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# Derivatives with Respect to Lifts of the Riemannian Metric of the Format ${}^{f}\tilde{G} = {}^{S}g_f + {}^{H}g$ on $TM$ Over a Riemannian Manifold $(M, g)$.

Haşim ÇAYIR
Department of Mathematics,
University of Giresun, Turkey.
Email: hasim.cayir@giresun.edu.tr

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**Abstract.** In this paper, we define the Riemannian metric of the format ${}^{f}\tilde{G} = {}^{S}g_f + {}^{H}g$ on $TM$ over $(M, g)$ Riemannian manifold , which is completely determined by vector fields $\beta^H$ and $\theta^V$. Later, we obtain the covarient and Lie derivatives applied to the Riemannian metric of the format ${}^{f}\tilde{G} = {}^{S}g_f + {}^{H}g$ with respect to the vertical $X^V$ and horizontal lifts $X^H$ of vector fields, respectively.

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**Key Words:** Covarient Derivative; Lie Derivative; Riemannian metrics; Horizontal Lift; Vertical Lift; Tangent Bundle.

## 1. Introduction

Riemannian manifolds and the tangent bundles of differentiable manifolds are very important in many areas of mathematics. This fields also studyed a lot of authors [1, 2, 5, 11, 12, 13, 14, 16, 17]. The geometry of tangent bundles goes back to the fundamental paper [15] of Sasaki published in 1958. Sasakian metrics (diagonal lifts of metrics) on tangent bundles were also studied in [10, 11, 19]

Let $n-$dimensional Riemannian manifold be $M$ with $g$ and its tangent bundle, denote by $\pi : TM \to M$. Then $TM$ is smooth manifold and have $2n-$dimensional. Also, local charts on $M$ may be used. Local coordinates $(U, x^i)$ in $M$ induces on $TM$ a system of $(\pi^{-1}(U), x^i, x^{\bar{\imath}} = y^i)$, where local coordinate system is $(x^i)$, $i = 1, ..., n$ in the neighborhood $U$ and Cartesian coordinates is $(y^i)$ the in $T_P M$ at a point $P$ in $U$ according to $\left\{ \frac{\partial}{\partial x^i} \middle| P \right\}$.

Let local expressions in $U$ of $\beta$ be $\beta = \beta^i \frac{\partial}{\partial x^i}$ be on $M$. The $\beta^V$, $\beta^C$and $\beta^H$ of $\beta$ are then given respectively by [7]

$$\beta^V = \beta^i \partial_{\bar{\imath}} \tag{1. 1}$$

$$\beta^C = \beta^i \partial_i + y^j \partial_j^i \beta \partial_{\bar{\imath}} \tag{1. 2}$$

and

$$\beta^H = \beta^i \partial_i - y^j \Gamma^i_{jk} \beta^k \partial_{\bar{i}} \tag{1. 3}$$

where the coefficients of Levi-Civita connection $\nabla$ of Riemannian metric $g$ are $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$, $\partial_i = \frac{\partial}{\partial x^i}$ and $\Gamma^i_{jk}$ .

For a tensor field $S \in \Im^p_q$, $\gamma S \in \Im^p_{q-1}(TM)$ on $\pi^{-1}(U)$ by

$$\gamma S = (y^s S^{j_1 \ldots j_p}_{s i_2 \ldots i_q}) \partial_{\overline{j_1}} \otimes \ldots \otimes \partial_{\overline{j_p}} \otimes dx^{i_2} \otimes \ldots \otimes dx^{iq}$$

where a tensor field $S \in \Im^p_q, q > 1$([19], p.12).

