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Extension of Morphisms in Geometry of Chain Complexes

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Abstract. In this article, the extension of homomorphisms have been defined to connect Grassmannian chain complex of projective configuration of points with variant of Cathelineau infinitesimal polylogarithmatic group complexes. Using these morphisms diagrams are constructed up to weight 5. The commutativity and bi-complex of the diagrams are shown.

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1. Introduction

The Grassmannian Complex is used as an important tool in the field of Motivic cohomology, algebraic K-theory, Algebraic geometry, manifold theory and a myriad of other theories. Grassmannian complex is naturally associated with polylog groups. Let $G_m(n)$ be a free abelian group generated by all possible projective configuration of m points in n-dimensional vector space $V_n(F)$, over a field F. These groups are connected by differential morphisms called Grassmannian complex initially introduced by Suslin [16]. Khalid et al. defined higher order Grassmannian complex connecting free abelian groups by 2^{nd} and 3^{rd}

order differential morphisms [12]. Khalid et al. also generalized differential morphisms between free abelian groups to introduce the N^{th} order Grassmannian complex [13]. Polylogarithms functions

$$Li_n(x) = \sum_{p=1}^{\infty} \frac{x^p}{p^n} \quad | x |< 1$$

are being studied for almost a decade, initially introduced by Liebniz in 16^{th} century. It is an absolutely convergent series defined in a unit disc. Dilogarithm functions for $n = 2$ were studied by many authors but the most important one is Able's functional equations called Able's five term relation. Trilogarithm for $n = 3$ and its group $\mathcal{B}_3(F)$ was first defined by Goncharov but the most important was his generalized triple cross ratio property of six points [5]. Goncharov generalized polylog complex called his own complex. Goncharov also connects Grassmannian complex with his own complex through morphisms only for weight 2 and 3. Goncharov showed that the associated diagrams are commutative [5]. Cathelineau [1, 2, 3] used derivation maps to define variant of Goncharov complex in both infinitesimal and tangential form called Cathelineau complex. Siddiqui introduced a v ariant of Cathelineau complex and connected Grassmannain complex with variant of Cathelineau complex (both infinitesimal and tangential) for weight 2 and 3 [8, 14]. Khalid et al. defined new morphisms between variant of Cathelineau infinitesimal complex and Grassmannian complex up to weight 4 [10]. Khalid et al. also generalized morphism in geometry of Grassmannian complex with variant of Cathelineau infinitesimal complex up to weight n [11].

In this article, the work of Khalid et al. is being followed to define extension of morphisms between Grassmannian and variant of Cathelineau infinitesimal polylogarithmic group complexes. Khalid et al. [11] generalized only two morphisms, so in this work morphism between higher order polylogarithmic groups of variant of Cathelineau infinitesimal complex and Grassmannian chain complex will be defined for the first time and the diagrams will be shown to be commutative and bi-complex.

2. Preliminary and Background

This section discusses the preliminary background relevant to this research and chain complexes in detail. It covers the Grassmannian complex, Goncharov complex, Cathelineau complex and variant of Cathelineau complex, all of which are crucial for this research study.

2.1. **Grassmannian Complex.** Consider a free abelian group $G_m(V_n)$ generated by elements $(q_1, ..., q_m) \in V^m$ for some set V_n with $q_i \in V_n$ then there is a simplicial complex $(G_m(V), d)$ having differential morphism defined as

$$d : G_m(V_n) \to G_{m-1}(V_n)$$

$$d(q_1, ..., q_m) \to \sum_{i=0}^{m} (-1)^i (q_1, ..., \hat{q}_i, ..., q_m)$$

$\hat{q}_i$ implies leaving-out. There exist another differential map called projection morphism denoted by p.

$$p : G_{m+1}(n) \to G_m(n-1)$$

$$p(q_1, ..., q_m) \to \sum_{i=0}^{m} (-1)^i (q_i | q_1, ..., \hat{q}_i, ..., q_m)$$

Where $(q_i|q_1, ..., \hat{q}_i, ..., q_m)$ is the configuration of vectors in $V_{n+1}/ < q_i >$ defined as the n-dimensional factor space got by the projection of vectors $q_j \in V_{n+1},\ j \neq i$ projected from $G_m(n)$ to $G_{m-1}(n-1)$, the following bi-complex is obtained.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{m+5}(n+2) & \xrightarrow{d} & G_{m+4}(n+2) & \xrightarrow{d} & G_{m+3}(n+2) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{m+4}(n+1) & \xrightarrow{d} & G_{m+3}(n+1) & \xrightarrow{d} & G_{m+2}(n+1) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{m+3}(n) & \xrightarrow{d} & G_{m+2}(n) & \xrightarrow{d} & G_{m+1}(n)
\end{array} \qquad \text{(A)}$$

Lemma 2.2. *Diagram A is bi-complex.*

Proof. For proof see [5, 7]

Lemma 2.3. $d \circ p = p \circ d$

Proof. For proof see [5, 7]

