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Algorithm of Difference Operator for Computing a Single Root of Nonlinear Equations

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Abstract: In this paper, we have developod an Algorithm of Difference Operator for finding the root of nonlinear equations. In fact, finding the root of nonlinear equations is a classical problem in numerical analysis, which arises in many branches of applied science and engineering. The proposed technique is derived from difference operator and Taylor series, and it is converging cubically. The aim of this paper to find the approximation root of the nonlinear equations with less iteration, good accuracy and without the evaluation of second derivatives. The new algorithm is comparable with the methods of variant of Newton Raphson Method and Halley Method. Few numerical examples are also presented in this paper in order to analyze the efficiency of the developed method with the assessment of existing cubic methods.

AMS (MOS) Subject Classification Codes: —
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1. Introduction

In recent years many researchers have developed several iterative methods for solving nonlinear equations. Actually, nonlinear equations are imperative problem in engineering and applied science [1-3]. For this purpose, numerous numerical methods had been offered and investigate under certain circumstances. Therefore, one of most effective technique is newton raphson technique, such as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1. 1}$$

Where $n = 0, 1, 2, ...$

This method is fast converging numerical technique but not reliable because it is keeping pitfall [4-5]. However, it is most useful and vigorous numerical techniques. Several modifications had been done in Newton Raphson Method by using taylor series and difference operator for finding a single root of a nonlinear equations in literatures [6-11]. Furthermore, increasing computational competence and rate of convergence a Variant of Newton Raphson Method had been proposed by using Quadrature rule [12], such as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + f'(y_n)} \tag{1. 2}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

Likewise, Homeier [14] derived the following cubically convergent numerical method.

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)} - \frac{f(x_n)}{2f'(y_n)} \tag{1. 3}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

Correspondingly, in this paper, we have been suggested an Algorithm of Difference Operator, which is free from second derivative. The efficiency of proposed numerical technique is experienced on few nonlinear problems with variant of Newton Raphson Method and Halley Method [12, 15]. C++ programming is used to defend the proposed method. For the duration of study, it has been detected that the Algorithm of Difference Operator is decent achievement with the assessment of existing cubic methods.

2. Proposed Method

In this segment, we have developed an Algorithm for solving nonlinear equations with the help of taylor series and difference operator, such as

Let Taylor series,

$$f(x_1) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) \tag{2. 4}$$

Where $f(x_1) = 0$, then (2.4) become

$$f(x) + hf'(x) + \frac{h^2}{2} f''(x) = 0 \tag{2. 5}$$

By using difference operator, such as

$$f''(x_n) = \frac{f'(x) + f'(r)}{h} \tag{2. 6}$$

Substitute (2.6) in (2.5), we get

$$f(x) + hf'(x) + \frac{h^2}{2} (\frac{f'(x) + f'(r)}{h}) = 0 \tag{2. 7}$$

by solving (2.7), we get

$$2f(x) + h(3f'(x) - f'(r)) = 0 \tag{2. 8}$$

As we know that $h = x^* - x$, where ‘h‘ is a step size, then (2.8) become

$$2f(x) + (x^* - x)(3f'(x) - f'(r)) = 0 \tag{2. 9}$$

or

$$x^* = x - \frac{2f(x)}{3f'(x) - f'(r)} \tag{2. 10}$$

Finally, we get

$$x_{n+1} = y_n - \frac{2f(y_n)}{3f'(y_n) - f'(x_n)} \tag{2. 11}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

Henceforth, (2.11) is an Algorithm of Difference Operator for finding the root of nonlinear equations.

3. Analysis Of Convergence

This section have presented the analysis of convergence by giving mathematical proof for the order of convergence of the proposed method, which is defined in (2.11).

Theorem 3.1: Let $f : D \subseteq R \rightarrow R$ be a suciently dierentiable function and $a \in D$ be a simple zero of an open interval D. If x_0 is suffciently close to a, then the proposed numerical iterative method has 2nd-order convergence and satisfies the following error equation:

$$e_{n+1} = -c^2 e_n^3 + o^4(h) \tag{3. 12}$$

Proof
Let a be a approximate root of function, for expanding $f(x_n)$ and $f'(x_n)$ by using Taylor series about a and taking only cubic order term about to a, such as

$$f(x_n) = f'(a)(e_n + ce_n^2 + c^2 e_n^3) \tag{3. 13}$$

Or

$$f'(x_n) = f'(a)(1 + 2ce_n + 3c^2 e_n^2 + 4c^3 e_n^3) \tag{3. 14}$$

Note $c = \frac{f'(a)}{2f'(a)}$

From (3.13) and (3.14), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - ce_n^2 + (c^2 - c)e_n^3 \tag{3. 15}$$

