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SEMT Labelings and Deficiencies of Forests with Two Components (I)

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Abstract. A set of nodes called *vertices* V accompanied with the lines that bridge these nodes called *edges* E compose an explicit figure termed as a *graph* $G(V, E)$. $|V(G)| = \nu$ and $|E(G)| = \varepsilon$ specify its *order* and *size* respectively. A (ν, ε)-graph G determines an *edge-magic total (EMT) labeling* when $\Gamma : V(G) \cup E(G) \rightarrow \{\overline{1, \nu + \varepsilon}\}$ is bijective so as the weights at every edge are the same constant (say) c i.e., for $x, y \in V(G)$; $\Gamma(x) + \Gamma(xy) + \Gamma(y) = c$, independent of the choice of any $xy \in E(G)$, such a number is interpreted as a *magic constant*. If all vertices gain the smallest of the labels then an EMT labeling is called a *super edge-magic total (SEMT) labeling*. If a graph G allows at least one SEMT labeling then the smallest of the magic constants for all possible distinct SEMT labelings of G describes *super edge-magic total (SEMT) strength*, $sm(G)$, of G. For any graph G, *SEMT deficiency* is the least number of isolated vertices which when uniting with G yields a SEMT graph. In this paper, we will find SEMT labeling and deficiency of forests consisting of two components, where one of the components for each forest is generalized comb $Cb_\tau(\underbrace{\ell, \ell, \ldots, \ell}_{\tau-times})$ and other component is a star, bistar, comb or path respectively, moreover, we will investigate SEMT strength of aforesaid generalized comb.

AMS (MOS) Subject Classification Codes: 05C78
Key Words: SEMT labeling, SEMT strength, SEMT Deficiency, generalized comb, star, bistar, comb, path.

1. Preliminaries

Labeling is a technique that allots labels to the components of a graph. *Total labeling* gives us both components (vertices and edges) labelled. A (ν, ε)-graph G determines an *edge-magic total (EMT) labeling* when $\Gamma : V(G) \cup E(G) \to \{\overline{1, \nu + \varepsilon}\}$ is bijective so as the weights at every edge are the same constant (say) c, such a number c is interpreted as a *magic constant*. If all vertices gain the smallest of the labels then an EMT labeling is called a *super edge-magic total (SEMT) labeling*. Kotzig and Rosa [17] and Enomoto *et al.* [7] first introduced the notions of EMT and SEMT graphs respectively and presented the conjectures: *every tree is EMT* [17], and *every tree is SEMT* [7].

If a graph G allows at least one SEMT labeling then the smallest of the magic constants for all possible distinct SEMT labelings of G describes *super edge-magic total (SEMT) strength*, $sm(G)$, of G. Avadayappan *et al.* first introduced the notion of SEMT strength [4] and found exact values of SEMT strength for some graphs.

In [17], the notion of *EMT deficiency* was proposed by authors and Figueroa-Centeno *et al.* [8] continued it to SEMT graphs. For any graph G, the *SEMT deficiency*, signified as $\mu_s(G)$, is the least number n of isolated vertices that we have to take in union with G so that the resulting graph $G \cup nK_1$ is SEMT, the case $+\infty$ will arise if no isolated vertex fulfils this criteria. More specifically,

$$\mu_s(G) = \begin{cases} minM(G) & if\, M(G) \neq \emptyset \\ +\infty & if\, M(G) = \emptyset \end{cases}$$

where $M(G) = \{n \geq 0 : G \cup nK_1 \textit{ is a SEMT graph}\}$.

Exact values for SEMT deficiencies of several classes of graphs are provided in [9, 8]. The authors also proposed a conjecture which tells us about the confined deficiencies of the forests. In [10], an assumption was made as a special case of a previous one that says, the deficiency of each two-tree forest is not more than 1. Baig *et al.* [6] determined SEMT deficiencies of various forests made up of banana trees, stars etc. In [13, 21], S. Javed *et al.* and Ngurah *et al.* gave some upper bounds for SEMT deficiency of forests composed of stars, fans, combs, double fans, wheels and generalized combs. The results in [1, 2, 3, 5, 14, 15, 16, 18, 19, 20] migth found useful in the aspect of examined labeling here. A general reference to graph theoretic terminologies can be found in [22]. For more review, see the recent survey of graph labelings by Gallian [12].

In this paper, we formulated the results on SEMT labeling and deficiency of forests consisting of two components, where one component in each forest is a generalized comb and the other component is a star, bistar, comb or path respectively. Moreover, SEMT strength of a generalized comb $Cb_\tau(\underbrace{\ell, \ell, \ldots, \ell}_{\tau-times})$ has also been discussed here. The values of parameters of the star, bistar, comb and path are totally dependant on the parameters involved in the generalized comb.