2.4. **Siegel Cross Ratio Property.** Siegel [15] defined the most important cross ratio of four points define as $r(q_0, ..., q_3) = \frac{\triangle(q_0,q_3)\triangle(q_1,q_2)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}$($\triangle$ is determinant of points), with property

$$1 = \frac{\triangle(q_0, q_3)\triangle(q_1, q_2)}{\triangle(q_0, q_2)\triangle(q_1, q_3)} + \frac{\triangle(q_0, q_1)\triangle(q_2, q_3)}{\triangle(q_0, q_2)\triangle(q_1, q_3)}$$

Or

$$\frac{\triangle(q_0, q_2)\triangle(q_1, q_3) - \triangle(q_0, q_3)\triangle(q_1, q_2)}{\triangle(q_0, q_2)\triangle(q_1, q_3)} = \frac{\triangle(q_0, q_1)\triangle(q_2, q_3)}{\triangle(q_0, q_2)\triangle(q_1, q_3)}$$

2.5. **Polylog Groups and its Complexes.** Let $Z[\mathbf{P}^1_F/\{0, 1, \infty\}]$ is $Z-$module called free abelian group generated by $[x] \in \mathbf{P}^1_F$ [4, 5] (where $[x]$ means $ln(x)$), in this work F will be considered as a field and set $F^{\cdot\cdot} = F - \{0, 1\}$ is doubly punctured.

Definition 2.6. *The Scissor congruence group is denoted by $\mathcal{B}(F)$, it is factor group of $Z[F^{\cdot\cdot}]$ and its subgroup generated by* $[x]-[y]+[\frac{y}{x}]-[\frac{1-y}{1-x}]+[\frac{1-y^{-1}}{1-x^{-1}}]$, *where $x \neq y, x, y \neq 0, 1$* ([6])

2.6.1. ***Weight 1***. Let the group $R_1(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ generated by 3 terms relation $\langle [xy]-[x]-[y]\rangle, x, y \in F^\times - \{1\}$. Define $\mathcal{B}_1(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_1(F)$ [5]. The morphism $\delta : \mathcal{B}_1(F) \to F^\times$, defined as $[x] \to x$, is an isomorphism, so $\mathcal{B}_1(F) = F^\times$.

2.6.2. ***Weight 2***. The subgroup $R_2(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ [5], generated by the cross ratio of five terms relations is defined as

$$R_2(F) = \Big\langle \sum_{i=0}^{4}(-1)^i[r(q_0, ..., \hat{q}_i, ..., q_4)], q_i \in \mathbf{P}^1_F \Big\rangle$$

where

$$r(q_0, ..., q_3) = \frac{\triangle(q_0, q_3)\triangle(q_1, q_2)}{\triangle(q_0, q_2)\triangle(q_1, q_3)}$$

It is called cross ratio of four points. Define a map $\delta_2 : Z[\mathbf{P}^1_F/\{0,1,\infty\}] \to \wedge^2 F^\times$, defined as $[x] \to (1-x)\wedge x$, where $\wedge^2 F^\times = F^\times \otimes F^\times/\langle x \otimes_Z x | x \in F^\times\rangle$. It has been proven that $\delta_2(R_2(F)) = 0$ [5]. Define group $\mathcal{B}_2(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_2(F)$. Now introduce a complex

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times$$

where δ is an induced morphism defined as $\delta : [x]_2 \to (1-x)\wedge x$, this complex is known as Bloch-Suslin complex. The functional equations in $\mathcal{B}_2(F)$ are

(1) $[q]_2 = -[1-q]_2$
(2) $[q]_2 = -[\frac{1}{q}]_2$
(3) A five term relation $\sum_{i=0}^{4}(-1)^i r(q_0, ..., \hat{q}_i, ..., q_4) = 0$

2.6.3. ***Weight 3***. As defined in [5]

$$r_3(q_0, ..., q_5) = Alt_6 \frac{\triangle(q_0, q_1, q_3)\triangle(q_1, q_2, q_4)\triangle(q_2, q_0, q_5)}{\triangle(q_0, q_1, q_4)\triangle(q_1, q_2, q_5)\triangle(q_2, q_0, q_3)}$$

is a triple cross ratio of 6 points $\in Z[\mathbf{P}^1_F]$ in generic position, where $\triangle(q_i, q_j, q_k) = \langle \omega, q_i \wedge q_j \wedge q_k\rangle$ and $\omega \in \det V_3$ is volume form in 3-dimensional vector space . Take $R_3(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ [5], defined as

$$R_3(F) = \Big\langle \sum_{i=0}^{6}(-1)^i r_3(q_0, ..., \hat{q}_i, ..., q_6) | \quad (q_0, ..., \hat{q}_i, ..., q_6) \in G_6(\mathbf{P}^2_F)\Big\rangle \qquad (2.\ 1)$$

Eq. (2. 1) is 7 term relation of triple cross ratio. Goncharov defined $\mathcal{B}_3(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_3(F)$ and the Goncharov's complex in weight 3 is given by

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^\times \xrightarrow{\delta} \wedge^3 F^\times$$

