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Semi-Analytical Solutions of Multilayer Flow of Viscous Fluids in a Channel

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Abstract. Multilayer unidirectional flows of viscous, immiscible fluids in a channel bounded by two infinite parallel plates are studied. The bottom plate is translating in its plane with a time-dependent velocity and the upper plate is stationary. A pressure gradient in the flow direction is applied. The solutions of the initial and boundary condition problem are obtained using the Laplace transform method. Numerical Stehfest's algorithm is used in order to obtain the inverse Laplace transforms. The case of two-layers with one fluid-fluid interface is completely studied and influence of the pressure gradient on the fluid behavior is analyzed.

AMS (MOS) Subject Classification Codes: 76D99
Key Words: Multilayer flow, Stehfest's algorithm, Semi-analytical solutions.

1. Introduction

The multilayer flow has gained much interest in the last years especially due to its applications in the design of cooling systems of electronic device, solar energy collection, nuclear reactors and other practical applications. Yih [27] was the first who studied the linear stability of two layer flow in channels with the help of long wave limit. Joseph & Renardy [12] discussed the stability of two layers Couette – Poiseuille flows. Later on, Tilley et al. [24, 25] gave detailed in sight views of linear and non- linear stability of two layers flow in an inclined channel. It is known that it is difficult task to improve quality of the multilayer coating on the different surfaces, but later on this difficulty was overcome by doing experiments on two layers flows of different fluids in channels [23, 18, 2].

Analysis of two-layer flows of viscous non-Newtonian fluids in the channels allow better understanding for defining the necessary parameters of technological processes for manufacturing the multilayer products. The multilayer flows in channels have much attraction due to their vast range of applications in science and technology. The modern technologies based upon micro fluids created a lot of revolutions in the world involving multilayer flows of fluids in micro channels. In the same way the multi layer micro scale got much attention in the field of modern biomedical, medical physics and other field of science.

More recently, Govindarajan [10] investigated three-layer Poiseuille channel flows and highlighted the fact that at higher Schmidt numbers these flows go unstable at lower Reynolds numbers. Matar et al. [15] considered the pressure-driven channel flow of two viscous, immiscible, density-matched fluids in the context of cleaning-type applications with large viscosity contrasts. Hormozi et al. [11] analyzed the problem of multilayer channel flows with yield stress. Gao et al. [7, 8] obtained analytical solutions for velocity profiles and flow rates of two-liquid flow in a micro channel which was driven both by electro osmotic force and pressure gradient. Li et al. [14] studied the steady laminar multilayer flow in microchannel driven by pressure and electro-static forces. They considered N fluid layers with known viscosities and have obtained analytical and numerical solutions for the fluid velocity and shear stress. The convective heat transfer of nanoparticles in multilayer fluid flow has been explored by Vajravelu et al. [26]. Papaefthymiou et al. [17] investigated the dynamics of viscous immiscible pressure driven multilayer flows in channels and studied in detail the system of three stratified layers with two internal fluid-fluid interfaces. Kalmykov et al. [13] studied the two layer flow of magnetic fluids between two horizontal rigid planes. The mechanism of layer distribution, modeling and numerical simulation for three-dimensional flow in the multilayer co-extrusion die were studied by Mun et al. [16]. The relevant literature are presented in $[1, 3, 4, 5, 6, 9, 19, 20, 21, 22, 28]$.

In the present study, we developed an analytical solution of unidirectional and fully developed multilayer flow of incompressible and immiscible viscous fluids in a horizontal channel between two infinite flat plates, with constant pressure gradient in the x- direction. The bottom plate has a translational motion with time-dependent velocity and the upper plate is stationary. Analytical solution of our problem is obtained using the Laplace transform coupled with the classical method of differential equations with constant coefficients. Due to complicated mathematical expressions of the Laplace transform, the inverse Laplace transforms are obtained numerically with the Stehfest's algorithm.

