]$, $\eta_{[m]} : [m]\to f^{-1}(f([m]))$, so $L(\eta_{[m]}) : L([m])\to L(f^{-1}(f([m])))$ and $\alpha_{L([m])}$: $L([m])\to f^{\sharp}(f^{-1}(L([m])))$. But from commutativity of above squares, we get $L(f^{-1}(f([m]))) = f^{\sharp}(f^{-1}(f([m]))) = f^{\sharp}(f^{-1}(L([m])))$ for any object $[m]$ of $Sub_{\mathcal{C}}[X]$.

So $L([m]) \underset{\alpha_{L([m])}}{\overset{L(\eta_{[m]})}{\rightrightarrows}} L(f^{-1}(f([m])))$ are a pair of parallel arrows.

As $Sub_{\mathcal{C}}[X]$ and $Sub_{\mathcal{C}}[Y]$ are partial order categories and in partial order categories there can be at most one arrow between any two objects. So $L(\eta_{[m]}) = \alpha_{L([m])}$ for any object $[m]$ of $Sub_{\mathcal{C}}[X]$. Similarly we can check $\beta_{f^{-1}([n])} = f^{-1}(\epsilon_{[n]})$ for any object $[n]$ of $Sub_{\mathcal{C}}[Y]$. □

The above theorem is an instance of the following

Definition 2.1. (Mapping of adjunction)[8]: Let $< F, G, \eta, \epsilon >$: $\mathcal{X}\to\mathcal{D}$ means $F\dashv G$ with η, ϵ are the unit and counit of adjunctions resp. and $< U, V, \alpha, \beta >$: $\mathcal{P}\to\mathcal{Q}$ with α, β are the unit and counit of adjunctions resp. A pair of functors $L : \mathcal{X} \to \mathcal{P}$ and $K : \mathcal{D} \to \mathcal{Q}$ is called a mapping between these adjunctions if the following squares are commutative:

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{F} & \mathcal{D} & \xrightarrow{G} & \mathcal{X} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle L} \\ \mathcal{P} & \xrightarrow[U]{} & \mathcal{Q} & \xrightarrow[V]{} & \mathcal{P} \end{array}$$

and $L(\eta_X) = \alpha_{L(X)}$, $\beta_{K(B)} = K(\epsilon_B)$ for any object X of $\mathcal{X}$ and B of $\mathcal{D}$.
We may denote this by $(L, K) : F \dashv G \to U \dashv V$.

Remark 2.1. With the definition 2.1 above, we have the following restatement of Theorem 2.1:
For a morphism $f : X \to Y$ in $\mathcal{C}$, there exists a functor $L : Sub_{\mathcal{C}}[X] \to Sub_{\mathcal{C}}[Y]$ such that $(L, f^{-1}) : f(-) \dashv f^{-1} \to f^{-1} \dashv f^{\sharp}$.

Corollary 2.1. *For any map $f : X \to Y$ in **Sets**, there exists a functor $L : \wp(X) \to \wp(Y)$ such that*
$(L, f^{-1}) : Im(f) \dashv f^{-1} \to f^{-1} \dashv f^{\sharp}$.

For our next results we need the following definition and Remark 2.2:

Definition 2.2. (Arrow Category)[4]: Let $\mathcal{C}$ be any category. Define the arrow category $\overrightarrow{\mathcal{C}}$ with objects as all arrows of category $\mathcal{C}$ and for any two objects $f : X \to Y$ and $g : W \to Z$ an arrow from $f \to g$ is a pair (h, k) such that the following square is commutative:

$$\begin{array}{ccc} X & \xrightarrow{h} & W \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & Z \end{array} \tag{2. 1}$$

The composition in $\overrightarrow{\mathcal{C}}$ is given componentwise.

Remark 2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two partial order categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors such that $F \cong G$ i.e. F is iso to G then $F = G$.