2.6.4. ***Weight 4***. Assume that

$$r_4(q_0, ..., q_7) = Alt_8 \frac{\triangle(q_0, q_1, q_2, q_7)\triangle(q_1, q_2, q_3, q_4)\triangle(q_2, q_3, q_0, q_5)\triangle(q_3, q_0, q_1, q_6)}{\triangle(q_0, q_1, q_2, q_4)\triangle(q_1, q_2, q_3, q_5)\triangle(q_2, q_3, q_0, q_6)\triangle(q_3, q_0, q_1, q_7)}$$

it is a cross ratio of 8 points $\in Z[\mathbf{P}^1_F]$ in generic position. Take $R_4(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$, defined as

$$R_4(F) = \Big\langle \sum_{i=0}^{8}(-1)^i r_4(q_0, ..., \hat{q}_i, ..., q_8)| \quad (q_0, ..., \hat{q}_i, ..., q_8) \in G_8(\mathbf{P}^3_F)\Big\rangle$$

Define group $\mathcal{B}_4(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_4(F)$. Therefore functional equation of $\mathcal{B}_4(F)$ is 9 terms relation.

2.6.5. ***Weight 5***. Assume that

$$r_5(q_0,..., q_9) = Alt_{10}\frac{\triangle(q_0, q_1, q_2, q_3, q_9)\triangle(q_1, q_2, q_3, q_4, q_5)...\triangle(q_4, q_0, q_1, q_2, q_8)}{\triangle(q_0, q_1, q_2, q_3, q_5)\triangle(q_1, q_2, q_3, q_4, q_6)...\triangle(q_4, q_0, q_1, q_2, q_9)}$$

is a cross ratio of 10 points $\in Z[\mathbf{P}^1_F]$ in generic position. Take $R_5(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$, defined as

$$R_5(F) = \Big\langle \sum_{i=0}^{10}(-1)^i r_5(q_0, ..., \hat{q}_i, ..., q_{10})| \quad (q_0, ..., \hat{q}_i, ..., q_{10}) \in G_{11}(\mathbf{P}^4_F)\Big\rangle$$

Define group $\mathcal{B}_5(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_5(F)$. Therefore functional equation of $\mathcal{B}_5(F)$ is 11 terms relation.

2.6.6. ***Weight n***. Now define a morphism $\delta_n : Z[\mathbf{P}^1_F] \to \mathcal{B}_{n-1}(F)$ then $A_n(F) = Ker\delta_n$. Goncharov [5] generalized the group $\mathcal{B}_n(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_n(F)$, where $R_n(F)$ is subgroup of $Z[\mathbf{P}^1_F]$ generated by $q(0) - q(1)$, 0 and ∞. Where $q(t)$ runs through all elements of $A_n(F(t))$ for indeterminate t. So generalized Goncharov's complex is given as

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F)\otimes F^\times \xrightarrow{\delta} \mathcal{B}_{n-2}(F)\otimes\wedge^2(F^\times) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2(F)\otimes\wedge^{n-2}(F^\times) \xrightarrow{\delta} \frac{\wedge^n F^\times}{2 - torsion} \tag{B}$$

2.7. **Cathelineau Infinitesimal Chain Complexes.** Cathlelineau [1, 2] introduced F- Vector space, it is an analogy of Goncharov's group $\mathcal{B}_n(F)$, defined as

(1) $\beta_1(F) = F$
(2) $\beta_2(F) = \frac{F[F^{\cdot\cdot}]}{r_2(F)}$
where $r_2(F)$ is generated by four elements $[x] - [y] + x[\frac{y}{x}] + (1 - x)[\frac{1-y}{1-x}], x, y \in F^{\cdot\cdot}, x \neq y$. Therefore a complex for weight-2 is obtained as

$$\beta_2(F) \xrightarrow{\partial} F \otimes_F F^\times$$

∂ is an induced morphism defined as

$$\partial : \langle q\rangle_2 \mapsto q \otimes (q) + (1 - q) \otimes (1 - q)$$

The functional equations in $\beta_2(F)$ are

(1) $\langle q\rangle_2 = \langle 1 - q\rangle_2$
(2) $\langle q\rangle_2 = -q\langle \frac{1}{q}\rangle_2$
(3) $\langle q\rangle_2 - \langle r\rangle_2 + q\langle \frac{r}{q}\rangle_2 + (1 - q)\langle \frac{1-r}{1-q}\rangle_2 = 0$
(4) A distribution relation $\langle q\rangle_2^m = \sum_{\zeta^m=1} \frac{1-q^m}{1-q\zeta}\langle \zeta q\rangle_2$