2. Description of the Problem

We consider the flow of *n* immiscibly and incompressible viscous fluids in a horizontal channel between two infinite flat plates (Fig. 1). The viscosity of fluid occupying the slab $h_j \leq y \leq h_{j+1}, j = 0, 1, 2, \cdots, n-1, h_0 = 0, \;\; h_n = h$ is assumed to be $\mu_j, j = 1, 2, \cdots, n.$

The interface between fluids with viscosities $\mu_j \, and \, \mu_{j+1}$ is the plane $y = h_j, j = 1, 2, 3, \cdots,$

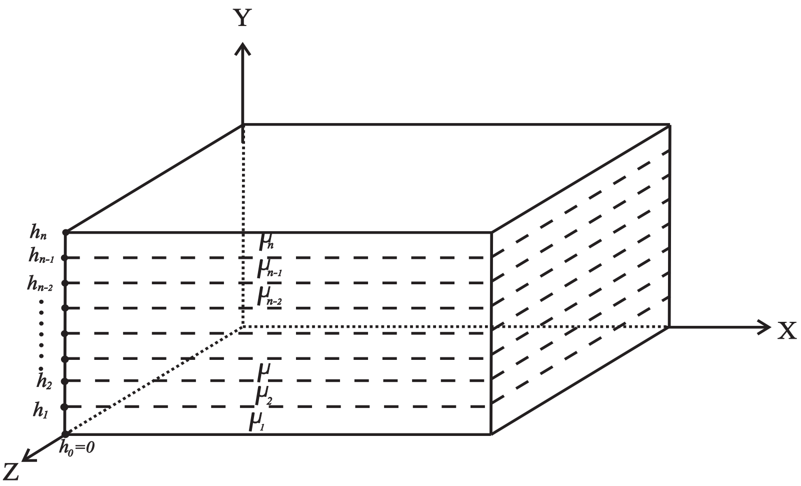

FIGURE 1. Flow geometry

$n-1$. The fluid of density μ_1 has the solid boundary the plate $y = h_0 = 0$ and the fluid with the viscosity μ_n has the solid boundary $y = h_n = h$. The flow is driven by a constant pressure gradient in the *x*- direction, $\frac{\partial p}{\partial x} = -p_0 = constant$. The bottom plate has a translational motion in its plane with the time dependent velocity $U_0 f(t)$ along the $x-$axis. $f(t)$ is a continuous function of exponential order to infinity with $f(0) = 0$. The upper plate is stationary. We assume that the flow is unsteady, unidirectional and fully developed. The flow geometry is shown in Fig.1.

The basic equations which govern the flow of incompressible fluids of constant density are:

- *the continuity equation*

$$div\overrightarrow{V_j} = 0, \qquad j = 1, 2, ..., n, \tag{2. 1}$$

- *the linear momentum equation*

$$\rho_j \frac{d\overrightarrow{V_j}}{dt} = \mu_j divA_{1j} - \nabla p + \rho_j \vec{b}, \tag{2. 2}$$

where the subscript $j = 1, 2, ..., n$ denotes the fluid layer number, with $j = 1$ is the lowest fluid layer that is in contact with the bottom plate, $j = 2$ the fluid layer adjacent to the first layer and so on. The top most layer is represented by $j = n$.

In the above equation $\overrightarrow{V_j}$ denotes the velocity vector of the j^{th} layer, ρ_j is the constant density of the j^{th} fluid layer, $\frac{d\overrightarrow{V_j}}{dt}$ is the material derivative of the velocity field, $\vec{b}$ is the body force vector, p is the fluid pressure and $A_{1j} = L_j + L_j^T$, $L_j = grad\overrightarrow{V_j}$ is the first Rivlin -Ericksen tensor.

In the studied problem, the velocities $\overrightarrow{V_j}$ are assumed to be of the form

$$\overrightarrow{V_j} = u_j(y,t)\overrightarrow{e_x}, \quad j = 1,2,...,n, \tag{2. 3}$$

Based on Eq. (2. 3), the continuity equation (2. 1) is satisfied and the linear momentum equation, in the absence of body force, becomes

$$\rho_j \frac{\partial u_j(y,t)}{\partial t} = \mu_j \frac{\partial^2 u_j(y,t)}{\partial y^2} + p_0, \tag{2. 4}$$

or, into equivalent form

$$\frac{\partial u_j(y,t)}{\partial t} = \nu_j \frac{\partial^2 u_j(y,t)}{\partial y^2} + p_j, \quad j = 1,2,...,n, \tag{2. 5}$$

where $\nu_j = \frac{\mu_j}{\rho_j}$ is the kinematic viscosity of the j^{th} fluid layer and $p_j = \frac{p_0}{\rho_j}$.

