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Generalized Fibonacci and k-Pell Matrix Sequences

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Abstract. In the present article first and foremost we define generalized Fibonacci sequence and k-Pell sequence. After that by using these sequences we delineate generalized Fibonacci matrix sequence and k-Pell matrix sequence. At the hindmost we obtain results by some matrix technique for both general sequences as well as for matrix sequences.

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1. Introduction

Fibonacci numbers have many applications and plays an important role in the field of biology, physics, chemistry, computer science etc. In other words, we can say that these numbers behaves just like a ruler for the whole universe. For the detailed information [1, 2] can be consulted. The main motivation of this article is to study the matrix sequences of generalized Fibonacci sequence and k-Pell sequence.

Many authors dedicated to study the generalizations of classical Fibonacci numbers.

Horadam [3] introduced the generalized Fibonacci sequence and obtained the various properties for this sequence and the author defined the generalized Fibonacci sequence by the following equation:

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3,\ H_1 = p,\ H_2 = p + q$$

where p and q are arbitrary integers. Bolat et al. [4] presented some properties for another generalized Fibonacci sequence given by

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2},\ n \geq 3,\ k \in \mathbb{R}^+,\ \text{with } L_{k,0} = 2,\ L_{k,1} = 1$$

Chong and Ho [5] derived some summation formulae for the generalized Fibonacci sequence, which is recurrently defined by

$$U_{n+2} = pU_{n+1} + qU_n\ \forall\ n \in \mathbb{Z}_0^+ \text{ and } p, q \in \mathbb{Z}^+$$

From several past years many authors studied Fibonacci sequence and its generalizations by matrix technique. In [6] Silvester shows that a number of properties of the Fibonacci sequence can be derived from a matrix representation. In doing so, he showed that if

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ then } A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

where F_n represents the n^{th} Fibonacci number. Catarino and Vasco [7] obtained some basic properties for k-Pell sequence with the help of a 2×2 matrix. Now the main aim of the present article is to study generalized Fibonacci matrix and k-Pell matrix sequences.

In 2008 Civic and Turkmen [8] studied Fibonacci numbers using another aspect by introducing the concept of Fibonacci matrix sequence in terms of (s, t)-Fibonacci matrix sequence and here the authors defined (s, t)-Fibonacci matrix sequence as

$$\mathcal{F}_{n+1}(s,t) = s\mathcal{F}_n(s,t) + t\mathcal{F}_{n-1}(s,t) \text{ for } n \geq 1$$

with $\mathcal{F}_0(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{F}_1(s,t) = \begin{bmatrix} s & 1 \\ t & 0 \end{bmatrix}$ and $s > 0,\ t \neq 0,\ s^2 + 4t > 0$.
Again Civic and Turkmen [9] delineated (s, t)-Lucas matrix sequence which is defined as follows:

$$\mathcal{L}_{n+1}(s,t) = s\mathcal{L}_n(s,t) + t\mathcal{L}_{n-1}(s,t) \text{ for } n \geq 1$$

with $\mathcal{L}_0(s,t) = \begin{bmatrix} s & 2 \\ 2t & -s \end{bmatrix}$, $\mathcal{F}_1(s,t) = \begin{bmatrix} s^2+2t & s \\ st & 2t \end{bmatrix}$ and $s > 0,\ t \neq 0,\ s^2 + 4t > 0$.
After that in 2015 Ipek et al. [10] introduced the generalization of (s, t)-Fibonacci matrix sequence and here the authors defined generalized (s, t)-Fibonacci matrix sequence by the following recurrence relation:

$$\mathfrak{R}_{n+1}(s,t) = s\mathfrak{R}_n(s,t) + t\mathfrak{R}_{n-1}(s,t) \text{ for } n \geq 1$$

with $\mathfrak{R}_0(s,t) = \begin{bmatrix} a_1 & a_0 \\ ta_0 & a_1 - sa_0 \end{bmatrix}$, $\mathfrak{R}_1(s,t) = \begin{bmatrix} sa_1 + ta_0 & a_1 \\ ta_1 & ta_0 \end{bmatrix}$, $s > 0,\ t \neq 0,\ s^2 + 4t > 0$ and $a_0,\ a_1 \in \mathbb{R}$. Similarly Uslu and Uygun [11] presented (s, t)-Jacobsthal