The Cathelineau infinitesimal polylog chain complex [2] for groups $\beta_n(F)$ and higher Bloch groups $\mathcal{B}_n(F)$ is given as

$$\beta_n(F) \xrightarrow{\partial_n} \begin{array}{c}\beta_{n-1}(F)\otimes F^\times\\ \oplus\\ F\otimes\mathcal{B}_{n-1}(F)\end{array} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \begin{array}{c}\beta_2(F)\otimes\wedge^{n-2}F^\times\\ \oplus\\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^\times\end{array} \xrightarrow{\partial_0} F\otimes\wedge^{n-1}F^\times \quad \text{(C)}$$

where ∂_n is given by

$$\partial_n : [q] \longmapsto \langle q\rangle_{n-1}\otimes q + (-1)^{n-1}(1-q)\otimes[q]_{n-1}$$

Lemma 2.8. $\partial_{n-1}\circ\partial_n = 0$

Proof. For proof see [2]

2.9. **Variant of Cathelineau Infinitesimal Chain Complex.** Siddiqui [14] defined a map $\partial^D : Z[F] \to F[F^{\cdot\cdot}]$, defined as $[q] = \frac{D(q)}{q(1-q)}[q]$. Let $\beta_2^D(F)$ is sub-vector space generated by $[\![q]\!]^D, q\in F^{\cdot\cdot}$ subject to five term relation $[\![q]\!]_2^D - [\![r]\!]_2^D + [\![\frac{r}{q}]\!]_2^D - [\![\frac{1-r}{1-q}]\!]_2^D + [\![\frac{1-r^{-1}}{1-q^{-1}}]\!]_2^D$ [14]. Where $[\![q]\!]^D = \frac{D(q)}{q(1-q)}[q]$. Now by $\partial_2^D : F[F^{\cdot\cdot}] \to F\otimes_F F$ with $\partial^D : [\![q]\!]_2^D \to -D\log(1-q)\otimes q + D\log(q)\otimes(1-q)$ where $D\log(1-q) = \frac{D(1-q)}{(1-q)}$. Define $\beta_2^D(F)$, it is a quotient group defined as $F[F^{\cdot\cdot}]/(\rho_2)$ where ρ_2 is subgroup generated by five term relation $[\![q]\!]_2^D - [\![r]\!]_2^D + [\![\frac{r}{q}]\!]_2^D - [\![\frac{1-r}{1-q}]\!]_2^D + [\![\frac{1-r^{-1}}{1-q^{-1}}]\!]_2^D$ where $q\neq r$, $1-q\neq 0$. Now define a chain Complex

$$\beta_2^D(F) \xrightarrow{\partial^D} F\otimes F^\times$$

where $\partial^D : [\![q]\!]_2^D \to -D\log(1-q)\otimes q + D\log(q)\otimes(1-q)$
The generators in $\beta_2^D(F)$ are

(1) $[\![q]\!]_2^D = -[\![1-q]\!]_2^D$
(2) $[\![q]\!]_2^D = -[\![\frac{1}{q}]\!]_2^D$
(3) $[\![q]\!]_2^D - [\![r]\!]_2^D + [\![\frac{r}{q}]\!]_2^D - [\![\frac{1-r}{1-r}]\!]_2^D + [\![\frac{1-r^{-1}}{1-q^{-1}}]\!]_2^D = 0$

Take subgroup $\beta_3^D(F) = F[F^{\cdot\cdot}]/(\rho_3)$, ρ_3 is subgroup of $F[F^{\cdot\cdot}]$ defined as

$$\rho_3 = \sum_{i=0}^{6}(-1)^i[\![r_3(q_0,...,\hat{q}_i,...,q_6)]\!]_2^D \quad (2.\ 2)$$

Eq. (2. 2) is a triple cross ratio of 6 points called 7 term relation. Functional equation of group $\beta_3^D(F)$ is a 7 terms relation. Now take subgroup $\beta_4^D(F) = F[F^{\cdot\cdot}]/(\rho_4)$, $\rho_4 \in F[F^{\cdot\cdot}]$ defined as

$$\rho_4 = \sum_{i=0}^{8}(-1)^i[\![r_4(q_0,...,\hat{q}_i,...,q_8)]\!]_2^D \quad (2.\ 3)$$

Eq. (2. 3) is a 9 term relation of cross ratio. Therefore functional equation of $\beta_4^D(F)$ is a 9 terms relation. Let subgroup $\beta_5^D(F) = F[F^{\cdot\cdot}]/(\rho_5)$, $\rho_5 \in F[F^{\cdot\cdot}]$ defined as

$$\rho_5 = \sum_{i=0}^{10}(-1)^i[\![r_5(q_0,...,\hat{q}_i,...,q_{10})]\!]_2^D \quad (2.\ 4)$$