Proof. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two iso functors then there exist natural transformation $\eta : F \Rightarrow G$ and natural transformation $\beta : G \Rightarrow F$ such that $\eta \circ \beta = 1_G$ and $\beta \circ \eta = 1_F$. This implies that for each object $C \in \mathcal{C}$, there exist $\eta_C : FC \to GC$ and $\beta_C : GC \to FC$ i.e.

$$FC \underset{\beta_C}{\overset{\eta_C}{\rightleftarrows}} GC$$

But $\mathcal{D}$ is a partial order category. Therefore, $FC = GC$ for each object $C \in \mathcal{C}$ and $F(f) = G(f)$ for all arrows of $\mathcal{C}$, since there exist at most one arrow between any two objects of partial order category. Hence $F = G$. □

Since in **Sets**, for any map $f : X \to Y$, $f^{\sharp} : \wp(X) \to \wp(Y)$ is given by $f^{\#}(A) = \{y \in Y : f^{-1}(y) \subseteq A\}$ for any subset A of X [3]. We now show that in general for a category $\mathcal{C}$, $f^{\sharp}$ is obtained by defining a functor $\sharp : \overrightarrow{\mathcal{C}} \to \overrightarrow{Pos}$.

Theorem 2.2. *For a morphism $f : X \to Y$ of category $\mathcal{C}$ there exists a functor $\sharp : \overrightarrow{\mathcal{C}} \to \overrightarrow{Pos}$ which takes f in $\overrightarrow{\mathcal{C}}$ to $f^\sharp$ in $\overrightarrow{Pos}$ or $\sharp(f : X \to Y) = f^\sharp : Sub_{\mathcal{C}}[X] \to Sub_{\mathcal{C}}[Y]$ on objects.*

Proof. The functor $f^\sharp$ gives us the definition of $\sharp$ on objects f of $\overrightarrow{\mathcal{C}}$. We need to extend this on arrows of $\overrightarrow{\mathcal{C}}$. For this let us define $\sharp((h,k) : f \to g)$= $(h^\sharp, k^\sharp)$ on arrows. First we will prove that it is well defined. $(h,k) : f \to g$ means that the above square 2. 1 is commutative i.e. $goh = kof$, so $(goh)^\sharp = (kof)^\sharp$, where $(goh)^\sharp : Sub_{\mathcal{C}}[X] \to Sub_{\mathcal{C}}[Z]$. Now for an arrow $g : W \to Z$, there is an adjunction $< g^{-1} : Sub_{\mathcal{C}}[Z] \rightleftarrows Sub_{\mathcal{C}}[W] : g^\sharp >$ i.e. $g^{-1} \dashv g^\sharp$ and for an arrow $h : X \to W$, there is an adjunction $< h^{-1} : Sub_{\mathcal{C}}[W] \rightleftarrows Sub_{\mathcal{C}}[X] : h^\sharp >$ i.e. $h^{-1} \dashv h^\sharp$. Also for an arrow $g \circ h : X \to Z$, there is an adjunction $< (g \circ h)^{-1} : Sub_{\mathcal{C}}[Z] \rightleftarrows Sub_{\mathcal{C}}[X] : (g \circ h)^\sharp >$ i.e. $(g \circ h)^{-1} \dashv (g \circ h)^\sharp$. Therefore, we have the following adjunctions:

$$Sub_{\mathcal{C}}[Z] \underset{g^\sharp}{\overset{g^{-1}}{\rightleftarrows}} Sub_{\mathcal{C}}[W] \quad Sub_{\mathcal{C}}[W] \underset{h^\sharp}{\overset{h^{-1}}{\rightleftarrows}} Sub_{\mathcal{C}}[X],$$

$$Sub_{\mathcal{C}}[Z] \underset{(g \circ h)^\sharp}{\overset{(g \circ h)^{-1}}{\rightleftarrows}} Sub_{\mathcal{C}}[X]$$