$\{J_n(s,t)\}_{n\in\mathbb{N}}$ and (s,t)- Jacobsthal-Lucas $\{C_n(s,t)\}_{n\in\mathbb{N}}$ matrix sequences and these sequences are recurrently defined by

$$J_n(s,t) = sJ_{n-1}(s,t) + 2tJ_{n-2}(s,t) \text{ for } n \geq 2$$

with $J_0(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ J_1(s,t) = \begin{bmatrix} s & 2 \\ t & 0 \end{bmatrix}$ and $s > 0,\ t \neq 0,\ s^2 + 8t \neq 0$ and

$$C_n(s,t) = sC_{n-1}(s,t) + 2tC_{n-2}(s,t) \text{ for } n \geq 2$$

with $C_0(s,t) = \begin{bmatrix} s & 4 \\ 2t & -s \end{bmatrix},\ J_1(s,t) = \begin{bmatrix} s^2+4t & 2s \\ st & 4t \end{bmatrix}$ and $s > 0,\ t \neq 0,\ s^2 + 8t \neq 0.$ Then Uygun and Uslu [12] generalized (s,t)-Jacobsthal matrix sequence which resulted in a sequence known as generalized (s,t)-Jacobsthal matrix sequence.

Now in the following sections we give some required definitions as well as main results of this article.

2. Generalized Fibonacci Sequence and k-Pell Sequence

Definition 2.1. *For $k \in \mathbb{R}^+$, the generalized Fibonacci sequence $\langle R_{k,n}\rangle$ defined by*

$$R_{k,n+1} = 2R_{k,n} + kR_{k,n-1},\ n \geq 1,\ R_{k,0} = 2,\ R_{k,1} = 1. \tag{2. 1}$$

Definition 2.2. [7] *For $k \in \mathbb{R}^+$, the k-Pell sequence $\langle P_{k,n}\rangle$ is defined recurrently by*

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1},\ n \geq 1,\ P_{k,0} = 0,\ P_{k,1} = 1. \tag{2. 2}$$

Both the recurrence relations (2. 1) and (2. 2) have same characteristic equation $x^2 - 2x - k = 0$ and suppose that a and b are the roots of this characteristic equation. Then by [7] the well known general form for k-Pell sequence known as Binet formula is given and write by

$$P_{k,n} = \frac{a^n - b^n}{a - b} \tag{2. 3}$$

where $a = 1 + \sqrt{1+k}$ and $b = 1 - \sqrt{1+k}$. Here we can easily see that a and b have the following properties:

i. $a + b = 2 \Rightarrow a - 1 = 1 - b$

ii. $a - b = 2\sqrt{1+k}$

iii. $ab = -k$

iv. $a^2 = 2a + k$ and $b^2 = 2b + k$

v. $a^n = aP_{k,n} + kP_{k,n-1}$ and $b^n = bP_{k,n} + kP_{k,n-1}$

3. Generalized Fibonacci Matrix Sequence and k-Pell Matrix Sequence

In this section by using equations (2. 1) and (2. 2) we introduce generalized Fibonacci matrix sequence and k-Pell matrix sequence respectively.

Definition 3.1. *For $k \in \mathbb{R}^+$, the generalized Fibonacci matrix sequence $\langle S_{k,n} \rangle$ is defined recurrently by*

$$S_{k,n} = 2S_{k,n-1} + kS_{k,n-2},\ n \geq 2 \tag{3. 1}$$

with $S_{k,0} = \begin{bmatrix} 1 & 2k \\ 2 & -3 \end{bmatrix}$, $S_{k,1} = \begin{bmatrix} 2+2k & k \\ 1 & 2k \end{bmatrix}$.