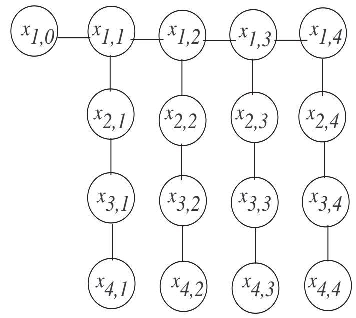

FIGURE 1. Generalized comb $Cb_4(4, 4, 4, 4)$

2. MAIN RESULTS

A star on n vertices is isomorphic to $K_{1,n-1}$. When we join two stars $K_{1,g}$, $K_{1,h}$ through a bridge, where $g, h \geq 1$ and $g + h = n - 2$, the resulting tree is termed a *bistar* $BS(g, h)$. The graph P_n denotes the *path* of order n and size $n - 1$, with vertices labelled from x_1 to x_n along P_n. The *comb* Cb_n [6] is an acyclic graph consisting of P_n together with $n - 1$ new pendant vertices $y_1, y_2, ..., y_{n-1}$ adjacent to $x_2, x_3, ..., x_n$ respectively, thus the new edges obtained are $\{x_{\imath+1}y_\imath : \imath \in \{\overline{1, n-1}\}\}$. A *generalized comb* [13] is basically a detailing (or subdivision) of a comb's pendant vertices hanging from the main horizontal path to form τ hanging paths of order $\ell_\imath$, this is denoted by $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau)$. When $\ell_1 = \ell_2 = \ldots = \ell_\tau = \ell$, then a generalized comb transforms into a balanced generalized comb $Cb_\tau(\underbrace{\ell, \ell, \ldots, \ell}_{\tau-times})$, which can precisely be denoted as $Cb_\tau(\ell, \ell, \ldots, \ell)$, as elaborated in fig. 1.

The following Lemma is an elementary tool for proving graphs to be SEMT. It will be used as a base in each result presented in this work.

Lemma 2.1. [11] *A (ν, ε)-graph G is SEMT if and only if $\exists$ a bijective map $\Gamma : V(G) \rightarrow \{\overline{1, \nu}\}$ s.t. the set of edge-sums*

$$S = \{\Gamma(l) + \Gamma(m) : lm \in E(G)\}$$

constructs ε consecutive Z^+. In that case, G can extend to a SEMT labeling of G with magic constant $c = \nu + \varepsilon + min(S)$ and

$$S = \{c - (\nu + \varepsilon), c - (\nu + \varepsilon) + 1, \ldots, c - (\nu + 1)\}.$$

To understand the lemma 2.1, we consider an example, see fig. 2, where it is shown that if a graph constitutes consecutive edge-sums then its super edge-magicness is assured.

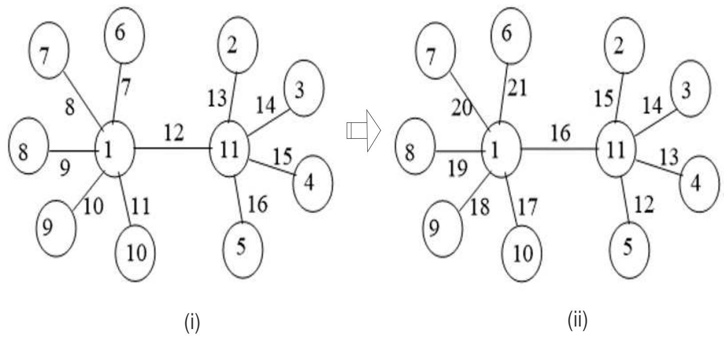

FIGURE 2. (i) A Bistar $BS(5,4)$ with consecutive edge-sums, (ii) A SEMT Bistar $BS(5,4)$ with magic constant $c = 28$

It can be seen easily that the following result about SEMT graphs also holds i.e.,

Note. [4] Let $c(\Gamma)$ be a magic constant of a SEMT labeling Γ of $G(V,E)$, then we end up on this statement:

$$\varepsilon\ c(\Gamma) = \sum_{v\in V} deg_G(v)\Gamma(v) + \sum_{p\in E}\Gamma(p), \qquad \varepsilon = |E(G)| \tag{2. 1}$$

For a single graph, many SEMT labelings might exist and of-course for a different labeling, there will be a different magic constant. Hereafter, we are going to find the bounds for magic constants of SEMT labelings of the generalized comb.

We can see that, $Cb_\tau(\ell_1,\ell_2,\ldots,\ell_\tau)$, $\ell_1 = \ell_2 = \ldots = \ell_\tau$; $\tau \geq 2$ has $\tau\ell + 1$ vertices and $\tau\ell$ edges. From these vertices, $\tau - 1$ vertices have degree 3, $\varepsilon + 1 - 2\tau$ vertices have degree 2, and the remaining $\tau + 1$ vertices have degree 1, see fig 1. Consider that $Cb_\tau(\ell_1,\ell_2,\ldots,\ell_\tau)$ has an EMT labeling with magic constant *"c"*, then εc where $\varepsilon = \tau\ell$, can not be less than the sum we achieve when we allocate the degree-3 vertices with lowest $\tau - 1$ labels, the $\varepsilon + 1 - 2\tau$ next lowest labels to degree-2 vertices, and $\tau + 1$ next lowest labels to degree-1 vertices; in other words:

$$\varepsilon\, c \quad \geq \quad 3\sum_{\imath=1}^{\tau-1}\imath + 2\sum_{\imath=\tau}^{\varepsilon-\tau}\imath + \sum_{\imath=\varepsilon-\tau+1}^{\varepsilon+1}\imath + \sum_{\imath=\varepsilon+2}^{2\varepsilon+1}\imath$$