and
But composition of adjunctions is also an adjunction [8]. Therefore, $h^{-1} \circ g^{-1} \dashv g^\sharp \circ h^\sharp$. Further, it is well known that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. So $(g \circ h)^{-1}$ has two right adjoints namely $(g \circ h)^\sharp$ and $g^\sharp \circ h^\sharp$. Therefore, $(g \circ h)^\sharp \cong g^\sharp \circ h^\sharp$, since adjoints are unique up to isomorphism [4] and so by Remark 2.2 $(g \circ h)^\sharp = g^\sharp \circ h^\sharp$
proving that the following square is commutative:

$$\begin{array}{ccc} Sub_{\mathcal{C}}[X] & \xrightarrow{h^\sharp} & Sub_{\mathcal{C}}[W] \\ \downarrow f^\sharp & & \downarrow g^\sharp \\ Sub_{\mathcal{C}}[Y] & \xrightarrow[k^\sharp]{} & Sub_{\mathcal{C}}[Z] \end{array}$$

So $(h^\sharp, k^\sharp) : f^\sharp \to g^\sharp$ or $(h,k) : f \to g$ implies $\sharp(h,k) : \sharp(f) \to \sharp(g)$. So $\sharp$ preserves domains and codomains. Also for any $(h,k) : f \to g$ and $(v,u) : g \to n$, $\sharp((v,u) \circ (h,k)) = \sharp(v \circ h, u \circ k) = ((v \circ h)^\sharp, (u \circ k)^\sharp) = (v^\sharp \circ h^\sharp, u^\sharp \circ k^\sharp) = ((v^\sharp, u^\sharp) \circ (h^\sharp, k^\sharp)) = \sharp(v,u) \circ \sharp(h,k)$. So $\sharp$ respects composition. Also it can be easily seen that it preserves identity. Hence $\sharp$ operator is a functor. □

Corollary 2.2. *In particular, for the category* **Sets***, there exists a functor $\sharp$: $\overrightarrow{Sets} \to \overrightarrow{Pos}$.*

In [9], the second author and Bala proved the Lemma $f^\#(T^c) = (f(T))^c$ and so $f^\#(T) = (f(T^c))^c$ and $f(T) = (f^\#(T^c))^c$ for any subset T of X, where T^c denotes the complement of a set T in X. Here we will give categorical version of this Lemma using string of adjunction $Imf \dashv f^{-1} \dashv f^\sharp$

Remark 2.3. **Categorical Meaning of** Lemma 2.3(vii) [9]:
$f^\#(T^c) = (f(T))^c$ and so $f^\#(T) = (f(T^c))^c$ and $f(T) = (f^\#(T^c))^c$.

The real content of this relationship between Imf and $f^\sharp$ lies in the fact that adjoints are unique up to isomorphism [4]. For this we note that taking complements is a functor. Define the functor $C_X : \wp(X) \to \wp(X)$ by $C_X(S) = X - S = S^c$ and for any arrow

$S \to T$ i.e. $S \subseteq T$ in domain category $\wp(X)$, $C_X(T) \to C_X(S)$ i.e. $X - T \to X - S$ in range category $\wp(X)$. So C_X is a contravariant functor. We show $C_Y \circ f^\sharp \circ C_X \dashv f^{-1}$. In view of equivalent condition of adjunction, we will prove that $C_Y \circ f^\sharp \circ C_X(T) \subseteq B$ if and only if $T \subseteq f^{-1}(B)$ for any subsets T of X and B of Y. Let $C_Y \circ f^\sharp \circ C_X(T) \subseteq B$ i.e. $Y - f^\sharp(X-T) \subseteq B$. Then $Y - B \subseteq f^\sharp(X-T)$ and so $f^{-1}(Y-B) \subseteq f^{-1}(f^\sharp(X-T))$. But $f^{-1} \dashv f^\sharp$, so counit of adjunction implies that $X - f^{-1}(B) \subseteq X - T$ i.e $T \subseteq f^{-1}(B)$. Similarly the reverse condition holds. Therefore, f^{-1} has two left adjoints namely Imf and $C_Y \circ f^\sharp \circ C_X$ and so by uniqueness of adjoints $Imf \cong C_Y \circ f^\sharp \circ C_X$. It follows that $f(T) = (f^\sharp(T^c))^c$ and so $f(T^c) = (f^\sharp(T))^c$ and $(f(T^c))^c = f^\sharp(T)$ for any subset T of X.