From (3.15), we get

$$y_n = ce_n^2 + (c^2 - c)e_n^3 \tag{3. 16}$$

for the need of $f(y_n)$ and $f'(y_n)$, so we will be expand $f(y_n)$ and $f'(y_n)$ in Taylors Series about 'a' and using (3.16), we have

$$f(y_n) = f'(a)(ce_n^2 + (c^2 - c)e_n^3) \tag{3. 17}$$

Or

$$f'(y_n) = f'(a)(1 + 2c^2e_n^2 + 4(c^2 - c^3)e_n^3) \tag{3. 18}$$

By using (3.14), (3.16), (3.17) and (3.18) in (2.11), we get

$$e_{n+1} = ce_n^2 + (c^2 - c)e_n^3 - \frac{2f'(a)(ce_n^2 + (c^2 - c)e_n^3)}{f'(a)(3 + 6c^2e_n^2 + 12(c^2 - c^3)e_n^3) - 1 - 2ce_n - 3c^2e_n^2 - 4c^3e_n^3)} \tag{3. 19}$$

$$e_{n+1} = ce_n^2 + (c^2 - c)e_n^3 - \frac{ce_n^2 + (c^2 - c)e_n^3)}{1 - ce_n - 2ce_n^2 + 4c^2e_n^3} \tag{3. 20}$$

$$e_{n+1} = ce_n^2 + (c^2 - c)e_n^3 - (ce_n^2 + (c^2 - c)e_n^3))[1 - ce_n - 2ce_n^2 + 4c^2e_n^3]^{-1} \tag{3. 21}$$

$$e_{n+1} = ce_n^2 + (c^2 - c)e_n^3 - (ce_n^2 + (c^2 - c)e_n^3))[1 + ce_n + 2ce_n^2 - 4c^2e_n^3] \tag{3. 22}$$

$$e_{n+1} = ce_n^2 + (c^2 - c)e_n^3 - (ce_n^2 + (c^2 - c)e_n^3) + c^2e_n^3) \tag{3. 23}$$

Finally, we get

$$e_{n+1} = -c^2e_n^3 + o^4(h) \tag{3. 24}$$

Hence, this has been proven that the Algorithm of Difference Operator is cubically convergence iterated method.

4. Numerical Results

This session the proposed iterative method is applied on few examples and compared with variant of Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + f'(y_n)} \tag{4. 25}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

and Halley Method

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)f''(x_n)} \tag{4. 26}$$

From numerical results in Table-1, it has been detected that the proposed third-order algorithm is reducing the number of iteration which is less than the number of iterations of the existing cubic methods and as well as accuracy perception. We calculate the numerical results by using the well-known MATLAB/C++ programming with absolute error $|x_{n+1} - x_n| < \epsilon$ where $\epsilon < 10^{10}$, which is obtained in Table-1, such as

TABLE-1

$f(x)$	METHODS	ITERATIONS	ROOT	$\lvert x_{n+1} - x_n\rvert$
Sinx-x+1	Variant Method	3	1.93456	$1.19209e^{-7}$
x_o=2	Halley Method	2		$4.33922e^{-5}$
	New Method	2		$3.21865e^{-6}$
2x-lnx-7	Variant Method	2	4.21991	$1.19209e^{-5}$
x_o=4	Halley Method	3		$3.10421e^{-4}$
	New Method	2		$1.66368e^{-3}$
xe^x–2	Variant Method	8	0.852605	$2.98023e^{-7}$
x_o=1	Halley Method	2		$7.42078e^{-4}$
	New Method	2		$2.50341e^{-6}$
x^3 -9x+1	Variant Method	4	0.111264	$1.49012e^{-7}$
x_o=0	Halley Method	2		$1.53050e^{-4}$
	New Method	2		$1.56462e^{-7}$
Cosx-x^3	Variant Method	6	0.865474	$1.78814e^{-7}$
x_o=1	Halley Method	2		$1.28573e^{-3}$
	New Method	2		$2.98023e^{-6}$

5. Conclusion

The problem of locating root of nonlinear equations occurs frequently in scientific work. In this paper, we have introduced cubic algorithm of difference operator for estimating nonlinear equations. The developed algorithm is free from second derivatives, and it is derived from Taylor series and difference operator. The competence of the proposed algorithm has tested on few nonlinear examples and the results accomplished are summarized in Table-1. Henceforth, analysis of efficiency shows that the developed method is well execution and superlative performance with the existing variant of Newton Raphson Method and Halley Method for solving nonlinear equations.

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