Definition 3.2. *For $k \in \mathbb{R}^+$, the k-Pell matrix sequence $\langle V_{k,n} \rangle$ is defined by the following equation:*

$$V_{k,n} = 2V_{k,n-1} + kV_{k,n-2},\ n \geq 2 \tag{3. 2}$$

with $V_{k,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $V_{k,1} = \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix}$.

Theorem 3.3. *For any integer $n \geq 1$, we obtain*

$$S_{k,n} = \begin{bmatrix} R_{k,n+1} & kR_{k,n} \\ R_{k,n} & kR_{k,n-1} \end{bmatrix} \text{ and} \tag{3. 3}$$

$$V_{k,n} = \begin{bmatrix} P_{k,n+1} & kP_{k,n} \\ P_{k,n} & kP_{k,n-1} \end{bmatrix}. \tag{3. 4}$$

Proof. To prove equation (3. 3), we shall use induction on n. For $n = 1$, by equation (3. 1), we have

$$S_{k,1} = \begin{bmatrix} 2+2k & k \\ k & 2k \end{bmatrix} = \begin{bmatrix} R_{k,2} & kR_{k,1} \\ R_{k,1} & kR_{k,0} \end{bmatrix}$$

Let us suppose that the result is true for all values i less than or equal n. Then

$$\begin{aligned} S_{k,n+1} &= 2S_{k,n} + kS_{k,n-1} \\ &= 2\begin{bmatrix} R_{k,n+1} & kR_{k,n} \\ R_{k,n} & kR_{k,n-1} \end{bmatrix} + k\begin{bmatrix} R_{k,n} & kR_{k,n-1} \\ R_{k,n-1} & kR_{k,n-2} \end{bmatrix} \\ &= \begin{bmatrix} 2R_{k,n+1} + kR_{k,n} & 2kR_{k,n} + k^2R_{k,n-1} \\ 2R_{k,n} + kR_{k,n-1} & 2kR_{k,n-1} + k^2R_{k,n-2} \end{bmatrix} \\ S_{k,n+1} &= \begin{bmatrix} R_{k,n+2} & kR_{k,n+1} \\ R_{k,n+1} & kR_{k,n} \end{bmatrix} \end{aligned}$$

as required.
The proof of equation (3. 4) is similar to the proof of equation (3. 3). □

The next result shows two properties which involve these matrix sequences.

Lemma 3.4.

$$V_{k,m+n} = V_{k,m}V_{k,n}, \forall\, m, n \geq 0 \tag{3. 5}$$

$$S_{k,n} = S_{k,1}V_{k,n-1}, n \geq 1 \tag{3. 6}$$

Proof. To prove equation (3. 5), let us use the induction on n. For $n = 0$, by equation (3. 4), we get

$$V_{k,m+0} = \begin{bmatrix} P_{k,m+1} & kP_{k,m} \\ P_{k,m} & kP_{k,m-1} \end{bmatrix} = \begin{bmatrix} P_{k,m+1} & kP_{k,m} \\ P_{k,m} & kP_{k,m-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = V_{k,m}V_{k,0}$$

Let us assume that the equation (3. 5) holds for all values i less than or equal n. Now we have to show that the result is true for $n+1$. Now from (3. 2)

$$\begin{aligned} V_{k,m+(n+1)} &= 2V_{k,m+n} + kV_{k,m+(n-1)} \\ &= 2V_{k,m}V_{k,n} + kV_{k,m}V_{k,n-1} \\ &= V_{k,m}\left(2V_{k,n} + kV_{k,n-1}\right) \\ &= V_{k,m}V_{k,n+1} \end{aligned}$$

Hence we obtain the result.
To prove equation (3. 6), we again use induction on n. Let $n = 1$, we get

$$S_{k,1} = \begin{bmatrix} 2+2k & k \\ k & 2k \end{bmatrix} = \begin{bmatrix} 2+2k & k \\ k & 2k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = S_{k,1}V_{k,0}$$

Let us assume that (3. 6) is true for all values i less than or equal n. Then

$$\begin{aligned} S_{k,n+1} &= \begin{bmatrix} R_{k,n+2} & kR_{k,n+1} \\ R_{k,n+1} & kR_{k,n} \end{bmatrix} \\ &= \begin{bmatrix} R_{k,n+1} & kR_{k,n} \\ R_{k,n} & kR_{k,n-1} \end{bmatrix} \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} \\ &= S_{k,n}V_{k,1} \\ &= S_{k,1}V_{k,n-1}V_{k,1} && \text{By Hyp.} \\ &= S_{k,1}V_{k,n} && \text{By Eq. (3. 5)} \end{aligned}$$

as required. □

4. Commutative Properties

In this section we present some commutative properties which involve these matrix sequences.