By using Lemma 2.1 above and Theorems 1.1 and 1.2, we are now able to give various characterizations of mono and epi nature of the image functor in our Theorems 2.3 and 2.4 below:

Theorem 2.3. *For a morphism $f : X \to Y$ of category $\mathcal{C}$, following conditions are equivalent:*

(a) $f(-)$ *is monic.*
(b) f^{-1} *is essentially surjective on objects.*
(c) *The unit of adjunction* $f(-) \dashv f^{-1}$ *is iso.*
(d) $f(-)$ *is full and faithful.*
(e) $f^\sharp$ *is full and faithful.*
(f) *The counit of adjunction* $f^{-1} \dashv f^\sharp$ *is iso.*
(g) $f^\sharp$ *is monic.*

Proof. (a)⇒(b): By Lemma 2.1(i), we have $f(-) \circ f^{-1} \circ f(-) = f(-)$ or

$$Sub_{\mathcal{C}}[X] \underset{1_{Sub_{\mathcal{C}}[X]}}{\overset{f^{-1} \circ f(-)}{\rightrightarrows}} Sub_{\mathcal{C}}[X] \xrightarrow{\text{f(-)}} Sub_{\mathcal{C}}[Y]$$

is commutative.
But $f(-)$ is monic implies $f^{-1} \circ f(-) = 1_{Sub_{\mathcal{C}}[X]}$ and therefore $[f^{-1}(f(m))] = [m]$ for any object $[m]$ of $Sub_{\mathcal{C}}[X]$. So for any object $[m]$ of $Sub_{\mathcal{C}}[X]$, there is $[f(m)]$ such that $f^{-1}[f(m)] = [m]$. Hence f^{-1} is essentially surjective on objects.
(b)⇒(c): Let $[m]$ be any object of $Sub_{\mathcal{C}}[X]$ then f^{-1} is essentially surjective on objects imply that there exist object $[n_1]$ of $Sub_{\mathcal{C}}[Y]$ such that $[f^{-1}(n_1)] = [m]$. So $f^{-1} \circ f(-)[f^{-1}(n_1)] = f^{-1} \circ f(-)[m]$ i.e. $[f^{-1} \circ f(-) \circ f^{-1}(n_1)] = [f^{-1} \circ f(-)(m)]$ But triangle identities above in Lemma 2.1(i) imply that $[f^{-1}(n_1)] = [f^{-1} \circ f(-)(m)]$. So $[m] = [f^{-1} \circ f(-)(m)]$ for any object $[m]$ of $Sub_{\mathcal{C}}[X]$. Therefore $f^{-1} \circ f(-) = 1_{Sub_{\mathcal{C}}[X]}$ and so $f^{-1} \circ f(-) \cong 1_{Sub_{\mathcal{C}}[X]}$ i.e. the identity natural transformation $1_{Sub_{\mathcal{C}}[X]} \Rightarrow f^{-1} \circ f(-)$ is iso. But $f^{-1} \circ f(-)$ is defined on partial order category and there exist at most one arrow in this category. Therefore the unit of adjunction $\eta : 1_{Sub_{\mathcal{C}}[X]} \Rightarrow f^{-1} \circ f(-)$ must be the identity natural transformation. Hence unit of adjunction $f(-) \dashv f^{-1}$ is iso.
(c)⇒(a): For proving $f(-)$ to be monic consider the following commutative diagram