Eq. (2. 4) is a 11 term relation of cross ratio. Therefore functional equation of polylog group $\beta_5^D(F)$ is a 11 terms relation. Similarly subgroup $\beta_n^D(F) = F[F^{\cdot\cdot}]/(\rho_n)$, ρ_n is a kernel of the morphism defined as $\partial_n^D : F[F] \to \beta_{n-1}^D \otimes F^\times \oplus F \otimes \mathcal{B}_{n-1}(F)$ [14]. Finally [14] generalized variant of Cathelineau's infinitesimal Complex for groups $\beta_n^D(F)$ and $\mathcal{B}_n(F)$ given by

$$\beta_n^D(F) \xrightarrow{\partial_n^D} \begin{matrix} \beta_{n-1}^D(F)\otimes F^\times \\ \oplus \\ nF\otimes\mathcal{B}_{n-1}(F) \end{matrix} \xrightarrow{\partial_{n-1}^D} \cdots \xrightarrow{\partial_1^D} \begin{matrix} \beta_2^D(F)\otimes\wedge^{n-2}F^\times \\ \oplus \\ (F)\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^\times \end{matrix} \xrightarrow{\partial_0^D} F\otimes\wedge^{n-1}F^\times \tag{D}$$

∂_n^D is a map given by

$$\partial^D : [q]_n^D \longmapsto -D\log[\![q]\!]_{n-1}\otimes q + (-1)^n D\log[q]_{n-1}\otimes(1-q)$$

Lemma 2.10. $\partial_{n-1}^D \circ \partial_n^D = 0$

Proof. For proof see [14]

3. Extension of Morphisms Between Grassmannian and Infinitesimal Chain Complexes

3.1. Weight 2 (Dilogarithm).

$$\begin{array}{ccc} G_5(2) & \xrightarrow{p} & G_4(1) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ G_4(2) & \xrightarrow{p} & G_3(1) \\ \downarrow{\scriptstyle h_1^2} & & \downarrow{\scriptstyle h_0^2} \\ \beta_2^D(F) & \xrightarrow{\partial^D} & F\otimes F^\times \end{array} \tag{E}$$

where

$$h_0^2 : (q_0, q_1, q_2) \to \sum_{i=0}^{2}(-1)^i \frac{D\triangle(q_i)}{\triangle(q_i)} \otimes \frac{\triangle(q_{i+1})}{\triangle(q_{i+2})} \quad (mod\ 3)$$

and

$$h_1^2(q_0, q_1, q_2, q_3) = \sum_{i=0}^{3}(-1)^i [\![r(q_0, ..., q_3)]\!]_2^D$$

Lemma 3.2. $h_0^2 \circ d = h_1^2 \circ d = 0$

Proof. For proof see [10]

Lemma 3.3. $h_0^2 \circ p = \partial^D \circ h_1^2$

Proof. For proof see [10]

3.4. **Weight 3 (Trilogarithm).**

$$\begin{array}{ccccc} G_7(3) & \xrightarrow{p} & G_6(2) & \xrightarrow{p} & G_5(1) \\ \downarrow d & & \downarrow d & & \downarrow d \\ G_6(3) & \xrightarrow{p} & G_5(2) & \xrightarrow{p} & G_4(1) \\ & & \downarrow h_1^3 & & \downarrow h_0^3 \\ & & \beta_2^D(F)\otimes F^\times \oplus F\otimes \mathcal{B}_2(F) & \xrightarrow{\partial^D} & F\otimes \wedge^2 F^\times \end{array} \tag{F}$$

where

$$h_0^3 : (q_0, q_1, q_2, q_3) \to \sum_{i=0}^{3}(-1)^i \mathrm{D}\log(q_i) \otimes \frac{\triangle(q_{i+1})}{\triangle(q_{i+2})} \wedge \frac{\triangle(q_{i+2})}{\triangle(q_{i+3})} \quad (mod\ 4)$$

and

$$h_1^3(q_0, ..., q_4) = -\frac{1}{3}\Big[\sum_{i=0}^{4}(-1)^i(\llbracket r(q_0, ..., \hat{q}_i, ..., q_4)\rrbracket_2^D \otimes \prod_{\substack{j\neq i\\ j=0}}^{4} \triangle(q_i, q_j)$$

$$- \mathrm{D}\log(\prod_{\substack{j\neq i\\ j=0}}^{4} \triangle(q_i, q_j)) \otimes [r(q_0, ..., \hat{q}_i, ..., q_4)]_2)\Big]$$

Lemma 3.5. $h_0^3 \circ d = h_1^3 \circ d = 0$

Proof. For proof see [10]

Lemma 3.6. $h_0^3 \circ p = \partial^D \circ h_1^3$

Proof. For proof see [10]

3.7. **Extension in Weight 3 (Tri-logarithm).** From [9] extension of weight 3 is defined as

$$\begin{array}{ccccc} G_7(3) & \xrightarrow{p} & G_6(2) & \xrightarrow{p} & G_5(1) \\ \downarrow d & & \downarrow d & & \downarrow d \\ G_6(3) & \xrightarrow{p} & G_5(2) & \xrightarrow{p} & G_4(1) \\ \downarrow h_2^3 & & \downarrow h_1^3 & & \downarrow h_0^3 \\ \beta_3^D(F) & \xrightarrow{\partial^D} & \beta_2^D(F)\otimes F^\times \oplus F\otimes \mathcal{B}_2(F) & \xrightarrow{\partial^D} & F\otimes \wedge^2 F^\times \end{array} \tag{G}$$

$$h_2^3(q_0, ..., q_5) = \frac{2}{45} Alt_6 \llbracket r(q_0, ..., q_5)\rrbracket_3^D$$