Theorem 4.1. *For $m, n \in \mathbb{N}_0$, the following result holds*

$$V_{k,m}V_{k,n} = V_{k,n}V_{k,m} \tag{4. 1}$$

Proof. It is easy to see from the equation (3. 5) that

$$V_{k,m}V_{k,n} = V_{k,m+n} = V_{k,n+m} = V_{k,n}V_{k,m}$$

Hence the theorem. □

Corollary 4.2. *For $m, n \in \mathbb{N}_0$, we get*

$$V_{k,m}S_{k,n} = S_{k,n}V_{k,m} \tag{4. 2}$$

Proof. Let us use the induction on m. For $m = 0$, from the (3. 3), we have

$$V_{k,0}S_{k,n} = \begin{bmatrix} R_{k,n+1} & kR_{k,n} \\ R_{k,n} & kR_{k,n} \end{bmatrix} = S_{k,n}V_{k,0}$$

Suppose that the result is true for all values i less than or equal m. Then

$$\begin{aligned} V_{k,m+1}S_{k,n} &= (V_{k,m}V_{k,1})(S_{k,1}V_{k,n-1}) && \text{By Eqs. (3. 5) and (3. 6)} \\ &= V_{k,m}(S_{k,1}V_{k,1})V_{k,n-1} && \text{By Hyp.} \\ &= (V_{k,m}S_{k,1})(V_{k,1}V_{k,n-1}) \\ &= (S_{k,1}V_{k,m})(V_{k,n-1}V_{k,1}) && \text{By Hyp. and Eq. (4. 1)} \\ &= S_{k,1}(V_{k,m}V_{k,n-1})V_{k,1} \\ &= (S_{k,1}V_{k,n-1})(V_{k,m}V_{k,1}) && \text{By Eq. (4. 1)} \\ &= S_{k,n}V_{k,m+1} && \text{By Eqs. (3. 5) and (3. 6)} \end{aligned}$$

as needed. □

Corollary 4.3. *For $m, n \in \mathbb{N}_0$, we get*

$$S_{k,m}S_{k,n} = S_{k,n}S_{k,m} \tag{4. 3}$$

Proof. To prove the result we shall use induction on m. Let $m = 0$, from the (3. 3), we get

$$S_{k,0}S_{k,n} = \begin{bmatrix} R_{k,n+1} + 2kR_{k,n} & kR_{k,n} + 2k^2R_{k,n-1} \\ 2R_{k,n+1} - 3R_{k,n} & 2R_{k,n} - 3kR_{k,n-1} \end{bmatrix} = S_{k,n}S_{k,0}$$

Let us assume that the equation (4. 3) is true for all i less than or equal m. Therefore

$$\begin{aligned} S_{k,m+1}S_{k,n} &= S_{k,1}V_{k,m}S_{k,n} && \text{By Eq. (3. 6)} \\ &= S_{k,1}S_{k,n}V_{k,m} && \text{By Eq. (4. 2)} \\ &= S_{k,n}S_{k,1}V_{k,m} && \text{By Hyp.} \\ &= S_{k,n}S_{k,m+1} && \text{By Eq. (3. 6)} \end{aligned}$$

as required. □

5. Fundamental Relations Between Generalized Fibonacci Matrix Sequence $\langle S_{k,n} \rangle$ and k-Pell Matrix Sequence $\langle V_{k,n} \rangle$

In this section we present some important results which establish relation between a Generalized Fibonacci matrix sequence and k-Pell matrix sequence in a simple and better way. So, for this we have the following theorems:

Theorem 5.1. *The following properties hold*

$$S_{k,n} = V_{k,n} + 2kV_{k,n-1}, \; n \geq 1 \tag{5. 1}$$

$$S_{k,n} = 2V_{k,n+1} - 3V_{k,n}, \; n \geq 0 \tag{5. 2}$$

Proof. For the proof of equation (5. 1), we use induction on n. If $n = 1$, we obtain

$$\begin{aligned} S_{k,1} &= \begin{bmatrix} 2+2k & k \\ 1 & 2k \end{bmatrix} \\ &= \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2k & 0 \\ 0 & 2k \end{bmatrix} \\ &= \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} 2k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= V_{k,1} + 2kV_{k,0} \end{aligned}$$

Let us assume that the result is true for all values i less than or equal n. Then

$$\begin{aligned} S_{k,n+1} &= S_{k,1}V_{k,n} && \text{By Eq. (3. 6)} \\ &= (V_{k,1} + 2kV_{k,0})\, V_{k,n} && \text{By Hyp.} \\ &= V_{k,1}V_{k,n} + 2kV_{k,0}V_{k,n} \\ &= V_{k,n+1} + 2kV_{k,n} && \text{By Eq. (3. 5)} \end{aligned}$$

as needed.
If a similar technique is used for equation (5. 2), the proof is obvious. □

Theorem 5.2. *For $n \geq 0$, the following properties hold*

$$S_{k,n+1}^2 = S_{k,1}^2 V_{k,2n} \tag{5. 3}$$
$$S_{k,2n+1} = V_{k,n}S_{k,n+1} \tag{5. 4}$$

Proof. Since

$$\begin{aligned} S_{k,n+1}^2 &= S_{k,n+1}S_{k,n+1} \\ &= S_{k,1}V_{k,n}S_{k,1}V_{k,n} && \text{By Eq. (3. 6)} \\ &= S_{k,1}S_{k,1}V_{k,n}V_{k,n} && \text{By Eq. (4. 2)} \\ &= S_{k,1}^2 V_{k,2n} && \text{By Eq. (3. 5)} \end{aligned}$$

and since

$$\begin{aligned} S_{k,2n+1} &= S_{k,1}V_{k,2n} && \text{By Eq. (3. 6)} \\ &= S_{k,1}V_{k,n}V_{k,n} && \text{By Eq. (3. 5)} \\ &= S_{k,n+1}V_{k,n} && \text{By Eq. (3. 6)} \\ &= V_{k,n}S_{k,n+1} && \text{By Eq. (4. 2)} \end{aligned}$$

Hence the proof. □

Corollary 5.3. *For $n \geq 0$, we get*

$$S_{k,n+1}^2 = S_{k,1}S_{k,2n+1} \tag{5. 5}$$

Proof. The proof is clearly seen by the equations (3. 6) and (5. 3). □

Theorem 5.4. *For $n \in \mathbb{N}_0$, the following result holds*

$$S_{k,n} = S_{k,0}V_{k,n} \tag{5. 6}$$

Proof. To prove the result we shall use induction on n. Let $n = 0$, we get

$$S_{k,0} = \begin{bmatrix} 1 & 2k \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2k \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = S_{k,0}V_{k,0}$$

Suppose that the result is true for all values of i less than or equal to n. Then by the equation (3. 3)

$$\begin{aligned} S_{k,n+1} &= \begin{bmatrix} R_{k,n+2} & kR_{k,n+1} \\ R_{k,n+1} & kR_{k,n} \end{bmatrix} \\ &= \begin{bmatrix} R_{k,n+1} & kR_{k,n} \\ R_{k,n} & kR_{k,n-1} \end{bmatrix} \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} \\ &= S_{k,n}V_{k,1} \\ &= S_{k,0}V_{k,n}V_{k,1} && \text{By Hyp.} \\ &= S_{k,0}V_{k,n+1} && \text{By Eq. (3. 5)} \end{aligned}$$

as required. □

Corollary 5.5. *For $m, n \geq 0$, we get*

$$\begin{aligned} S_{k,n+m} &= S_{k,n}V_{k,m} \\ &= V_{k,n}S_{k,m} \end{aligned} \tag{5. 7}$$