We can get the upper bound for εc by assigning highest $\tau - 1$ labels to vertices of degree 3, $\varepsilon - 2\tau + 1$ next highest labels to vertices of degree 2, and $\tau + 1$ next highest labels to vertices of degree 1, in other words:

$$\varepsilon\, c \quad \leq \quad 3\sum_{\imath=2\varepsilon-\tau+3}^{2\varepsilon+1}\imath + 2\sum_{\imath=\varepsilon+\tau+2}^{2\varepsilon-\tau+2}\imath + \sum_{\imath=\varepsilon+1}^{\varepsilon+\tau+1}\imath + \sum_{\imath=1}^{\varepsilon}\imath$$

Consequently, we end up with the following result;

Lemma 2.2. *If $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau)$; $\tau \geq 2$ is EMT graph, then magic constant " c" is in the following interval:*

$$\frac{1}{2\varepsilon}(5\varepsilon^2 + 2\tau^2 - 2\varepsilon\tau + 7\varepsilon - 2\tau + 2) \leq c \leq \frac{1}{2\varepsilon}(7\varepsilon^2 - 2\tau^2 + 2\varepsilon\tau + 5\varepsilon + 2\tau - 2);\ \varepsilon = \tau\ell$$

By similar process, we can get following result for SEMT graphs:

Lemma 2.3. *If $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau)$; $\tau \geq 2$ is SEMT graph, then magic constant " c" is in the following interval:*

$$\frac{1}{2\varepsilon}(5\varepsilon^2 + 2\tau^2 - 2\varepsilon\tau + 7\varepsilon - 2\tau + 2) \leq c \leq \frac{1}{2\varepsilon}(5\varepsilon^2 - 2\tau^2 + 2\varepsilon\tau + 7\varepsilon + 2\tau - 2);\ \varepsilon = \tau\ell$$

3. SEMT Strength of Generalized Comb

From SEMT labeling for a generalized comb $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau)$, $\ell_1 = \ell_2 = \ldots = \ell_\tau$; $\tau \geq 2$, [13], we have magic constant $c = 2\tau\ell + \lceil \frac{\tau\ell}{2} \rceil + 3$ and by the given lower bound of magic constants in Lemma 2.3, we have:

Theorem 3.1. *The SEMT strength for generalized comb $G \cong Cb_\tau(\ell, \ell, \ldots, \ell)$, $\tau \geq 2$ is, for $\varepsilon = \tau\ell$:*

$$\frac{5\varepsilon^2 + 2\tau^2 - 2\varepsilon\tau + 7\varepsilon - 2\tau + 2}{2\varepsilon} \leq sm(G) \leq 2\varepsilon + \left\lceil \frac{\varepsilon}{2} \right\rceil + 3$$

4. SEMT Labeling and Deficiency of Forests formed by Generalized Comb and Star, Generalized Comb and Bistar

In this section, it is shown that the two forests, made up of two components i.e., generalized comb and star, generalized comb and bistar, are SEMT. The first result of this section can be concluded as follows:

Theorem 4.1. *For $\ell \geq 2$, $\tau \geq 2$*
(a): $Cb_\tau(\ell, \ell, \ldots, \ell) \cup K_{1,\varpi}$ is SEMT.
(b): $\mu_s(Cb_\tau(\ell, \ell, \ldots, \ell) \cup K_{1,\varpi-1}) \leq 1$; $(\tau, \ell) \neq (2, 2)$,
where $\varpi \geq 1$ and is given by $\varpi = \lfloor \frac{\tau\ell - 1}{2} \rfloor$.

Proof. **(a):** Consider the graph $G \cong Cb_\tau(\ell, \ell, \ldots, \ell) \cup K_{1,\varpi}$.
Here $V(K_{1,\varpi}) = \{y_p; 1 \leq p \leq \varpi + 1\}$ and $E(K_{1,\varpi}) = \{y_1 y_p; 2 \leq p \leq \varpi + 1\}$.
Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so $\nu = \tau\ell + \varpi + 2$ and $\varepsilon = \tau\ell + \varpi$.
Valuation $\Gamma : V(Cb_\tau(\ell, \ell, \ldots, \ell)) \to \{\overline{1, \ell\tau + 1}\}$ is described as follows:

$$\Gamma(x_{\imath,\jmath}) = \begin{cases} \frac{\imath+1}{2} + \frac{\ell(\jmath-1)}{2} & ; \imath, \jmath \equiv 1 (mod\, 2) \\ \frac{\ell\jmath}{2} - \frac{\imath}{2} + 1 & ; \imath, \jmath \equiv 0 (mod\, 2) \end{cases}$$