Along with the partial differential equations (2. 5) we consider the following condition:

- *the initial condition*

$$u_j(y,0) = 0, \;\; y \in [0,h], \;\; j = 1,2,...,n, \tag{2. 6}$$

- *the boundary condition (no- slip condition)*

$$\left.\begin{array}{ll} u_1(0,t) = U_0 f(t), & t \geq 0, \\ u_n(h,t) = 0, & t \geq 0; \end{array}\right\}, \tag{2. 7}$$

- *the conditions on the fluid interfaces*

$$\left.\begin{array}{l} u_j(h_j,t) = u_{j+1}(h_j,t), \qquad j = 1,2,...,n-1, \;\; t \geq 0 \\ \mu_j \left.\frac{\partial u_j(y,t)}{\partial y}\right|_{y=h_j} = \mu_{j+1} \left.\frac{\partial u_{j+1}(y,t)}{\partial y}\right|_{y=h_j}, \;\; j = 1,2,...,n-1, \;\; t \geq 0 \end{array}\right\}, \tag{2. 8}$$

where $f(t)$ is paisewise continuous function. The interfaces condition (2. 8) denotes the continuity of the velocities and shear stresses of the fluids at the interfaces. It is observed that condition (2. 6)-(2. 8) are sufficient to find the solution of the partial differential e-quation (2. 5).

3. Solution of the problem

In order to determine the solution of Eq. (2. 5), along with the conditions (2. 6)-(2. 8), we use the Laplace transform to eliminate the variable t. Applying the Laplace transform [6] to Eq. (2. 5) and using the initial condition (2. 6), we obtain the transformed equation

$$s\bar{u}_j(y,s) = \nu_j \frac{\partial^2 \bar{u}_j(y,s)}{\partial y^2} + \frac{p_j}{s}, j = 1,2,...,n. \tag{3. 9}$$

The transformed forms of the condition (2. 7) and (2. 8) are

$$\left.\begin{array}{l} \bar{u}_1(0,s) = U_0 F(s), \quad \bar{u}_n(h,s) = 0, \\ \bar{u}_j(y,s) = \bar{u}_{j+1}(h_j,s), \quad j = 1,2,...,n-1, \\ \mu_j \left.\frac{\partial \bar{u}_j(y,s)}{\partial y}\right|_{y=h_j} = \mu_{j+1} \left.\frac{\partial \bar{u}_{j+1}(y,s)}{\partial y}\right|_{y=h_j}, j = 1,2,...,n-1, \end{array}\right\}. \tag{3. 10}$$

A particular solution of Eq. (3. 9) is

$$\bar{u}_{jp} = \frac{p_j}{s^2},\ j = 1, 2, ..., n, \tag{3. 11}$$

and, the general solution of the homogeneous equation associated with Eq. (3. 9) is

$$\bar{u}_{jh} = C_{1j}(s)e^{-y\sqrt{\frac{s}{\nu_j}}} + C_{2j}(s)e^{y\sqrt{\frac{s}{\nu_j}}}, j = 1, 2, ..., n. \tag{3. 12}$$

Now, we obtain the general solution of the Eq. (3. 9) given by

$$\bar{u}_j(y, s) = C_{1j}(s)e^{-y\sqrt{\frac{s}{\nu_j}}} + C_{2j}(s)e^{y\sqrt{\frac{s}{\nu_j}}} + \frac{p_j}{s^2},\ j = 1, 2, ..., n, \tag{3. 13}$$

where $C_{1j}(s),\ C_{2j}(s)$ are $2n$ functions independent of variable y, which are determined by the conditions gives by Eq. (3. 10).