Diagram G is commutative see [9]

4. Weight 4

As defined in [10]

$$\begin{array}{ccccc}
G_8(3) & \xrightarrow{p} & G_7(2) & \xrightarrow{p} & G_6(1) \\
\downarrow d & & \downarrow d & & \downarrow d \\
G_7(3) & \xrightarrow{p} & G_6(2) & \xrightarrow{p} & G_5(1) \\
 & & \downarrow h_1^4 & & \downarrow h_0^4 \\
 & & \beta_2^D(F)\otimes\wedge^2F^\times \oplus F\otimes\mathcal{B}_2(F)\otimes F^\times & \xrightarrow{\partial^D} & F\otimes\wedge^3F^\times
\end{array} \tag{H}$$

where

$$h_0^4 : (q_0, q_1, q_2, q_3, q_4) \to \sum_{i=0}^{4}(-1)^i D\log(q_i) \otimes \frac{\triangle(q_{i+1})}{\triangle(q_{i+2})} \wedge \frac{\triangle(q_{i+2})}{\triangle(q_{i+3})} \wedge \frac{\triangle(q_{i+3})}{\triangle(q_{i+4})} \quad (mod\ 5)$$

and h_1^4 is given below

$$\begin{aligned}
h_1^4(q_0, ..., q_5) = &\frac{1}{6}\sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{5}(-1)^{i+j}\Big[([\![r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_5)]\!]_2^D \otimes \prod_{\substack{j\neq m\\ m=0}}^{5}\triangle(q_j, q_m) \wedge \prod_{\substack{i\neq m\\ m=0}}^{5}\triangle(q_i, q_m) - \\
&D\log(\prod_{\substack{i\neq m\\ m=0}}^{5}\triangle(q_i, q_m)) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_5)]_2 \wedge \prod_{\substack{j\neq m\\ m=0}}^{5}\triangle(q_j, q_m) + \\
&D\log(\prod_{\substack{j\neq m\\ m=0}}^{5}\triangle(q_j, q_m)) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_5)]_2 \wedge \prod_{\substack{i\neq m\\ m=0}}^{5}\triangle(q_i, q_m))\Big]
\end{aligned}$$

Lemma 4.1. $h_0^4 \circ d = h_1^4 \circ d = 0.$

Proof. For proof see [10]

Lemma 4.2. $h_0^4 \circ p = \partial^D \circ h_1^4$

Proof. For proof see [10]

4.3. **Extension in Weight 4.** Introduce 2 new morphisms h_2^4 and h_3^4 to extend the chain complex H as

$$\begin{array}{ccccc}
G_9(4) & \xrightarrow{d} & G_8(4) & \xrightarrow{h_3^4} & \beta_4^D(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_8(3) & \xrightarrow{d} & G_7(3) & \xrightarrow{h_2^4} & \beta_3^D(F) \otimes F^\times \oplus F \otimes \mathcal{B}_3(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_7(2) & \xrightarrow{d} & G_6(2) & \xrightarrow{h_1^4} & \beta_2^D(F) \otimes \wedge^2 F^\times \oplus F \otimes \mathcal{B}_2(F) \otimes F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_6(1) & \xrightarrow{d} & G_5(1) & \xrightarrow{h_0^4} & F \otimes \wedge^3 F^\times
\end{array} \tag{I}$$

where

$$h_2^4(q_0, ..., q_6) = -\frac{1}{28}\sum_{i=0}^{6}(-1)^i\Big[(\llbracket r(q_0, ..., \hat{q}_i, ..., q_6)\rrbracket_3^D \otimes \prod_{\substack{i\neq m\\ m=0}}^{6} \triangle(q_i, q_m) -$$

$$D\log(\prod_{\substack{i\neq m\\ m=0}}^{6} \triangle(q_i, q_m)) \otimes [r(q_0, ..., \hat{q}_i, ..., q_6)]_3)\Big]$$

and

$$h_3^4(q_0, ..., q_7) = \frac{1}{66} Alt_8 \llbracket r(q_0, ..., q_7)\rrbracket_4^D$$

Lemma 4.4. $h_2^4 \circ d = 0$

Proof. Let $(q_0, ..., q_7) \in G_8(3)$, then apply morphism d

$$d(q_0, ..., q_7) = \sum_{i=0}^{7}(-1)^i(q_0, ..., \hat{q}_i, ..., q_7)$$

now by applying morphism h_1^4

$$h_2^4 \circ d(q_0, ..., q_7) = -\frac{1}{28}\Big[\Big(\sum_{\substack{i=0\\ j=i+1}}^{7}(-1)^i\llbracket r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_7)\rrbracket_3^D\Big) \otimes \prod_{\substack{j\neq m\\ m=j+1}}^{7} \triangle(q_i, q_m) -$$

$$D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{7} \triangle(q_i, q_m)) \otimes \Big(\sum_{\substack{i=0\\ j=i+1}}^{7}(-1)^j[r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_7)]_3\Big)\Big] \tag{4. 5}$$