Now consider the labeling $\Omega : V(G) \to \{\overline{1, \nu}\}$.
For $1 \leq p \leq \varpi + 1$

$$\Omega(y_p) = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + 1 & ; p = 1 \\ \tau\ell + p & ; p \neq 1 \end{cases}$$

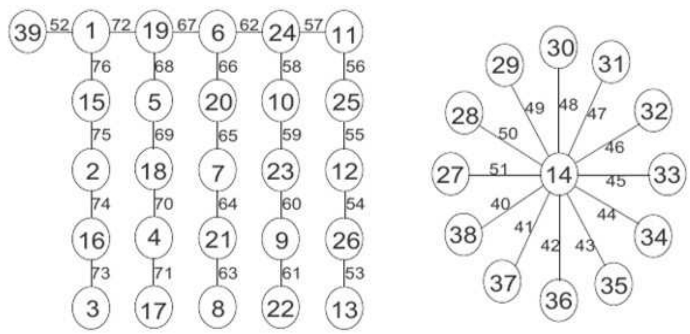

FIGURE 3. SEMT labeling of $Cb_5(5,5,5,5,5) \cup K_{1,12}$

Let $A = \lceil \frac{\tau\ell}{2} \rceil + 1$ and $B = \tau\ell + \varpi + 1$, then

$$\Gamma(x_{\imath,\jmath}) = \begin{cases} A + \frac{\ell(\jmath-1)}{2} + \frac{\imath}{2} & ; \imath \equiv 0 (mod\, 2), \jmath \equiv 1 (mod\, 2) \\ A + \frac{\ell\jmath}{2} - \frac{\imath-1}{2} & ; \imath \equiv 1 (mod\, 2), \jmath \equiv 0 (mod\, 2) \end{cases}$$

$$\Gamma(x_{1,0}) = B + 1 = \tau\ell + \varpi + 2$$

$$\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \ell, 0 \leq \jmath \leq \tau.$$

The edge-sums of G induced by the above labeling Ω form consecutive integers starting from $\hbar + 1$ and ending on $\hbar + \varepsilon$, where $\hbar = \lceil \frac{\tau\ell}{2} \rceil + 2$. Hence from Lemma 2.1, we end up on a SEMT graph with $c = \lceil \frac{\tau\ell}{2} \rceil + 2\tau\ell + 2\varpi + 5$.

(b): Let $\acute{G} \cong Cb_\tau(\ell, \ell, \ldots, \ell) \cup K_{1,\varpi-1} \cup K_1$; $(\tau, \ell) \neq (2,2)$
so,

$$V(\acute{G}) = V(Cb_\tau(\ell, \ell, \ldots, \ell)) \cup V(K_{1,\varpi-1}) \cup \{z\}$$

$$V(K_{1,\varpi-1}) = \{y_p; 1 \leq p \leq \varpi\}$$

$$E(K_{1,\varpi-1}) = \{y_1 y_p; 2 \leq p \leq \varpi\}$$

Let $\acute{\nu} = |V(\acute{G})| = \tau\ell + \varpi + 2$ and $\acute{\varepsilon} = |E(\acute{G})| = \tau\ell + \varpi - 1$. Keeping in mind the valuation Γ defined in (a), we describe the labeling $\acute{\Omega} : V(\acute{G}) \to \{\overline{1, \acute{\nu}}\}$ as

$$\acute{\Omega}(x_{1,0}) = \tau\ell + \varpi + 2$$

$$\acute{\Omega}(z) = \tau\ell + \varpi + 1$$

$$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \ell, 0 \leq \jmath \leq \tau.$$

The edge-sums of $\acute{G}$ induced by the above labeling $\acute{\Omega}$ form consecutive integers starting from $\acute{\hbar} + 1$ and ending on $\acute{\hbar} + \acute{\varepsilon}$, where $\acute{\hbar} = \lceil \frac{\tau\ell}{2} \rceil + 2$. Hence from Lemma 2.1, we end up on a SEMT graph with $\acute{c} = \acute{\nu} + \acute{\varepsilon} + \acute{\hbar} + 1$. □

In the formulation of next results, we will use the labeling Γ provided in previous theorem 4.1.

Theorem 4.2. *For* $\ell, \tau \geq 2; \omega, \varpi \geq 1$,
(a)*:* $Cb_\tau(\ell, \ell, \dots, \ell) \cup BS(\omega, \varpi)$ *is SEMT,* $(\ell, \tau) \neq (2, 2)$.
(b)*:* $\mu_s(Cb_\tau(\ell, \ell, \dots, \ell) \cup BS(\omega, \varpi - 1)) \leq 1; (\ell, \tau) \notin \{(3, 2), (2, 3)\}$ *and* $\varpi \geq 2$.
Where ϖ *is given by* $\varpi = \lfloor \frac{\tau\ell - 3}{2} \rfloor$.