Using conditions (3. 10) into Eq. (3. 13), we obtain a system of *2n* algebraic equations with *2n* unknown functions of *s*, $C_{1j}(s),\ C_{2j}(s),\ j = 1, 2, ...n$,

$$\begin{cases} C_{11} + C_{21} + \frac{p_1}{s^2} = U_0F(s), \\ C_{1j}e^{-h_j\sqrt{\frac{s}{\nu_j}}} + C_{2j}e^{h_j\sqrt{\frac{s}{\nu_j}}} + \frac{p_j}{s^2} = C_{1j+1}e^{-h_j\sqrt{\frac{s}{\nu_{j+1}}}} + C_{2j+1}e^{h_j\sqrt{\frac{s}{\nu_{j+1}}}} + \frac{p_{j+1}}{s^2}, \\ \mu_j\left(-C_{1j}\sqrt{\frac{s}{\nu_j}}e^{-h_j\sqrt{\frac{s}{\nu_j}}} + C_{2j}\sqrt{\frac{s}{\nu_j}}e^{h_j\sqrt{\frac{s}{\nu_j}}}\right) = \\ \qquad \mu_{j+1}\left(-C_{1j+1}\sqrt{\frac{s}{\nu_{j+1}}}e^{-h_j\sqrt{\frac{s}{\nu_{j+1}}}} + C_{2j+1}\sqrt{\frac{s}{\nu_{j+1}}}e^{h_j\sqrt{\frac{s}{\nu_{j+1}}}}\right), \\ C_{1n}e^{-h_n\sqrt{\frac{s}{\nu_n}}} + C_{2n}e^{h_n\sqrt{\frac{s}{\nu_n}}} + \frac{p_n}{s^2} = 0. \end{cases} \tag{3. 14}$$

Introducing notations

$$C_{1j} = K_{2j-1},\ \ C_{2j} = K_{2j},\ j = 1, 2, ..., n, \tag{3. 15}$$

the above linear algebraic system is written in the form:

$$\begin{cases} K_1 + K_2 = U_0F(s) - \frac{p_1}{s^2}, \\ K_{2j-1}e^{-h_j\sqrt{\frac{s}{\nu_j}}} + K_{2j}e^{h_j\sqrt{\frac{s}{\nu_j}}} - K_{2j+1}e^{-h_j\sqrt{\frac{s}{\nu_{j+1}}}} - K_{2j+2}e^{h_j\sqrt{\frac{s}{\nu_{j+1}}}} = \frac{p_{j+1}-p_j}{s^2}, \\ -\mu_j\sqrt{\frac{s}{\nu_j}}K_{2j-1}e^{-h_j\sqrt{\frac{s}{\nu_j}}} + \mu_j\sqrt{\frac{s}{\nu_j}}K_{2j}e^{h_j\sqrt{\frac{s}{\nu_j}}} + \mu_{j+1}\sqrt{\frac{s}{\nu_{j+1}}}K_{2j+1}e^{-h_j\sqrt{\frac{s}{\nu_{j+1}}}} - \\ \qquad \mu_{j+1}\sqrt{\frac{s}{\nu_{j+1}}}K_{2j+2}e^{h_j\sqrt{\frac{s}{\nu_{j+1}}}} = 0, \\ K_{2n-1}e^{-h\sqrt{\frac{s}{\nu_n}}} + K_{2n}e^{h\sqrt{\frac{s}{\nu_j}}} = -\frac{p_n}{s^2} \end{cases} \tag{3. 16}$$

In the present paper we will study the particular case of two immiscible fluids. In this case, the system (3. 16) becomes

$$\begin{cases} K_1 + K_2 = U_0F(s) - \frac{p_1}{s^2}, \\ K_1e^{-h_1\sqrt{\frac{s}{\nu_1}}} + K_2e^{h_1\sqrt{\frac{s}{\nu_1}}} - K_3e^{-h_1\sqrt{\frac{s}{\nu_2}}} - K_4e^{h_1\sqrt{\frac{s}{\nu_2}}} = \frac{p_2-p_1}{s^2}, \\ -\frac{\mu_1}{\sqrt{\nu_1}}K_1e^{-h_1\sqrt{\frac{s}{\nu_1}}} + \frac{\mu_1}{\sqrt{\nu_1}}K_2e^{h_1\sqrt{\frac{s}{\nu_1}}} + \frac{\mu_2}{\sqrt{\nu_2}}K_3e^{-h_1\sqrt{\frac{s}{\nu_2}}} - \frac{\mu_2}{\sqrt{\nu_2}}K_4e^{h_1\sqrt{\frac{s}{\nu_2}}} = 0, \\ K_3e^{-h\sqrt{\frac{s}{\nu_2}}} + K_4e^{h\sqrt{\frac{s}{\nu_2}}} = -\frac{p_2}{s^2}. \end{cases} \tag{3. 17}$$