$$\mathcal{Z} \overset{G}{\underset{H}{\rightrightarrows}} Sub_{\mathcal{C}}[X] \xrightarrow{\text{f(-)}} Sub_{\mathcal{C}}[Y]$$

where $\mathcal{Z}$ is any category and G, H are functors.
Therefore $f(-) \circ G = f(-) \circ H$ and so $f^{-1} \circ f(-) \circ G = f^{-1} \circ f(-) \circ H$. But by (c) the unit of adjunction is iso and so $f^{-1} \circ f(-) \cong 1_{Sub_{\mathcal{C}}[X]}$. Therefore by Remark 2.2, $f^{-1} \circ f(-) = 1_{Sub_{\mathcal{C}}[X]}$. Hence $G = H$ and so $f(-)$ is monic.
(c)⇔(d)⇔(e)⇔(f) follows using Theorems 1.1 and 1.2.
(f)⇒(g): The counit of adjunction is iso implies $f^\sharp$ is monic can be proved similarly as (c)⇒(a) by using $f^\sharp$ in place of $f(-)$.
(g)⇒(f): Also follows from the triangle identity $f^\sharp \circ f^{-1} \circ f^\sharp = f^\sharp$. □

Theorem 2.4. *For a morphism $f : X \to Y$ of category $\mathcal{C}$, following conditions are equivalent:*

(a) $f(-)$ *is epi.*
(b) $f(-)$ *is essentially surjective on objects.*
(c) *Counit of adjunction* $f(-) \dashv f^{-1}$ *is iso.*
(d) $f^\sharp[m] = f(-) \circ f^{-1} \circ f^\sharp[m]$ *for any object* $[m]$ *of* $Sub_{\mathcal{C}}[X]$.
(e) $f^\sharp$ *is essentially surjective on objects.*
(f) $f^\sharp$ *is epi.*
(g) *Unit of adjunction* $f^{-1} \dashv f^\sharp$ *is iso.*
(h) f^{-1} *is full and faithful.*

Proof. (a)⇒(b): By Lemma 2.1(i), we have $f(-) \circ f^{-1} \circ f(-) = f(-)$ or

$$Sub_{\mathcal{C}}[X] \xrightarrow{f(-)} Sub_{\mathcal{C}}[Y] \overset{f(-) \circ f^{-1}}{\underset{1_{Sub_{\mathcal{C}}[Y]}}{\rightrightarrows}} Sub_{\mathcal{C}}[Y]$$

is commutative.
Further proof is similar to (a)⇒(b) part of Theorem 2.3 and hence is omitted.
(b)⇒(c)⇒(a) : Proof is similar to (b)⇒(c)⇒(a) part of Theorem 2.3 and hence is omitted.
(c) ⇒(d): Counit of adjunction $f(-) \dashv f^{-1}$ is iso implies $f(-) \circ f^{-1}[n_1] = [n_1]$ for any object $[n_1]$ of $Sub_{\mathcal{C}}[Y]$. In particular, for $[n_1] = f^\sharp[m]$ we have $f(-) \circ f^{-1} \circ f^\sharp[m] = f^\sharp[m]$ for any object $[m]$ of $Sub_{\mathcal{C}}[X]$.
(d)⇒(e): Let $[n_1]$ be any object of $Sub_{\mathcal{C}}[Y]$. For $[m] = f^{-1}[n_1] = [f^{-1}(n_1)]$ in (d) implies $f(-) \circ f^{-1} \circ f^\sharp[f^{-1}(n_1)] = f^\sharp[f^{-1}(n_1)]$. So $f(-) \circ f^{-1}[n_1] = f^\sharp \circ f^{-1}[n_1]$ using Lemma 2.1(ii). Now counit of adjunction $f(-) \dashv f^{-1}$ and unit of adjunction $f^{-1} \dashv f^\sharp$ imply that there is arrow $f(-) \circ f^{-1}[n_1] \to [n_1] \to f^\sharp \circ f^{-1}[n_1]$ i.e. $[f(-) \circ f^{-1}(n_1)] \to [n_1] \to [f^\sharp \circ f^{-1}(n_1)]$ for any object $[n_1]$ of $Sub_{\mathcal{C}}[Y]$. So $[f(-) \circ f^{-1}(n_1)] = [f^\sharp \circ f^{-1}(n_1)] = [n_1]$. Hence $f^\sharp$ is essentially surjective on objects.
(e)⇔(f) ⇔ (g) : It can be proved similarly as we have proved (a)⇔(b)⇔(c) by taking $f^\sharp$ in place of $f(-)$ and using the triangle identity $f^\sharp \circ f^{-1} \circ f^\sharp = f^\sharp$.
Also using Theorem 1.1 and Theorem 1.2 it follows that (c)⇔ (g)⇔ (h). □