Proof. **(a):** Consider the graph $G \cong Cb_\tau(\ell, \ell, \dots, \ell) \cup BS(\omega, \varpi)$; $\ell, \tau \geq 2$, $\omega, \varpi \geq 1$. $V(BS(\omega, \varpi)) = \{z_{ut} : u = 1, 2; 0 \leq t \leq \rho\}$, where

$$\rho = \begin{cases} \omega & ; u = 1 \\ \varpi & ; u = 2 \end{cases}$$

and $E(BS(\omega, \varpi)) = \{z_{10}z_{1\ell}; 1 \leq \ell \leq \omega\} \cup \{z_{10}z_{20}\} \cup \{z_{20}z_{2m}; 1 \leq m \leq \varpi\}$.
Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \tau\ell + \omega + \varpi + 3$ and $\varepsilon = \tau\ell + \omega + \varpi + 1$. Keeping in mind the valuation Γ defined in Theorem 4.1 with $A = \lceil \frac{\tau\ell}{2} \rceil + \omega + 1$ and $B = \tau\ell + \omega + \varpi + 2$, we describe the labeling $\Omega : V(G) \to \{\overline{1, \nu}\}$ as

$$\Omega(z_{ut}) = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + t & ; u = 1, t = r, 1 \leq r \leq \omega \\ \lceil \frac{\tau\ell}{2} \rceil + \omega + 1 & ; u = 2, t = 0 \\ \tau\ell + \omega + 2 & ; u = 1, t = 0 \\ \tau\ell + \omega + 2 + t & ; u = 2, t = r, 1 \leq r \leq \varpi \end{cases}$$

$$\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \ell, 0 \leq \jmath \leq \tau$$

$$\Omega(x_{1,0}) = B + 1 = \tau\ell + \omega + \varpi + 3.$$

The edge-sums of G induced by the above labeling Ω form consecutive integers starting from $\hbar + 1$ and ending on $\hbar + \varepsilon$, where $\hbar = \lceil \frac{\tau\ell}{2} \rceil + \omega + 2$. Hence from Lemma 2.1, we end up on a SEMT graph.

(b): Let $\acute{G} \cong Cb_\tau(\ell, \ell, \dots, \ell) \cup BS(\omega, \varpi - 1) \cup K_1$; $\ell, \tau \geq 2$, $\varpi \geq 2$. Here $V(\acute{G}) = V(Cb_\tau(\ell, \ell, \dots, \ell)) \cup V(BS(\omega, \varpi - 1)) \cup \{z\}$ and where $V(BS(\omega, \varpi - 1)) = \{z_{ut} : u = 1, 2; 0 \leq t \leq \rho\}$, where

$$\rho = \begin{cases} \omega & ; u = 1 \\ \varpi - 1 & ; u = 2 \end{cases}$$

and $E(BS(\omega, \varpi - 1)) = \{z_{10}z_{1t}; 1 \leq t \leq \omega\} \cup \{z_{10}z_{20}\} \cup \{z_{20}z_{2t}; 1 \leq t \leq \varpi - 1\}$. Let $\acute{\nu} = |V(\acute{G})|$ and $\acute{\varepsilon} = |E(\acute{G})|$, so we get $\acute{\nu} = \tau\ell + \omega + \varpi + 3$ and $\acute{\varepsilon} = \tau\ell + \omega + \varpi$. Keeping in mind the valuation Γ defined in Theorem 4.1 with A and B the same as in part (a), we describe the labeling $\acute{\Omega} : V(\acute{G}) \to \{\overline{1, \acute{\nu}}\}$ as

$$\acute{\Omega}(x_{1,0}) = B + 2, \acute{\Omega}(z) = B + 1$$

$$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \ell, 0 \leq \jmath \leq \tau.$$

The edge-sums of $\acute{G}$ induced by the above labeling $\acute{\Omega}$ form consecutive integers starting from $\acute{\hbar} + 1$ and ending on $\acute{\hbar} + \acute{\varepsilon}$, where $\acute{\hbar} = \lceil \frac{\tau\ell}{2} \rceil + \omega + 2 = \hbar$. Hence from Lemma 2.1, we end up on a SEMT graph. $\square$