Using relationships

$$K_2 = G(s) - K_1, \;\; K_4 = -\frac{p_2}{s^2} e^{-h\sqrt{\frac{s}{\nu_2}}} - K_3 e^{-2h\sqrt{\frac{s}{\nu_2}}}, \;\; G(s) = U_0 F(s) - \frac{p_1}{s^2}, \tag{3. 18}$$

the system (3. 17) can be written in the form

$$\begin{cases} K_1 \sin h\left(h_1\sqrt{\frac{s}{\nu_1}}\right) + K_3 \frac{\sin h\left[(h-h_1)\sqrt{\frac{s}{\nu_2}}\right]}{e^{h\sqrt{\frac{s}{\nu_2}}}} = A_1(s), \\ K_1 \frac{\mu_1}{\sqrt{\nu_1}} \cosh\left(h_1\sqrt{\frac{s}{\nu_1}}\right) - K_3 \frac{\mu_2}{\sqrt{\nu_2}} \frac{\cosh\left[(h-h_1)\sqrt{\frac{s}{\nu_2}}\right]}{e^{h\sqrt{\frac{s}{\nu_2}}}} = B_1(s), \end{cases} \tag{3. 19}$$

where,

$$\begin{aligned} A_1(s) &= \frac{1}{2}\left[\frac{p_1-p_2}{s^2} + G(s) e^{h_1\sqrt{\frac{s}{\nu_1}}} + \frac{p_2}{s^2} e^{(h_1-h)\sqrt{\frac{s}{\nu_2}}}\right], \\ B_1(s) &= \frac{1}{2}\left[G(s)\frac{\mu_1}{\sqrt{\nu_1}} e^{h_1\sqrt{\frac{s}{\nu_1}}} + \frac{\mu_2}{\sqrt{\nu_2}}\frac{p_2}{s^2} e^{(h_1-h)\sqrt{\frac{s}{\nu_2}}}\right]. \end{aligned} \tag{3. 20}$$

Finally, we obtain

$$\left.\begin{aligned} K_1(s) = C_{11}(s) = \frac{A(s)}{C(s)}, \;\; K_2(s) = C_{21}(s) = G(s) - \frac{A(s)}{C(s)} \\ K_3(s) = C_{12}(s) = \frac{B(s)}{C(s)}, \;\; K_4(s) = C_{22}(s) = -\frac{p_2}{s^2} e^{-h\sqrt{\frac{s}{\nu_2}}} - \frac{B(s)}{C(s)} e^{-2h\sqrt{\frac{s}{\nu_2}}} \end{aligned}\right\}, \tag{3. 21}$$

with

$$A(s) = -\frac{\mu_2}{\sqrt{\nu_2}} \frac{\cosh\left[(h-h_1)\sqrt{\frac{s}{\nu_2}}\right]}{\exp\left(h\sqrt{\frac{s}{\nu_2}}\right)} A_1(s) - \frac{\sinh\left[(h-h_1)\sqrt{\frac{s}{\nu_2}}\right]}{\exp\left(h\sqrt{\frac{s}{\nu_2}}\right)} B_1(s), \tag{3. 22}$$

$$B(s) = B_1(s) \sinh\left(h_1\sqrt{\frac{s}{\nu_1}}\right) - \frac{\mu_1}{\sqrt{\nu_1}} A_1(s) \cosh\left(h_1\sqrt{\frac{s}{\nu_1}}\right), \tag{3. 23}$$