Remark 2.4. In particular in view of Remarks 1.1 and 1.2, for the category **Sets**, results of Theorem 2.3 and Theorem 2.4 hold. Since in **Sets** f is one-one(onto) if and only if f is monic(epi) [4]. So in **Sets** all the conditions of Theorem 2.3 are equivalent to f is one-one and of Theorem 2.4 are equivalent to f is onto.

Corollary 2.3. *Let $f(-) \dashv f^{-1}$ corresponding to a morphism $f : X \to Y$ of $\mathcal{C}$. Then following are equivalent conditions:*

(a) $f(-)$ *is epi.*
(b) $f(-)$ *is left-adjoint left inverse of f^{-1}.*

Proof. (a)⇒(b): Firstly, let $f(-)$ be epi. Then by the equivalence of (a) and (c) of Theorem 2.4, it follows that counit of adjunction is iso i.e. $\epsilon : f(-) \circ f^{-1} \Rightarrow 1_{Sub_{\mathcal{C}}[Y]}$ is iso. But $Sub_{\mathcal{C}}[Y]$ is a partial order category. Therefore, $f(-) \circ f^{-1} = 1_{Sub_{\mathcal{C}}[Y]}$ and so counit of adjunction is identity. Hence $f(-)$ is left-adjoint left-inverse of f^{-1}.
(b)⇒(a): Let $f(-)$ be left-adjoint left inverse of f^{-1} then counit of adjunction be identity. Therefore, $\epsilon : f(-) \circ f^{-1} \Rightarrow 1_{Sub_{\mathcal{C}}[Y]}$ is an isomorphism. Hence $f(-)$ is epi. □

Corollary 2.4. *For the adjunction $f^{-1} \dashv f^{\sharp}$ following are equivalent conditions:*

(a) $f(-)$ *is monic or* $f^{\sharp}$ *is monic.*
(b) f^{-1} *is left-adjoint left-inverse of* $f^{\sharp}$.

Proof. (a)⇒(b): Let $f^{-1} \dashv f^{\sharp}$ and $f(-)$ be monic or $f^{\sharp}$ be monic. Then by the equivalence of (a) and (f) or by (f) and (g) of Theorem 2.3, it follows that counit of adjunction is iso i.e. $\beta : f^{-1} \circ f^{\sharp} \Rightarrow 1_{Sub_{\mathcal{C}}[X]}$ is iso. But $Sub_{\mathcal{C}}[X]$ is a partial order category. Therefore, $f^{-1} \circ f^{\sharp} = 1_{Sub_{\mathcal{C}}[X]}$ and so counit of adjunction is identity. Hence f^{-1} is left-adjoint left inverse of $f^{\sharp}$.
(b)⇒(a): Let f^{-1} be left-adjoint left inverse of $f^{\sharp}$ and so counit of adjunction is identity. Therefore, $\beta : f^{-1} \circ f^{\sharp} \Rightarrow 1_{Sub_{\mathcal{C}}[X]}$ is an isomorphism. Hence $f(-)$ as well as $f^{\sharp}$ is monic. □

For our further results, firstly we will define saturated objects.