Proof. Since

$$\begin{aligned} S_{k,n+m} &= S_{k,0}V_{k,n+m} && \text{By Eq. (5. 6)} \\ &= S_{k,0}V_{k,n}V_{k,m} && \text{By Eq. (3. 5)} \\ &= S_{k,n}V_{k,m} && \text{By Eq. (5. 6)} \end{aligned}$$

Hence the result. □

6. Some Relations Between Generalized Fibonacci Sequence $\langle R_{k,n} \rangle$ and k-Pell Sequence $\langle P_{k,n} \rangle$

In the present section we investigate the relations between the generalized Fibonacci sequence and k-Pell sequence by using their respective matrix sequences.

Theorem 6.1. *For $n \in \mathbb{N}_0$, we have the following result*

$$\begin{aligned} R^2_{k,n+2} + kR^2_{k,n+1} &= P_{k,2n+3} + 4kP_{k,2n+2} + 4k^2P_{k,2n+1} \\ &= R_{k,2n+3} + 2kR_{k,2n+2} \end{aligned} \tag{6. 1}$$

Proof. By using equations (3. 4) and (5. 3), we get

$$\begin{aligned}S_{k,n+1}^2 &= S_{k,1}^2 V_{k,2n}\\ &= \begin{bmatrix} 2+2k & k \\ 1 & 2k\end{bmatrix}^2 \begin{bmatrix} P_{k,2n+1} & kP_{k,2n} \\ P_{k,2n} & kP_{k,2n-1}\end{bmatrix}\\ &= \begin{bmatrix} 4k^2+9k+4 & 4k^2+2k \\ 4k+2 & 4k^2+k\end{bmatrix} \begin{bmatrix} P_{k,2n+1} & kP_{k,2n} \\ P_{k,2n} & kP_{k,2n-1}\end{bmatrix}\\ &= \begin{bmatrix} \left(4k^2+9k+4\right) P_{k,2n+1} + \left(4k^2+2k\right) P_{k,2n} & d_1 \\ d_2 & d_3\end{bmatrix}\end{aligned} \tag{6. 2}$$

where d_1, d_2 and d_3 represent corresponding terms of the matrix.
Since

$$\begin{aligned}S_{k,n+1}^2 &= S_{k,n+1} S_{k,n+1}\\ &= \begin{bmatrix} R_{k,n+2}^2 + kR_{k,n+1}^2 & e_1 \\ e_2 & e_3\end{bmatrix}\end{aligned} \tag{6. 3}$$

where e_1, e_2 and e_3 are the corresponding terms of the matrix.
Now from the equations (6. 2) and (6. 3), we obtain

$$\begin{aligned}R_{k,n+2}^2 + kR_{k,n+1}^2 &= 4k^2 P_{k,2n+1} + 9kP_{k,2n+1} + 4k^2 P_{k,2n} + 4P_{k,2n+1} + 2kP_{k,2n}\\ &= 4k^2 P_{k,2n+1} + 9kP_{k,2n+1} + 4k^2 P_{k,2n} + 2P_{k,2n+2}\\ &= 8kP_{k,2n+1} + 4k^2 P_{k,2n} + 4k^2 P_{k,2n+1} + P_{k,2n+3}\\ &= P_{k,2n+3} + 4kP_{k,2n+2} + 4k^2 P_{k,2n+1}\end{aligned}$$