$$\begin{aligned} C(s) = &-\frac{\mu_2}{\sqrt{\nu_2}} \sinh\left(h_1\sqrt{\frac{s}{\nu_1}}\right) \frac{\cosh\left[(h-h_1)\sqrt{\frac{s}{\nu_2}}\right]}{\exp\left(h\sqrt{\frac{s}{\nu_2}}\right)} \\ &-\frac{\mu_1}{\sqrt{\nu_1}} \cosh\left(h_1\sqrt{\frac{s}{\nu_1}}\right) \frac{\sinh\left[(h-h_1)\sqrt{\frac{s}{\nu_2}}\right]}{\exp\left(h\sqrt{\frac{s}{\nu_2}}\right)}. \end{aligned} \tag{3. 24}$$

In the case of two layers of fluid, the velocities are given by

$$\begin{aligned} \bar{u}_1(y,s) &= C_{11} e^{-y\sqrt{\frac{s}{\nu_1}}} + C_{21} e^{y\sqrt{\frac{s}{\nu_1}}} + \frac{p_1}{s^2} = \frac{A(s)}{C(s)} e^{-y\sqrt{\frac{s}{\nu_1}}} + \left[G(s) - \frac{A(s)}{C(s)}\right] e^{y\sqrt{\frac{s}{\nu_1}}} \\ &= G(s) e^{y\sqrt{\frac{s}{\nu_1}}} - \frac{2A(s)}{C(s)} \sinh\left(y\sqrt{\frac{s}{\nu_1}}\right) + \frac{p_1}{s^2}, \end{aligned} \tag{3. 25}$$

respectively,

$$\begin{aligned} \bar{u}_2(y,s) &= C_{12} e^{-y\sqrt{\frac{s}{\nu_2}}} + C_{22} e^{y\sqrt{\frac{s}{\nu_2}}} + \frac{p_2}{s^2} \\ &= \frac{B(s)}{C(s)} e^{-y\sqrt{\frac{s}{\nu_2}}} - \frac{p_2}{s^2} e^{-h\sqrt{\frac{s}{\nu_2}}} e^{y\sqrt{\frac{s}{\nu_2}}} - \frac{B(s)}{C(s)} e^{-h\sqrt{\frac{s}{\nu_2}}} e^{y\sqrt{\frac{s}{\nu_2}}} + \frac{p_2}{s^2} \\ &= 2\frac{B(s)}{C(s)} \frac{\sinh\left[(h-y)\sqrt{\frac{s}{\nu_2}}\right]}{e^{h\sqrt{\frac{s}{\nu_2}}}} + \frac{p_2}{s^2}\left[1 - e^{-(h-y)\sqrt{\frac{s}{\nu_2}}}\right] \end{aligned} \tag{3. 26}$$

The corresponding shear stresses are

$$\bar{\tau}_1(y,s) = \mu_1 \frac{\partial \bar{u}_1(y,s)}{\partial y} = \mu_1 \left[G(s)\sqrt{\frac{s}{\nu_1}} e^{y\sqrt{\frac{s}{\nu_1}}} - \frac{2A(s)}{C(s)}\sqrt{\frac{s}{\nu_1}} \cosh\left(y\sqrt{\frac{s}{\nu_1}} \right) \right], \tag{3. 27}$$

$$\bar{\tau}_2(y,s) = -2\frac{B(s)}{C(s)}\sqrt{\frac{s}{\nu_2}} \frac{\cosh\left[(h-y)\sqrt{\frac{s}{\nu_2}} \right]}{e^{h\sqrt{\frac{s}{\nu_2}}}} - \frac{p_2}{s^2}\sqrt{\frac{s}{\nu_2}} e^{-(h-y)\sqrt{\frac{s}{\nu_2}}}. \tag{3. 28}$$

The Laplace transform (3. 25)-(3. 28) are complicated, therefore, the inverse Laplace transforms $u_1(y,t),\ u_2(y,t),\ \tau_1(y,t),\ \tau_2(y,t)$ will be obtained with the numerical algorithm proposed by Stehfest [9, 21].