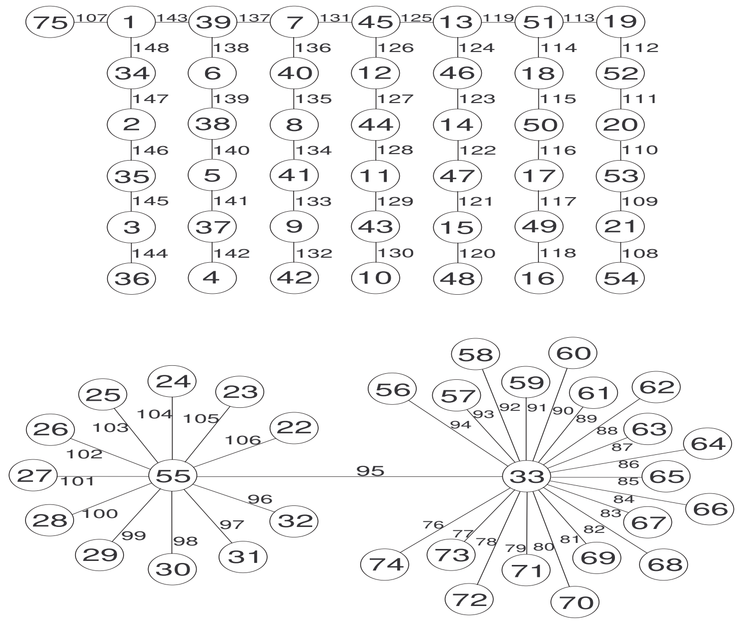

FIGURE 4. SEMT labeling of $Cb_7(6,6,6,6,6,6,6) \cup BS(11,19)$

5. SEMT Forests formed by Generalized Comb and Comb, Generalized Comb and Path

The motivation of this section is to continue the work of exploring forests with two components that are SEMT. In the previous section, we have determined two forests that were SEMT and also provided the situations for their SEMT deficiencies. The next two results of this section give us SEMT labeling for the disjoint union of the generalized combs $Cb_\tau(\ell,\ell,\dots,\ell)$ with comb Cb_ω and path P_ϖ respectively.

Theorem 5.1. *For* $\tau, \ell \geq 2; \omega \geq 1$
(a)*:* $Cb_\tau(\ell,\ell,\dots,\ell) \cup Cb_\omega$ *is SEMT.*
(b)*:* $\mu_s(Cb_\tau(\ell,\ell,\dots,\ell) \cup Cb_{\omega-1}) \leq 1$*;* $\omega \geq 2$*,* $(\ell,\tau) \neq (2,2)$*, where*

$$\omega = \left\lfloor \frac{\tau\ell - 1}{2} \right\rfloor.$$

Proof. **(a):** Consider the graph $G \cong Cb_\tau(\ell,\ell,\dots,\ell) \cup Cb_\omega$, where
$V(Cb_\omega) = \{x_p; 0 \leq p \leq \omega\} \cup \{y_q; 1 \leq q \leq \omega\}$,
$E(Cb_\omega) = \{x_p x_{p+1}; 0 \leq p \leq \omega - 1\} \cup \{x_p y_p; 1 \leq p \leq \omega\}$.
Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \tau\ell + 2\omega + 2$ and $\varepsilon = \tau\ell + 2\omega$. Keeping in mind the valuation Γ defined in Theorem 4.1 with $A = \lceil \frac{\tau\ell}{2} \rceil + \omega + 1$ and $B = \tau\ell + 2\omega + 1$,

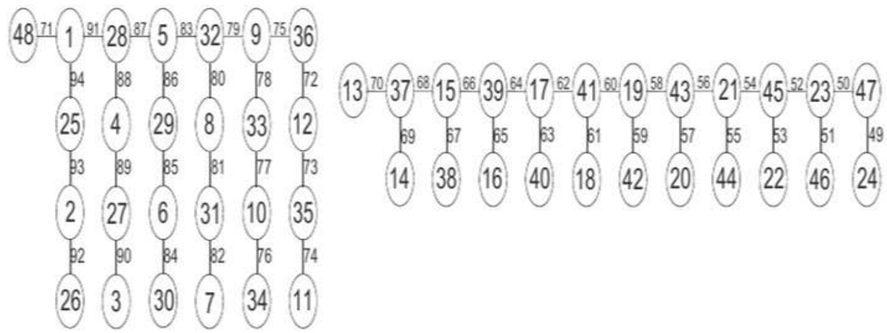

FIGURE 5. SEMT labeling of $Cb_6(4,4,4,4,4,4) \cup Cb_{11}$

we describe the labeling $\Omega : V(G) \rightarrow \{\overline{1,\nu}\}$ as

For $0 \leq p \leq \omega$, $1 \leq q \leq \omega$,

$$\Omega(x_p) = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + p + 1 & ; p \ is \ even \\ \tau\ell + \omega + 1 + p & ; p \ is \ odd \end{cases}$$

and

$$\Omega(y_q) = \begin{cases} \tau\ell + \omega + 1 + q & ; q \ is \ even \\ \lceil \frac{\tau\ell}{2} \rceil + q + 1 & ; q \ is \ odd \end{cases}$$

$$\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \ell, 0 \leq \jmath \leq \tau$$

and

$$\Omega(x_{1,0}) = B + 1.$$

The edge-sums of G induced by the above labeling Ω form consecutive integers starting from $\hbar + 1$ and ending on $\hbar + \varepsilon$, where $\hbar = \lceil \frac{\tau\ell}{2} \rceil + \omega + 2$. Hence from Lemma 2.1, we end up on a SEMT graph.

(b): Let $\acute{G} \cong Cb_\tau(\ell,\ell,\ldots,\ell) \cup Cb_{\omega-1} \cup K_1$, where $V(K_1) = \{z\}$, $V(Cb_{\omega-1}) = \{x_p; 0 \leq p \leq \omega - 1\} \cup \{y_q; 1 \leq q \leq \omega - 1\}$, $E(Cb_{\omega-1}) = \{x_p x_{p+1}; 0 \leq p \leq \omega - 2\} \cup \{x_p y_p; 1 \leq p \leq \omega - 1\}$.