Definition 2.3. An object $[m_1]$ of $Sub_{\mathcal{C}}[X]$ is called **saturated** if f^{-1} is essentially surjective on that object i.e. there exist object $[n_1]$ of $Sub_{\mathcal{C}}[Y]$ such that $f^{-1}[n_1] = [m_1]$.

Theorem 2.5. *For the adjunction $< f^{-1}, f^{\sharp}, \alpha, \beta >: Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[X]$ and an object $[m_1]$ of $Sub_{\mathcal{C}}[X]$, following are equivalent conditions:*

(a) $[m_1]$ *is saturated.*
(b) $f^{-1} \circ f^{\sharp}[m_1] = [m_1]$.

Proof. (a)⇒(b) : Firstly, let $[m_1]$ be saturated object of $Sub_{\mathcal{C}}[X]$. Then $f^{-1}[n_1] = [m_1]$ for some object $[n_1]$ of $Sub_{\mathcal{C}}[Y]$. Therefore, there exists $1_{[m_1]} : f^{-1}[n_1] = [m_1] \to [m_1]$ in $Sub_{\mathcal{C}}[X]$.
Now, $f^{-1} \dashv f^{\sharp}$ with counit of adjunction $\beta : f^{-1} \circ f^{\sharp} \Rightarrow 1_{Sub_{\mathcal{C}}[X]}$ and so $\beta_{[m_1]}$: $f^{-1} \circ f^{\sharp}[m_1] \to [m_1]$ is the natural component.
Therefore, for the objects $[m_1]$ of $Sub_{\mathcal{C}}[X]$ and $[n_1]$ of $Sub_{\mathcal{C}}[Y]$ and an arrow $1_{[m_1]}$: $f^{-1}[n_1] \to [m_1]$, basic properties of adjunction implies that there exists unique arrow

$[n_1] \to f^\sharp[m_1]$ such that the following triangle is commutative:

$$\begin{array}{ccc} f^{-1}[n_1] & \longrightarrow & f^{-1} \circ f^\sharp[m_1] \\ & \searrow^{1_{[m_1]}} & \downarrow \beta_{[m_1]} \\ & & [m_1] \end{array}$$

and so there is an arrow $f^{-1}[n_1] \to f^{-1} \circ f^\sharp[m_1]$ i.e. there is an arrow $[m_1] \to f^{-1} \circ f^\sharp[m_1]$.
Therefore, $f^{-1} \circ f^\sharp[m_1] = [m_1]$. Hence (b) holds.

Conversely, let $f^{-1} \circ f^\sharp[m_1] = [m_1]$ then for the object $[n_1] = f^\sharp[m_1]$ of $Sub_{\mathcal{C}}[Y]$, $f^{-1}[n_1] = [m_1]$ and so $[m_1]$ is saturated. Hence (a) holds. □

Theorem 2.6. *For the adjunction* $< f(-), f^{-1}, \eta, \epsilon >$: $Sub_{\mathcal{C}}[X] \to Sub_{\mathcal{C}}[Y]$ *and an object* $[m_1]$ *of* $Sub_{\mathcal{C}}[X]$*, following are equivalent conditions:*

(a) $[m_1]$ *is saturated.*
(b) $f^{-1} \circ f(-)[m_1] = [m_1]$.

Proof. Proof is similar to Theorem 2.5 and follows from the basic properties of adjunction $f(-) \dashv f^{-1}$

$$\begin{array}{ccc} [m_1] & & \\ \eta_{[m_1]} \downarrow & \searrow^{1_{[m_1]}} & \\ f^{-1} \circ f(-)[m_1] & \longrightarrow & f^{-1}[n_1] \end{array}$$

where $f^{-1}[n_1] = [m_1]$. □

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