$\beta^H$ of $\beta \in \Im^1_0(M)$ defined by [19]

$$\beta^H = \beta^C - \nabla_\gamma \beta \,,\ (\nabla_\gamma \beta = \gamma \nabla \beta) \tag{1. 4}$$

in $T(M)$,

From ( 1. 2 ) and ( 1. 3 ), we get

$$\beta^H = (\hat{\nabla}_\beta)^C$$

for any $\beta \in \Im^1_0(M)$, where an affine connection $\hat{\nabla}$ in $M$ defined by [19]

$$\hat{\nabla}_\beta \theta = \nabla_\theta \beta + [\beta, \theta] \text{ or } (\nabla_\theta \beta)^v = (\hat{\nabla}_\beta \theta)^v + [\theta, \beta]^v$$

The three classical constructions of metrics are given as [8]:

(a) ${}^S g$ Sasaki metric on $TM$ .

$$\begin{aligned} {}^S g\left(\beta^H, \theta^H\right) &= g\left(\beta, \theta\right) \\ {}^S g\left(\beta^V, \theta^V\right) &= g\left(\beta, \theta\right) \\ {}^S g\left(\beta^H, \theta^V\right) &= {}^S g\left(\beta^V, \theta^H\right) = 0 \end{aligned} \tag{1. 5}$$

where $\beta, \theta \in \Im^1_0\left(M\right)$.

(b) The lift $g^H$ ( pseudo-Riemannian metric) on $TM$.

$$\begin{aligned} g^H\left(\beta^V, \theta^V\right) &= 0 \\ g^H\left(\beta^V, \theta^H\right) &= g^H\left(\beta^H, \theta^V\right) = g\left(\beta, \theta\right), \\ g^H\left(\beta^H, \theta^H\right) &= 0 \end{aligned} \tag{1. 6}$$

for all $\beta, \theta \in \Im^1_0\left(M\right)$.

(c) The lift $g^V$ (degenerate metric) on $TM$ .

$$\begin{aligned} g^V\left(\beta^V, \theta^V\right) &= g\left(\beta, \theta\right) \\ g^V\left(\beta^V, \theta^H\right) &= g^V\left(\beta^H, \theta^V\right) = 0 \\ g^V\left(\beta^H, \theta^H\right) &= 0 \end{aligned} \tag{1. 7}$$

for all $\beta, \theta \in \Im^1_0\left(M\right)$.

In [20], Riemannian metric ${}^S\overline{g}$ on $TM$ introduced by B. V. Zayatuev ([8]see also [21, 22])

$$\begin{aligned} {}^S g_f\left(\beta^V, \theta^V\right) &= g\left(\beta, \theta\right), \\ {}^S g_f\left(\beta^V, \theta^H\right) &= {}^S g_f\left(\beta^H, \theta^V\right) = 0, \\ {}^S g_f\left(\beta^H, \theta^H\right) &= f g\left(\beta, \theta\right), \end{aligned} \tag{1. 8}$$

where $f\rangle 0,\ f \in C^{\infty}(M)$ (see also, [9, 18]). If $f = 1$, we get ${}^{S}g_f = {}^{S}g$, i.e. ${}^{S}g_f$ is a generalization of the ${}^{S}g$.

${}^{f}\widetilde{G} = {}^{S}g_f + {}^{H}g$ Riemannian metric defined by [8]

$$\begin{aligned} {}^{f}\widetilde{G}\left(\beta^V, \theta^V\right) &= g\left(\beta, \theta\right) \\ {}^{f}\widetilde{G}\left(\beta^V, \theta^H\right) &= {}^{f}\widetilde{G}\left(\beta^H, \theta^V\right) = g\left(\beta, \theta\right) \\ {}^{f}\widetilde{G}\left(\beta^H, \theta^H\right) &= fg\left(\beta, \theta\right) \end{aligned} \tag{1. 9}$$

for all $\beta, \theta, \xi \in \Im_0^1(M)$, $f\rangle 1$, $f \in C^{\infty}(M)$, $\beta^H(fg(\theta,\xi)) = f\beta(g(\theta,\xi)) + \beta(f)g(\theta,\xi)$ and $\beta^V(fg(\theta,\xi)) = 0$.