It Eq. (4. 5), the summand $\sum\llbracket r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_7)\rrbracket_3^D \in \beta_3^D(F)$ and $\sum[r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_7$ $)]_3 \in \mathcal{B}_3(F)$ are functional equations of these groups. Therefore $h_2^4 \circ d = 0$

Lemma 4.5. $h_3^4 \circ d = 0$

Proof. Let $(q_0, ..., q_8) \in G_9(4)$, then by applying morphism d

$$d(q_0, ..., q_8) = \sum_{i=0}^{8} (-1)^i (q_0, ..., \hat{q}_i, ..., q_8)$$

now by applying morphism h_3^4

$$h_3^4 \circ d(q_0, ..., q_8) = \frac{1}{66} \sum_{i=0}^{8} (-1)^i \frac{1}{56} Alt_8([\![r(q_0, ..., \hat{q}_i, ..., q_8)]\!]_4^D \tag{4. 6}$$

Eqs. (4. 6) is a 9 terms relation and functional equation of $\beta_4^D(F)$. Therefore $h_3^4 \circ d = 0$

Lemma 4.6. $h_1^4 \circ p = \partial^D \circ h_2^4$

Proof. Let $(q_0, ..., q_6) \in G_7(3)$, then

$$p(q_0, ..., q_6) = \sum_{i=0}^{6} (-1)^i (q_i | q_0, ..., \hat{q}_i, ..., q_6) \tag{4. 7}$$

Eq. (4. 7) have 7 terms with single projected point. Now by applying a map h_1^4, then

$$\begin{aligned} h_1^4 \circ p(q_0, ..., q_6) =& \frac{1}{6} \sum_{\substack{i=0\\ j=i+1\\ k=i+2}}^{6} (-1)^i \Big[([\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]\!]_2^D \otimes \prod_{\substack{k\neq m\\ m=k+1}}^{6} \triangle(q_i|q_k, q_m) \wedge \\ & \prod_{\substack{j\neq m\\ m=j+1}}^{6} \triangle(q_i|q_j, q_m) - \\ & D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{6} \triangle(q_i|q_j, q_m)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2 \wedge \\ & \prod_{\substack{k\neq m\\ m=k+1}}^{6} \triangle(q_i|q_k, q_m) + \\ & D\log(\prod_{\substack{k\neq m\\ m=k+1}}^{6} \triangle(q_i|q_k, q_m)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2 \wedge \\ & \prod_{\substack{i\neq m\\ m=i+1}}^{6} \triangle(q_i|q_j, q_m)) \Big] \end{aligned} \tag{4. 8}$$

Assume again $(q_0, ..., q_6) \in G_7(3)$ and apply morphism h_2^4

$$h_2^4(q_0, ..., q_6) = -\frac{1}{28} \Big[\sum_{\substack{i=0\\ j=i+1}}^{6} (-1)^i ([\![r(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_6)]\!]_3^D \otimes \prod_{\substack{j\neq m\\ m=j+1}}^{6} \triangle(q_j, q_m) \wedge \prod_{\substack{i\neq m\\ m=i+1}}^{6} \triangle(q_i, q_m) -$$

$$D\log(\prod_{\substack{i\neq m\\ m=i+1}}^{6} \triangle(q_i,q_m))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_6)]_3)\Big]$$

after applying morphism ∂^D and all properties of tensor, wedge, odd cycle and Siegel cross ratio properties, it becomes

$$\begin{aligned}\partial^D\circ h_2^4(q_0,...,q_8) =& \frac{1}{6}\Big[\sum_{\substack{i=0\\ j=i+1}}^{6}(-1)^i\Big([\![r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]\!]_2^D\otimes\prod_{\substack{k\neq m\\ m=k+1}}^{6}\triangle(q_i|q_k,q_m)\wedge\\ &\prod_{\substack{j\neq m\\ m=j+1}}^{6}\triangle(q_i|q_j,q_m)-\\ &D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{6}(q_i|q_j,q_m))\otimes[r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\wedge\\ &\prod_{\substack{k\neq m\\ m=k+1}}^{6}\triangle(q_i|q_k,q_m))+\\ &D\log(\prod_{\substack{k\neq m\\ m=k+1}}^{6}\triangle(q_i|q_k,q_m)\otimes[r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\wedge\\ &\prod_{\substack{i\neq m\\ m=i+1}}^{6}\triangle(q_i|q_j,q_m)\Big)\Big]\end{aligned} \tag{4. 9}$$