Let $\acute{\nu} = |V(\acute{G})|$ and $\acute{\varepsilon} = |E(\acute{G})|$, so we get $\acute{\nu} = \tau\ell + 2\omega + 1$ and $\acute{\varepsilon} = \tau\ell + 2\omega - 2$. Keeping in mind the valuation Γ defined in Theorem 4.1 with $A = \lceil \frac{\tau\ell}{2} \rceil + \omega$ and $B = \tau\ell + 2\omega - 1$, we describe the labeling $\acute{\Omega} : V(\acute{G}) \rightarrow \{\overline{1,\acute{\nu}}\}$ as

For $0 \leq p \leq \omega - 1$, $1 \leq q \leq \omega - 1$

$$\acute{\Omega}(x_p) = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + p + 1 = \Omega(x_p) & ; p \ is \ even \\ \tau\ell + \omega + p = \Omega(x_p) - 1 & ; p \ is \ odd \end{cases}$$

also

$$\acute{\Omega}(y_q) = \begin{cases} \tau\ell + \omega + q = \Omega(y_q) - 1 & ; q \ is \ even \\ \lceil \frac{\tau\ell}{2} \rceil + q + 1 = \Omega(y_q) & ; q \ is \ odd \end{cases}$$

$$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}), 1 \leq \imath \leq \ell, 0 \leq \jmath \leq \tau$$

$$\acute{\Omega}(x_{1,0}) = B + 2, \acute{\Omega}(z) = B + 1.$$

The edge-sums of $\acute{G}$ induced by the above labeling $\acute{\Omega}$ form consecutive integers starting from $\acute{\hbar} + 1$ and ending on $\acute{\hbar} + \acute{\varepsilon}$, where $\acute{\hbar} = \lceil \frac{\tau\ell}{2} \rceil + \omega + 1$. Hence from Lemma 2.1, we end up on a SEMT graph. □

Theorem 5.2. *For $\tau \geq 2$, $\ell \geq 2$*
$(a)(i)$*: $Cb_\tau(\ell, \ell, \ldots, \ell) \cup P_\varpi$ is SEMT.*
$(a)(ii)$*: $Cb_\tau(\ell, \ell, \ldots, \ell) \cup P_{\varpi-1}$ is SEMT.*
$(b)(i)$*: $\mu_s(Cb_\tau(\ell, \ell, \ldots, \ell) \cup P_{\varpi-2}) \leq 1$; $(\ell, \tau) \neq (2, 2)$.*
$(b)(ii)$*: $\mu_s(Cb_\tau(\ell, \ell, \ldots, \ell) \cup P_{\varpi-3}) \leq 1$; $(\ell, \tau) \neq (2, 2)$. where*

$$\varpi = \begin{cases} \tau\ell & ; \tau, \ell \equiv 1 (mod\, 2) \\ \tau\ell - 1 & ; otherwise \end{cases}$$

Proof. **(a):** Consider the graph $G \cong Cb_\tau(\ell, \ell, \ldots, \ell) \cup P_t$, where $V(P_t) = \{x_p; 1 \leq p \leq t\}$ and $E(P_t) = \{x_p x_{p+1}; 1 \leq p \leq t-1\}$. Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \tau\ell + t + 1$ and $\varepsilon = \tau\ell + t - 1$, where

$$t = \begin{cases} \varpi & ; for(a(i)) \\ \varpi - 1 & ; for(a(ii)) \end{cases}$$

Keeping in mind the valuation Γ defined in Theorem 4.1 with

$$A = \begin{cases} 2\lceil \frac{\tau\ell}{2} \rceil & ; for(a(i)) \\ \lceil \frac{\tau\ell}{2} \rceil + \lfloor \frac{\tau\ell - 1}{2} \rfloor & ; for(a(ii)) \end{cases}$$

and for $a(i)$

$$B = \begin{cases} 2\tau\ell & ; \ell \ is \ odd, \ \tau \ is \ odd \\ 2\tau\ell - 1 & ; otherwise \end{cases}$$

and for $a(ii)$

$$B = \begin{cases} 2\tau\ell - 1 & ; \ell \ is \ odd, \ \tau \ is \ odd \\ 2\tau\ell - 2 & ; otherwise \end{cases}$$

We describe the labeling $\Omega : V(G) \to \{\overline{1, \nu}\}$ as

$$\Omega(x_p) = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + r & ; p = 2r - 1; 1 \leq r \leq \lfloor \frac{t+1}{2} \rfloor \\ \tau\ell + \lceil \frac{\tau\ell}{2} \rceil + \acute{t} & ; p = 2\acute{t}, 1 \leq \acute{t} \leq \lfloor \frac{t}{2} \rfloor, for(a(i)) \\ \tau\ell + \lceil \frac{\tau\ell}{2} \rceil + \acute{t} - 1 & ; p = 2\acute{t}, 1 \leq \acute{t} \leq \lfloor \frac{t}{2} \rfloor, for(a(ii)) \end{cases}$$

furthermore,

$$\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \ell, 0 \leq \jmath \leq \tau$$

$$\Omega(x_{1,0}) = B + 1.$$

The edge-sums of G induced by above labeling Ω form consecutive integers
For $(a)(i)$ $\{\overline{\hbar + 1, \hbar + \varepsilon}\}$