## 2. Main Results

**Definition 2.1.** *The transformation of $D = L_\beta$ is called as Lie derivation according to $\beta \in \Im_0^1(M)$ if*

$$\begin{aligned} L_\beta\theta &= [\beta, \theta], \forall\beta, \theta \in \Im_0^1(M^n), \\ L_\beta f &= \beta f, \forall f \in \Im_0^0(M^n). \end{aligned} \tag{2. 10}$$

*$[\beta, \theta]$ is the Lie bracked. The Lie derivative $L_\beta F$ of $F$ $\Im_1^1(M)$ according to $\beta$ is defined by* [3, 4, 19]

$$(L_\beta F)\theta = [\beta, F\theta] - F[\beta, \theta]. \tag{2. 11}$$

**Definition 2.2.** *The bracket operation for horizontal and vertical vector fields is defined by*

$$\begin{aligned} \beta^H f^V &= (\beta f)^V, [\beta^V, \theta^V] = 0, \\ [\beta^H, \theta^V] &= (\nabla_\beta\theta)^V, \\ [\beta^H, \theta^H] &= [\beta, \theta]^H - (R(\beta, \theta)u)^V, \end{aligned} \tag{2. 12}$$

where $f \in \Im_0^0(M), \beta, \theta \in \Im_0^1(M)$, Riemannian curvature $R$ [6]

$$R(\beta, \theta) = [\nabla_\beta, \nabla_\theta] - \nabla_{[\beta,\theta]}.$$

**Theorem 2.3.** *The format ${}^{f}\tilde{G} = {}^{S} g_f + {}^{H} g$ is the Riemannian metric on $TM$, defined by ( 1. 9 ). From ( 1. 9 ), Definition (2.1) and Definition (2.2), we have the following results*

$$\begin{aligned} i)\ (L_{\beta^V}{}^{f}\tilde{G})(\theta^V, \xi^V) &= 0, \\ ii)\ (L_{\beta^V}{}^{f}\tilde{G})(\theta^V, \xi^H) &= g(\theta, \hat{\nabla}_\xi\beta), \\ iii)\ (L_{\beta^V}{}^{f}\tilde{G})(\theta^H, \xi^V) &= g(\hat{\nabla}_\theta\beta, \xi), \\ iv)\ (L_{\beta^H}{}^{f}\tilde{G})(\theta^V, \xi^V) &= (\hat{\nabla}_\beta g)(\theta, \xi), \\ v)\ (L_{\beta^H}{}^{f}\tilde{G})(\theta^H, \xi^V) &= (L_\beta g)(\theta, \xi) - g(\theta, (\nabla_Z\beta)) + g(R(\beta, \theta)U, \xi), \\ vi)\ (L_{\beta^V}{}^{f}\tilde{G})(\theta^H, \xi^H) &= g((\hat{\nabla}_\theta\beta), \xi) + g(\theta, (\hat{\nabla}_Z\beta)), \\ vii)\ (L_{\beta^H}{}^{f}\tilde{G})(\theta^V, \xi^H) &= (L_\beta g)(\theta, \xi) + g(\theta, R(\beta, \xi)U) - g((\nabla_\theta\beta), \xi), \\ viii)\ (L_{\beta^H}{}^{f}\tilde{G})(\theta^H, \xi^H) &= (L_\beta fg)(\theta, \xi) + g(\theta, (R(\beta, \xi)U)) + g((R(\beta, \theta)U, \xi), \end{aligned}$$

where the lifts of $\beta^V, \beta^C, \beta^H \in \Im_0^1(TM)$ of vector field $\beta, \theta, \xi \in \Im_0^1(M)$, defined by ( 1. 1 ),( 1. 2 ),( 1. 3 ), respectively.