4. Results and discussions

The flow of n layers of incompressible and immiscible viscous fluids in the channel bounded of two parallel plates was modeled and studied. Such flows have significance both theoretically and in applications to the chemical and petroleum industries. The bottom plate of channel has a translational motion in its plane with a time-dependent velocity along the x-axis. The upper plate is stationary and the distance between parallel plates is h. The non-slip condition on boundaries was considered. The solutions for the velocity $\vec{V}_j = u_j(y,t)\vec{e}_x$ and for shear stress $\tau_j = \mu_j \frac{\partial u_j(y,t)}{\partial y},\ j = 1,2,...,n$, have been obtained using the Laplace transform coupled with the classical method of differential equations with constant coefficients. Due to complicated forms of the Laplace transforms of the velocities and shear stresses, the inverse Laplace transforms were obtained using the numerical algorithm proposed by Stehfest.

Using the Stehfest's algorithm, the inverse Laplace transform of the function $\bar{h}(y,s) = \int_0^\infty h(y,t)\exp(-st)dt$is approximated by

$$\begin{aligned} &h(y,t) \approx \tfrac{\ln 2}{t} \textstyle\sum_{k=1}^{N} X_k \bar{h}\left(y, k\tfrac{\ln 2}{t}\right), \\ &X_k = (-1)^{k+N/2} \textstyle\sum_{j=\left[\frac{k+1}{2}\right]}^{\min(k,N/2)} \tfrac{j^{N/2}(2j)!}{(N/2-j)!j!(j-1)!(k-j)!(2j-k)!} \end{aligned} \tag{4. 29}$$

where *N* is an even number and [*x*] denotes the integer part of the number *x*.

In the present paper we analyzed the flow of two-layers. The flow of fluids is generated by a constant pressure gradient and the motion of bottom plate. In order to incorporate several types of the plate translation, we have considered for the velocity of bottom plate a general form described by a time-dependent function $f(t)$, which is a continuous function with $f(0) = 0$. In the general expression of the velocity field it can replace $f(t)$ to study fluid flows for a given motion of the bottom plate (e.g. translation with constant velocity $f(t) = 1$, oscillatory motion $f(t) = sin(\omega t)$, etc.). In the numerical case analyzed in this paper we considered a more complicated expression for velocity, $U_0 f(t) = \frac{p_0}{\rho_1}t + U_0 erfc\left(\frac{h_1}{2\sqrt{\nu_1 t}}\right)$ which, for large values of the time t can be approximated with $\frac{p_0}{\rho_1}t + U_0$, therefore the bottom plate is moving almost uniform accelerated.

For other constants we have used the following values: $\rho_1 = 1000(Kg/m^3),\ \rho_2 = 899(Kg/m^3),\ \mu_1 = 10^{-3}(Ns/m^2),\ \mu_2 = 0.319(Ns/m^2), h = 0.5(m),\ h_1 = h/2.$

Using the software Mathcad, numerical calculations have carried out for velocities and shear stresses given by Eqs. (3. 25)-(3. 28). The numerical results are plotted in graphs from Fig. 2 which shows the velocity and shear stress profiles versus the spatial coordinate y for different values of the constant pressure gradient $p_0$and for different time instants. It is observed from Fig. 2 that fluid velocity and the absolute values of the shear stress increase with the pressure gradient and with the time t. Also numerical results are plotted in graphs from Fig. 3 which shows the velocity and shear stress profiles versus the spatial coordinate y for different values of pressure gradient p_0 and constant time ($t = 60$). The fluid situated close of the bottom plate moves with an almost constant velocity. In the vicinity of the interface h_1, the velocity of first fluid decreases due to interaction with the second fluid. The velocity of the second fluid is decreasing with y and has zero velocity on the upper plate.