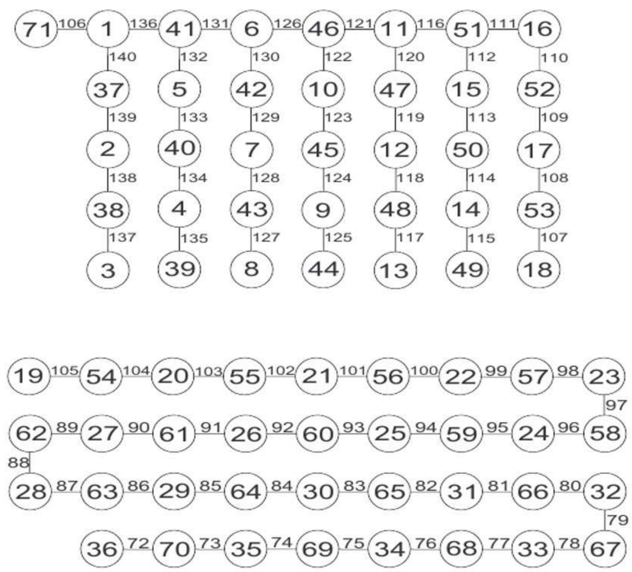

FIGURE 6. SEMT labeling of $Cb_7(5,5,5,5,5,5,5) \cup P_{35}$

For $(a)(ii)$ $\{\overline{\hbar, \hbar+\varepsilon-1}\}$
where

$$\hbar = \begin{cases} \tau\ell+2 & ;\tau \ is\ odd,\ \ell\ is\ odd \\ \tau\ell+1 & ;otherwise \end{cases}$$

Hence from Lemma 2.1, we end up on a SEMT graph.

(b): Let $\acute{G} \cong Cb_\tau(\ell,\ell,\ldots,\ell) \cup P_t \cup K_1$, where $V(P_t) = \{x_p; 1 \le p \le t\}$, $V(K_1) = \{z\}$ and $E(P_t) = \{x_p x_{p+1}; 1 \le p \le t-1\}$. Let $\acute{\nu} = |V(\acute{G})|$ and $\acute{\varepsilon} = |E(\acute{G})|$, so we get $\acute{\nu} = \tau\ell + t + 2$ and $\acute{\varepsilon} = \tau\ell + t - 1$, where

$$t = \begin{cases} \varpi - 2 & ;for(b(i)) \\ \varpi - 3 & ;for(b(ii)) \end{cases}$$

Keeping in mind the valuation Γ defined in Theorem 4.1 with

$$A = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + \lfloor \frac{\tau\ell-1}{2} \rfloor & ;for(b(i)) \\ \lceil \frac{\tau\ell}{2} \rceil + \lfloor \frac{\tau\ell-3}{2} \rfloor & ;for(b(ii)) \end{cases}$$

and for $b(i)$

$$B = \begin{cases} 2\tau\ell-2 & ;\tau \ is\ odd,\ \ell\ is\ odd \\ 2\tau\ell-3 & ;otherwise \end{cases}$$

and for $b(ii)$

$$B = \begin{cases} 2\tau\ell-3 & ;\tau \ is\ odd,\ \ell\ is\ odd \\ 2\tau\ell-4 & ;otherwise \end{cases}$$

We describe the labeling $\acute{\Omega} : V(\acute{G}) \to \{\overline{1, \acute{\nu}}\}$ as

$$\acute{\Omega}(x_p) = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + r & ; p = 2r-1; 1 \le r \le \lfloor \frac{t+1}{2} \rfloor \\ \tau\ell - 1 + \lceil \frac{\tau\ell}{2} \rceil + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, for(b(i)) \\ \tau\ell - 2 + \lceil \frac{\tau\ell}{2} \rceil + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, for(b(ii)) \end{cases}$$

Furthermore,

$$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \le \imath \le \ell, 0 \le \jmath \le \tau$$
$$\acute{\Omega}(x_{1,0}) = B + 2, \acute{\Omega}(z) = B + 1.$$

The edge-sums of $\acute{G}$ induced by above labeling $\acute{\Omega}$ form consecutive integers
For $(b)(i)$ $\{\overline{\acute{\hbar}+1, \acute{\hbar}+\acute{\varepsilon}}\}$
For $(b)(ii)$ $\{\overline{\acute{\hbar}, \acute{\hbar}+\acute{\varepsilon}-1}\}$
where

$$\acute{\hbar} = \begin{cases} \tau\ell + 1 & ; \tau \ is \ odd, \ \ell \ is \ odd \\ \tau\ell & ; otherwise \end{cases}$$

Hence from Lemma 2.1, we end up on a SEMT graph. □

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