*Proof.* From ( 1. 9 ), Definition 2.1 and Definition 2.2, we get the following results

$i)$

$$\begin{aligned}
(L_{\beta^V}{}^f\tilde{G})(\theta^V,\xi^V) &= L_{\beta^V}{}^f\tilde{G}(\theta^V,\xi^V) -{}^f \tilde{G}(L_{\beta^V}\theta^V,\xi^V) -{}^f \tilde{G}(\theta^V, L_{\beta^V}\xi^V),\\
&= L_{\beta^V}{}^f\tilde{G}(\theta^V,\xi^V), \quad (from\ (1.9))\\
&= L_{\beta^V}g(\theta,\xi),\\
&= 0.
\end{aligned}$$

$ii)$

$$\begin{aligned}
(L_{\beta^V}{}^f\tilde{G})(\theta^V,\xi^H) &= L_{\beta^V}{}^f\tilde{G}(\theta^V,\xi^H) -{}^f \tilde{G}(L_{\beta^V}\theta^V,\xi^H) -{}^f \tilde{G}(\theta^V, L_{\beta^V}\xi^H),\\
&= \beta^V g(\theta,\xi) -{}^f \tilde{G}(\theta^V,[\beta,\xi]^V-(\nabla_\beta\xi)^V), \ \ (from\ (1.9)\ )\\
&= -{}^f\tilde{G}(\theta^V,[\beta,\xi]^V) +{}^f \tilde{G}(\theta^V,(\nabla_\beta\xi)^V),\\
&= -g(\theta,[\beta,\xi])+g(\theta,\nabla_\beta\xi),\\
&= g(\theta,-[\beta,\xi]+\nabla_\beta\xi),\\
&= g(\theta,[\xi,\beta]+\nabla_\beta\xi), \ \ (from\ (\hat{\nabla}_\beta\theta=\nabla_\theta\beta+[\beta,\theta]\ )\\
&= g(\theta,\hat{\nabla}_\xi\beta).
\end{aligned}$$

$iii)$

$$\begin{aligned}
(L_{\beta^V}{}^f\tilde{G})(\theta^H,\xi^V) &= L_{\beta^V}{}^f\tilde{G}(\theta^H,\xi^V) -{}^f \tilde{G}(L_{\beta^V}\theta^H,\xi^V) -{}^f \tilde{G}(\theta^H, L_{\beta^V}\xi^V),\\
&= \beta^V g(\theta,\xi) -{}^f \tilde{G}([\beta,\theta]^V-(\nabla_\beta\theta)^V,\xi^V), \ \ (from\ (1.9)\ )\\
&= -{}^f\tilde{G}([\beta,\theta]^V,\xi^V) +{}^f \tilde{G}((\nabla_\beta\theta)^V,\xi^V),\\
&= g(-[\beta,\theta]+\nabla_\beta\theta,\xi),\\
&= g([\theta,\beta]+\nabla_\beta\theta,\xi), \quad (from\ (\hat{\nabla}_\beta\theta=\nabla_\theta\beta+[\beta,\theta]\ )\\
&= g(\hat{\nabla}_\theta\beta,\xi)
\end{aligned}$$

$iv)$

$$\begin{aligned}
(L_{\beta^H}{}^f\tilde{G})(\theta^V,\xi^V) &= L_{\beta^H}{}^f\tilde{G}(\theta^V,\xi^V) -{}^f \tilde{G}(L_{\beta^H}\theta^V,\xi^V) -{}^f \tilde{G}(\theta^V, L_{\beta^H}\xi^V),\\
&= \beta^H g(\theta,\xi) -{}^f \tilde{G}((\hat{\nabla}_\beta\theta)^V,\xi^V) -{}^f \tilde{G}(\theta^V,(\hat{\nabla}_\beta\xi)^V),\\
&= \beta g(\theta,\xi)-g((\hat{\nabla}_\beta\theta),\xi)-g(\theta,(\hat{\nabla}_\beta\xi)),\\
&= (\hat{\nabla}_\beta g)(\theta,\xi),
\end{aligned}$$

where $\hat{\nabla}_\beta g(\theta,\xi)=(\hat{\nabla}_\beta g)(\theta,\xi)+g((\hat{\nabla}_\beta\theta),\xi)+g(\theta,(\hat{\nabla}_\beta\xi))$.