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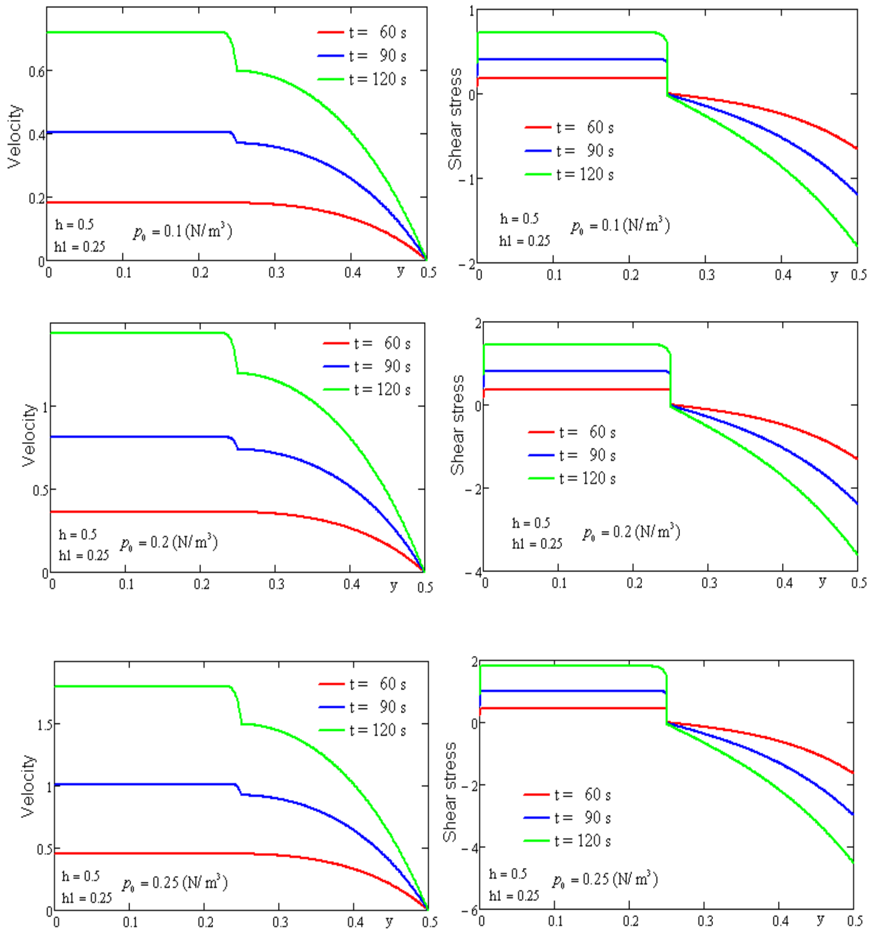

FIGURE 2. Velocity and shear stress profiles for two-layers flow for different values of constant pressure gradient p_0 at different time instants

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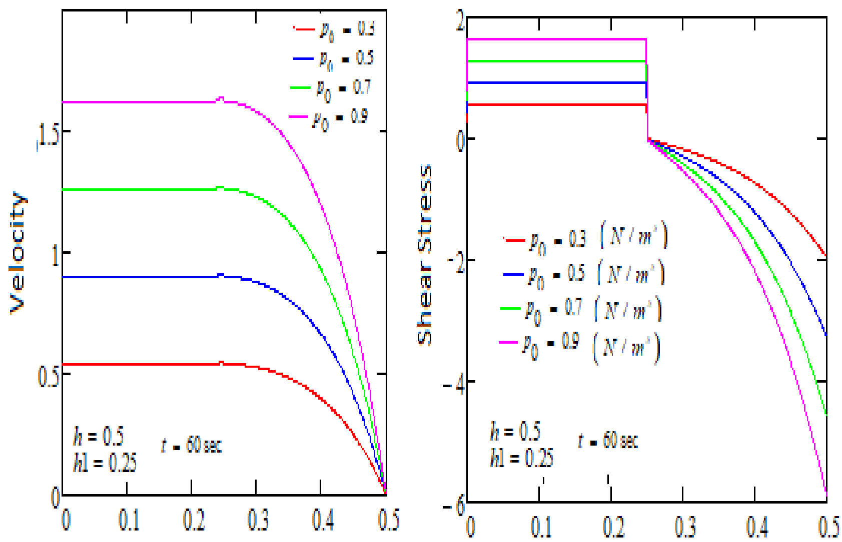

FIGURE 3. Velocity and shear stress profiles for two-layers flow for different values of pressure gradient p_0 at time $t = 60$

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