This proves the first part of the theorem (6.1).
Now to prove second part of this theorem considering the equation (5. 5), we have

$$\begin{aligned}S_{k,n+1}^2 &= S_{k,1} S_{k,2n+1}\\ &= \begin{bmatrix} 2+2k & k \\ 1 & 2k\end{bmatrix} \begin{bmatrix} R_{k,2n+2} & kR_{k,2n+1} \\ R_{k,2n+1} & kR_{k,2n}\end{bmatrix}\\ &= \begin{bmatrix} (2+2k)\, R_{k,2n+2} + kR_{k,2n+1} & f_1 \\ f_2 & f_3\end{bmatrix}\end{aligned}$$

where f_1, f_2 and f_3 are the corresponding terms of the matrix.
Thus from the equation (6. 3), we obtain

$$\begin{aligned}R_{k,n+2}^2 + kR_{k,n+1}^2 &= (2+2k)\, R_{k,2n+2} + kR_{k,2n+1}\\ &= R_{k,2n+3} + 2kR_{k,2n+2}\end{aligned}$$

This proves the second part of the theorem (6.1). □

Theorem 6.2. *For $n \in \mathbb{Z}^+$, we obtain*

$$\begin{aligned} R_{k,2n} &= R_{k,n+1}P_{k,n} + kR_{k,n}P_{k,n-1} \\ &= R_{k,n}P_{k,n+1} + kR_{k,n-1}P_{k,n} \end{aligned} \tag{6. 4}$$

Proof. Its proof can be easily seen by using the equations (5. 7), (3. 3) and (3. 4). □

7. Some Other Results

Lemma 7.1. *For $n \geq 0$, we obtain*

$$R_{k,n} = 2P_{k,n+1} - 3P_{k,n} \tag{7. 1}$$

Proof. If we equate the corresponding terms of matrices in the equation (5. 2), we get the desired result.

Theorem 7.2. (Binet's Formula for Generalized Fibonacci Sequence $\langle R_{k,n}\rangle$) *For $n \in \mathbb{N}_0$, the n^{th} term of $\langle R_{k,n}\rangle$ is given by*

$$R_{k,n} = \left(\frac{1-2b}{a-b}\right)a^n + \left(\frac{2a-1}{a-b}\right)b^n \tag{7. 2}$$

where a and b are the roots of the characteristic equation, $x^2 - 2x - k = 0$.

Proof. By lemma (7.1), we have

$$\begin{aligned} R_{k,n} &= 2P_{k,n+1} - 3P_{k,n} \\ &= \frac{1}{a-b}\left[2\left(a^{n+1} - b^{n+1}\right) - 3\left(a^n - b^n\right)\right] \\ &= \frac{1}{a-b}\left(a^n - 4a^n + 2a^{n+1} + 4b^n - 2b^{n+1} - b^n\right) \\ &= \frac{1}{a-b}\left[a^n - 2\left(2-a\right)a^n + 2\left(2-b\right)b^n - b^n\right] \\ &= \frac{1}{a-b}\left(a^n - 2ba^n + 2ab^n - b^n\right) && \text{By (i.)} \\ &= \left(\frac{1-2b}{a-b}\right)a^n + \left(\frac{2a-1}{a-b}\right)b^n \end{aligned}$$

as required. □

Theorem 7.3. *For the 2×2 matrices R_1 and R_2, we have*

$$S_{k,n} = \left(\frac{1-2b}{a-b}\right)R_1a^n + \left(\frac{2a-1}{a-b}\right)R_2b^n,\ n \geq 0 \tag{7. 3}$$

where $R_1 = \begin{pmatrix} a & k \\ 1 & -b \end{pmatrix}$ and $R_2 = \begin{pmatrix} b & k \\ 1 & -a \end{pmatrix}$.

Proof. Let

$$\frac{1}{a-b}\left[(1-2b)\begin{pmatrix} a & k \\ 1 & -b \end{pmatrix}a^n + (2a-1)\begin{pmatrix} b & k \\ 1 & -a \end{pmatrix}b^n\right]$$

$$= \frac{1}{a-b}\begin{bmatrix} a(1-2b)a^n + b(2a-1)b^n & k(1-2b)a^n + k(2a-1)b^n \\ (1-2b)a^n + (2a-1)b^n & -b(1-2b)a^n - a(2a-1)b^n \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1-2b}{a-b}\right)a^{n+1} + \left(\frac{2a-1}{a-b}\right)b^{n+1} & k\left[\left(\frac{1-2b}{a-b}\right)a^n + \left(\frac{2a-1}{a-b}\right)b^n\right] \\ \left(\frac{1-2b}{a-b}\right)a^n + \left(\frac{2a-1}{a-b}\right)b^n & -(ab)\left[\left(\frac{1-2b}{a-b}\right)a^{n-1} + \left(\frac{2a-1}{a-b}\right)b^{n-1}\right] \end{bmatrix}$$