$v)$

$$\begin{aligned}
(L_{\beta^H}{}^f\tilde{G})(\theta^H,\xi^V) &= L_{\beta^H}{}^f\tilde{G}(\theta^H,\xi^V) -{}^f \tilde{G}(L_{\beta^H}\theta^H,\xi^V) -{}^f \tilde{G}(\theta^H, L_{\beta^H}\xi^V),\\
&= \beta g(\theta,\xi) - {}^f\tilde{G}([\beta,\theta]^H-(R(\beta,\theta)U)^V,\xi^V)\\
&\quad -{}^f\tilde{G}(\theta^H,[\beta,\xi]^V+(\nabla_\xi\beta)^V), \ (from\ (1.9)\ and\ (2.12))\\
&= -g([\beta,\theta],\xi)+\beta g(\theta,\xi)+g(R(\beta,\theta)U,\xi)\\
&\quad -g(\theta,[\beta,\xi])-g(\theta,(\nabla_\xi\beta)),\\
&= (L_\beta g)(\theta,\xi)-g(\theta,(\nabla_\xi\beta))+g(R(\beta,\theta)U,\xi).
\end{aligned}$$

where $L_\beta g(\theta,\xi) = (L_\beta g)(\theta,\xi) + g((L_\beta\theta),\xi) + g(\theta,(L_\beta\xi)))$.

$vi)$

$$\begin{aligned}
(L_{\beta^V}{}^f\tilde{G})(\theta^H,\xi^H) &= L_{\beta^V}{}^f\tilde{G}(\theta^H,\xi^H) -{}^f\tilde{G}(L_{\beta^V}\theta^H,\xi^H) -{}^f\tilde{G}(\theta^H, L_{\beta^V}\xi^H)\\
&= \beta^V(fg(\theta,\xi)) -{}^f\tilde{G}(-(\nabla_\beta\theta)^V + [\beta,\theta]^V,\xi^H)\\
&\quad -{}^f\tilde{G}(\theta^H,[\beta,\xi]^V - (\nabla_\beta\xi)^V)\\
&= -{}^f\tilde{G}([\beta,\theta]^V,\xi^H) +{}^f\tilde{G}((\nabla_\beta\theta)^V,\xi^H) -{}^f\tilde{G}(\theta^H,[\beta,\xi]^V)\\
&\quad +{}^f\tilde{G}(\theta^H,(\nabla_\beta\xi)^V) \quad (from\ (1.9)\ and\ (2.12))\\
&= g(-[\beta,\theta] + (\nabla_\beta\theta),\xi) + g(\theta, -[\beta,\xi] + (\nabla_\beta\xi))\\
&= g([\theta,\beta] + (\nabla_\beta\theta),\xi) + g(\theta,(\nabla_\beta\xi) + [\xi,\beta])\\
&= g((\hat{\nabla}_\theta\beta),\xi) + g(\theta,(\hat{\nabla}_\xi\beta)) \quad (from\ (\hat{\nabla}_\beta\theta = \nabla_\theta\beta + [\beta,\theta]\ )
\end{aligned}$$

$vii)$

$$\begin{aligned}
(L_{\beta^H}{}^f\tilde{G})(\theta^V,\xi^H) &= L_{\beta^H}{}^f\tilde{G}(\theta^V,\xi^H) -{}^f\tilde{G}(L_{\beta^H}\theta^V,\xi^H) -{}^f\tilde{G}(\theta^V, L_{\beta^H}\xi^H),\\
&= \beta^H g(\theta,\xi) -{}^f\tilde{G}([\beta,\theta]^V + (\nabla_\beta\theta)^V,\xi^H)\\
&\quad -{}^f\tilde{G}(\theta^V,[\beta,\xi]^H - (R(\beta,\xi)U)^V),\ (from\ (2.10)\ and\ (2.12))\\
&= \beta g(\theta,\xi) -{}^f\tilde{G}([\beta,\theta]^V,\xi^H) -{}^f\tilde{G}((\nabla_\theta\beta)^V,\xi^H)\\
&\quad -{}^f\tilde{G}(\theta^V,[\beta,\xi]^H) +{}^f\tilde{G}(\theta^V,(R(\beta,\xi)U)^V),\\
&= \beta g(\theta,\xi) - g([\beta,\theta],\xi) - g((\nabla_\theta\beta),\xi) - g(\theta,[\beta,\xi])\\
&\quad + g(\theta, R(\beta,\xi)U),\ \ (from\ (1.9))\\
&= (L_\beta g)(\theta,\xi) - g((\nabla_\theta\beta),\xi) + g(\theta, R(\beta,\xi)U).
\end{aligned}$$

$viii)$

$$\begin{aligned}
(L_{\beta^H}{}^f\tilde{G})(\theta^H,\xi^H) &= L_{\beta^H}{}^f\tilde{G}(\theta^H,\xi^H) -{}^f\tilde{G}(L_{\beta^H}\theta^H,\xi^H) -{}^f\tilde{G}(\theta^H, L_{\beta^H}\xi^H),\\
&= -{}^f\tilde{G}(-(R(\beta,\theta)U)^V + [\beta,\theta]^H,\xi^H) + \beta^H(fg(\theta,\xi))\\
&\quad -{}^f\tilde{G}(\theta^H,[\beta,\xi]^H - (R(\beta,\xi)U)^V),\ (from\ (1.9)\ and\ (2.12))\\
&= \beta(f)g(\theta,\xi) - fg([\beta,\theta],\xi) + g(R(\beta,\theta)U,\xi) + f\beta g(\theta,\xi)\\
&\quad - fg(\theta,[\beta,\xi]) + g(\theta, R(\beta,\xi)U),\\
&= (L_\beta fg)(\theta,\xi) + g(\theta,(R(\beta,\xi)U) + g((R(\beta,\theta)U,\xi),
\end{aligned}$$

where $L_\beta g(\theta,\xi) = (L_\beta g)(\theta,\xi) + g((L_\beta\theta),\xi) + g(\theta,(L_\beta\xi))$. □

**Definition 2.4.** *Differential transformation according to vector field $\beta$, defined by*

$$D = \nabla_\beta : T(M) \to T(M), \beta \in \Im_0^1(M),$$

*is called as covarient derivation if*

$$\begin{aligned}
\nabla_\beta f &= \beta f, \\
\nabla_{f\beta+g\theta} t &= f\nabla_\beta t + g\nabla_\theta t,
\end{aligned} \tag{2. 13}$$

*where $\forall f, g \in \Im_0^0(M)\forall\beta,\theta \in \Im_0^1(M), \forall t \in \Im(M)$.*

Also, a transformation defined by

$$\nabla : \Im_0^1(M) \times \Im_0^1(M) \to \Im_0^1(M),$$

is called as affin connection [14, 19]. For any $\beta, \theta \in \Im_0^1(M)$, the lift $\nabla^H$ of $\nabla$ in $M$ to $T(M)$, defined by

$$\begin{aligned} \nabla^H_{\beta^H}\theta^H &= (\nabla_\beta \theta)^H,\ \nabla^H_{\beta^V}\theta^H = 0, \\ \nabla^H_{\beta^H}\theta^V &= (\nabla_\beta \theta)^V, \nabla^H_{\beta^V}\theta^V = 0, \end{aligned} \tag{2. 14}$$

**Theorem 2.5.** *The format ${}^f\tilde{G} = {}^S g_f + {}^H g$, defined by ( 1. 9 ) is the Riemannian metric on $TM$. The lift $\nabla^H$ of $\nabla$ in $M$ to $T(M)$. From ( 1. 9 ) and Definition 2.4, we obtain*

$$\begin{aligned} i)\ (\nabla^H_{\beta^V}{}^f\tilde{G})(\theta^V,\xi^V) &= 0, \\ ii)\ (\nabla^H_{\beta^V}{}^f\tilde{G})(\theta^V,\xi^H) &= 0, \\ iii)\ (\nabla^H_{\beta^V}{}^f\tilde{G})(\theta^H,\xi^V) &= 0, \\ iv)\ (\nabla^H_{\beta^V}{}^f\tilde{G})(\theta^H,\xi^H) &= 0, \\ v)\ (\nabla^H_{\beta^H}{}^f\tilde{G})(\theta^V,\xi^V) &= (\nabla_\beta\, g)(\theta,\xi), \\ vi)\ (\nabla^H_{\beta^H}{}^f\tilde{G})(\theta^V,\xi^H) &= (\nabla_\beta g)(\theta,\xi), \\ vii)\ (\nabla^H_{\beta^H}{}^f\tilde{G})(\theta^H,\xi^V) &= (\nabla_\beta g)(\theta,\xi), \\ viii)\ (\nabla^H_{\beta^H}{}^f\tilde{G})(\theta^H,\xi^H) &= (\nabla_\beta f g)(\theta,\xi), \end{aligned}$$