$$= \begin{bmatrix} R_{k,n+1} & kR_{k,n} \\ R_{k,n} & kR_{k,n-1} \end{bmatrix} \qquad \text{By Eq. (7. 2) and (iii.)}$$

$$= S_{k,n}$$

as needed. □

Theorem 7.4. *For $n \in \mathbb{N}_0$, the following results hold.*

$$S_{k,n} = \left(\frac{S_{k,1} - bS_{k,0}}{a-b}\right)a^n - \left(\frac{S_{k,1} - aS_{k,0}}{a-b}\right)b^n \tag{7. 4}$$

$$V_{k,n} = \left(\frac{V_{k,1} - bV_{k,0}}{a-b}\right)a^n - \left(\frac{V_{k,1} - aV_{k,0}}{a-b}\right)b^n \tag{7. 5}$$

Proof.

$$\left(\frac{S_{k,1} - bS_{k,0}}{a-b}\right)a^n - \left(\frac{S_{k,1} - aS_{k,0}}{a-b}\right)b^n$$

$$= \frac{1}{a-b}\left[(a^n - b^n)\begin{pmatrix} 2+2k & k \\ 1 & 2k \end{pmatrix} - (ba^n - ab^n)\begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix}\right]$$

$$= \frac{1}{a-b}\begin{bmatrix} (a^n - b^n)(2+2k) - (ba^n - ab^n) & k(a^n - b^n) - 2k(ba^n - ab^n) \\ (a^n - b^n) - 2(ba^n - ab^n) & 2k(a^n - b^n) + 3(ba^n - ab^n) \end{bmatrix}$$

$$= \frac{1}{a-b}\begin{bmatrix} a^n(2-b) & k[a^n(1-2b) - b^n(1-2a)] \\ -b^n(2-a) + 2ka^n - 2kb^n & \\ a^n(1-2b) + b^n(2a-1) & 2k(a^n - b^n) - 3k\left(a^{n-1} - b^{n-1}\right) \end{bmatrix} \text{By (iii.)}$$

$$= \frac{1}{a-b}\begin{bmatrix} a^{n+1} - b^{n+1} + 2ka^n - 2kb^n & k[(1-2b)a^n - (1-2a)b^n] \\ (1-2b)a^n + (2a-1)b^n & k\left[2(a^n - b^n) - 3\left(a^{n-1} - b^{n-1}\right)\right] \end{bmatrix} \text{By (i.)}$$

$$= \frac{1}{a-b}\begin{bmatrix} a^{n+1} - 2ba^{n+1} - b^{n+1} + 2ab^{n+1} & k[(1-2b)a^n - (1-2a)b^n] \\ (1-2b)a^n + (2a-1)b^n & k\left[2(a^n - b^n) - 3\left(a^{n-1} - b^{n-1}\right)\right] \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1-2b}{a-b}\right)a^{n+1} + \left(\frac{2a-1}{a-b}\right)b^{n+1} & k\left[\left(\frac{1-2b}{a-b}\right)a^n + \left(\frac{2a-1}{a-b}\right)b^n\right] \\ \left(\frac{1-2b}{a-b}\right)a^n + \left(\frac{2a-1}{a-b}\right)b^n & k\left[\left(\frac{1-2b}{a-b}\right)a^{n-1} + \left(\frac{2a-1}{a-b}\right)b^{n-1}\right] \end{bmatrix}$$

$$= \begin{bmatrix} R_{k,n+1} & kR_{k,n} \\ R_{k,n} & kR_{k,n-1} \end{bmatrix}$$
$$= S_{k,n}$$

Hence we obtain the result (7. 4).
Proof of (7. 5) is similar to the proof of (7. 4). □

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