where the vertical,complete and horizontal lifts $\beta^V, \beta^C, \beta^H \in \Im_0^1(TM)$ of vector field $\beta, \theta, \xi \in \Im_0^1(M)$, defined by ( 1. 1 ),( 1. 2 ),( 1. 3 ), respectively.

*Proof.* From ( 1. 9 ), ( 2. 13 ) and ( 2. 14 ), we get the following results

$i)$

$$\begin{aligned} (\nabla^H_{\beta^V}\ {}^f\tilde{G})(\theta^V,Z^V) &= \nabla^H_{\beta^V}\ {}^f\tilde{G}(\theta^V,\xi^V) - {}^f\ \tilde{G}(\nabla^H_{\beta^V}\theta^V,\xi^V) - {}^f\ \tilde{G}(\theta^V,\nabla^H_{\beta^V}\xi^V), \\ &= \beta^V\ g(\theta,\xi),\ (from\ (1.9)\ and\ (2.14)) \\ &= 0. \end{aligned}$$

$ii)$

$$\begin{aligned} (\nabla^H_{\beta^V}{}^f\tilde{G})(\theta^V,\xi^H) &= \nabla^H_{\beta^V}\ {}^f\tilde{G}(\theta^V,\xi^H) - {}^f\ \tilde{G}(\nabla^H_{\beta^V}\theta^V,\xi^H) - {}^f\ \tilde{G}(\theta^V,\nabla^H_{\beta^V}\xi^H), \\ &= \beta^V g(\theta,\xi),\ (from\ (1.9)\ and\ (2.14)) \\ &= 0. \end{aligned}$$

$iii)$

$$\begin{aligned} (\nabla^H_{\beta^V}\ {}^f\tilde{G})(\theta^H,\xi^V) &= \nabla^H_{\beta^V}\ {}^f\tilde{G}(\theta^H,\xi^V) - {}^f\ \tilde{G}(\nabla^H_{\beta^V}\theta^H,\xi^V) - {}^f\ \tilde{G}(\theta^H,\nabla^H_{\beta^V}\xi^V), \\ &= \beta^V g(\theta,\xi), \\ &= 0. \end{aligned}$$

$iv)$

$$
\begin{aligned}
(\nabla^H_{\beta^V}\ {}^f\tilde{G})(\theta^H, \xi^H) &= \nabla^H_{\beta^V}\ {}^f\tilde{G}(\theta^H, \xi^H) -^f \tilde{G}(\nabla^H_{\beta^V}\theta^H, \xi^H) -^f \tilde{G}(\theta^H, \nabla^H_{\beta^V}\xi^H),\\
&= \beta^V fg(\theta, \xi),\\
&= 0.
\end{aligned}
$$

$v)$

$$
\begin{aligned}
(\nabla^H_{\beta^H}\ {}^f\tilde{G})(\theta^V, \xi^V) &= \nabla^H_{\beta^H}\ {}^f\tilde{G}(\theta^V, \xi^V) -^f \tilde{G}(\nabla^H_{\beta^H}\theta^V, \xi^V) -^f \tilde{G}(\theta^V, \nabla^H_{\beta^H}\xi^V),\\
&= \beta^H g(\theta, \xi) -^f \tilde{G}((\nabla_\beta\theta)^V, \xi^V) -^f \tilde{G}(\theta^V, (\nabla_\beta\xi)^V),\\
&= \beta g(\theta, \xi) - g((\nabla_\beta\theta), \xi) - g(\theta, (\nabla_\beta\xi)),\ \ (from\ (1.9)\\
&= (\nabla_\beta\ g)(\theta, \xi),
\end{aligned}
$$

where $\nabla_\beta g(\theta, \xi) = (\nabla_\beta g)(\theta, \xi) + g((\nabla_\beta\theta), \xi) + g(\theta, (\nabla_\beta\xi)))$

$vi)$

$$
\begin{aligned}
(\nabla^H_{\beta^H}\ {}^f\tilde{G})(\theta^V, \xi^H) &= \nabla^H_{\beta^H}\ {}^f\tilde{G}(\theta^V, \xi^H) -^f \tilde{G}(\nabla^H_{\beta^H}\theta^V, \xi^H) -^f \tilde{G}(\theta^V, \nabla^H_{\beta^H}\xi^H),\\
&= \beta g(\theta, \xi) -^f \tilde{G}((\nabla_\beta\theta)^V, \xi^H) -^f \tilde{G}(\theta^V, (\nabla_\beta\xi)^H),\\
&= \beta g(\theta, \xi) - g(\theta, (\nabla_\beta\xi)) - g((\nabla_\beta\xi), Z),\\
&= (\nabla_\beta g)(\theta, \xi).(from\ (1.9)\ and\ (2.14))
\end{aligned}
$$

$vii)$

$$
\begin{aligned}
(\nabla^H_{\beta^H}\ {}^f\tilde{G})(\theta^H, \xi^V) &= \nabla^H_{\beta^H}\ {}^f\tilde{G}(\theta^H, \xi^V) -^f \tilde{G}(\nabla^H_{\beta^H}\theta^H, \xi^V) -^f \tilde{G}(\theta^H, \nabla^H_{\beta^H}\xi^V),\\
&= \beta g(\theta, \xi) - g(\theta, (\nabla_\beta\xi)) - g((\nabla_\beta\theta), \xi),\\
&= (\nabla_\beta g)(\theta, \xi).
\end{aligned}
$$

$viii)$

$$
\begin{aligned}
(\nabla^H_{\beta^H}\ {}^f\tilde{G})(\theta^H, \xi^H) &= \nabla^H_{\beta^H}\ {}^f\tilde{G}(\theta^H, \xi^H) -^f \tilde{G}(\nabla^H_{\beta^H}\theta^H, \xi^H) -^f \tilde{G}(\theta^H, \nabla^H_{\beta^H}\xi^H),\\
&= \beta(f)g(\theta, \xi) -^f \tilde{G}((\nabla_\beta\theta)^H, \xi^H) + f\beta g(\theta, \xi)\\
&\quad -^f\tilde{G}(\theta^H, (\nabla_\beta\xi)^H),\ (from\ (1.9)\ and\ (2.14))\\
&= \beta(f)g(\theta, \xi) - fg((\nabla_\beta\theta), \xi) + f\beta g(\theta, \xi) - g(\theta, (\nabla_\beta\xi)),\\
&= (\nabla_\beta fg)(\theta, \xi).
\end{aligned}
$$

where $\nabla_\beta fg(\theta, \xi) = (\nabla_\beta fg)(\theta, \xi) + fg((\nabla_\beta\theta), \xi) + g(\theta, (\nabla_\beta\xi))$ and $(\nabla_\beta fg)(\theta, \xi) = \beta(f)g(\theta, \xi) + f\beta g(\theta, \xi)$. □

## 3. Conclusion

In this paper, studyed on the Riemannian metric of the format ${}^f\tilde{G} =^S g_f +^H g$ on $TM$ over $(M, g)$ Riemannian manifold , which is completely determined by vector fields $\beta^H$ and $\theta^V$. Later, obtained the covarient and Lie derivatives applied to the Riemannian metric of the format ${}^f\tilde{G} =^S g_f +^H g$ with respect to the vertical $X^V$ and horizontal lifts $X^H$ of vector fields, respectively. It can also work on vertical and complete lifts.

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