From Eqs. (4. 8) and (4. 9), $h_1^4\circ p=\partial^D\circ h_2^4$

Lemma 4.7. $h_2^4\circ p=\partial^D\circ h_3^4$

Proof. Let $(q_0,...,q_7)\in G_8(4)$, then

$$p(q_0,.....,q_7)=\sum_{i=0}^{7}(-1)^i(q_i|q_0,...,\hat{q}_i,...,q_7)$$

now by applying map h_2^4, then

$$\begin{aligned}h_2^4\circ p(q_0,...,q_7) =& -\frac{1}{28}\sum_{\substack{i=0\\ j=i+1}}^{7}(-1)^i\Big[([\![r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,...,q_7)]\!]_3^D\otimes\prod_{\substack{j\neq m\\ m=j+1}}^{7}\triangle(q_i|q_j,q_m)-\\ &D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{7}\triangle(q_i|q_j,q_m))\otimes[r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,...,q_7)]_3)\Big]\end{aligned} \tag{4. 10}$$

Assume again $(q_0, ..., q_6) \in G_8(4)$ and apply morphism h_3^4

$$h_3^4(q_0, ..., q_7) = \frac{1}{66} Alt_8 [\![r(q_0, ..., q_7)]\!]_4^D \tag{4. 11}$$

after applying morphism ∂^D and all properties of tensor, wedge, odd cycle and Siegel cross ratio, it becomes

$$\partial^D \circ h_3^4(q_0, ..., q_7) = -\frac{1}{28} \sum_{\substack{i=0\\ j=i+1}}^{7} (-1)^i \Big[([\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_7)]\!]_3^D \otimes \prod_{\substack{j\neq m\\ m=j+1}}^{7} \triangle(q_i|q_j, q_m) - D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{7} \triangle(q_i|q_j, q_m)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_7)]_3) \Big] \tag{4. 12}$$

From Eqs. (4. 10) and (4. 12), it is observed that $h_2^4 \circ p = \partial^D \circ h_3^4$

5. Weight 5

As defined in [11]

$$\begin{array}{ccccc} G_9(3) & \xrightarrow{p} & G_8(2) & \xrightarrow{p} & G_7(1) \\ \downarrow d & & \downarrow d & & \downarrow d \\ G_9(3) & \xrightarrow{p} & G_7(2) & \xrightarrow{p} & G_6(1) \\ & & \downarrow h_1^5 & & \downarrow h_0^5 \\ & & \beta_2^D(F) \otimes \wedge^3 F^\times \oplus F \otimes \mathcal{B}_2(F) \otimes \wedge^2 F^\times & \xrightarrow{\partial^D} & F \otimes \wedge^4 F^\times \end{array} \tag{J}$$

where

$$h_0^5 : (q_0, ..., q_5) \to \sum_{i=0}^{4} (-1)^i D\log(q_i) \otimes \frac{\triangle(q_{i+1})}{\triangle(q_{i+2})} \wedge \frac{\triangle(q_{i+2})}{\triangle(q_{i+3})} \wedge \frac{\triangle(q_{i+3})}{\triangle(q_{i+4})} \wedge \frac{\triangle(q_{i+4})}{\triangle(q_{i+5})} \quad (mod\ 6)$$

and h_1^5 as given below

$$h_1^5(q_0,..., q_6) = -\frac{1}{10} \Big[\sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{6} (-1)^i \Big([\![r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]\!]_2^D \otimes \prod_{\substack{k\neq m\\ m=0}}^{6} \triangle(q_k, q_m) \wedge \prod_{\substack{j\neq m\\ m=0}}^{6} \triangle(q_j, q_m) \wedge \prod_{\substack{i\neq m\\ m=0}}^{6} \triangle(q_i, q_m)$$

$$- D\log(\prod_{\substack{i\neq m\\ m=0}}^{6} \triangle(q_i, q_m)) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2 \otimes \prod_{\substack{j\neq m\\ m=0}}^{6} \triangle(q_j, q_m) \wedge \prod_{\substack{k\neq m\\ m=0}}^{6} \triangle(q_k, q_m)$$

$$+ D\log\Big(\prod_{\substack{j\neq m\\ m=0}}^{6}\triangle(q_j,q_m)\Big)\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{k\neq m\\ m=0}}^{6}\triangle(q_k,q_m)\wedge\prod_{\substack{i\neq m\\ m=0}}^{6}\triangle(q_i,q_m)$$

$$- D\log\Big(\prod_{\substack{k\neq m\\ m=0}}^{6}\triangle(q_k,q_m)\Big)\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{i\neq m\\ m=0}}^{6}\triangle(q_i,q_m)\wedge\prod_{\substack{j\neq m\\ m=0}}^{6}\triangle(q_j,q_m)\Big)\Big]$$

Lemma 5.1. $h_0^5\circ d = h_1^5\circ d = 0$

Proof. For Proof see [11]

Lemma 5.2. $\partial^D\circ h_1^5 = h_0^5\circ p$

Proof. For Proof see [11]

5.3. **Extension in Weight 5.** Introduce 3 new morphisms h_2^5, h_3^5 and h_4^5 to extend the chain Complex J as

$$\begin{array}{ccccc}
G_{11}(5) & \xrightarrow{d} & G_{10}(5) & \xrightarrow{h_4^5} & \beta_5^D(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_{10}(4) & \xrightarrow{d} & G_9(4) & \xrightarrow{h_3^5} & \beta_4^D(F)\otimes F^\times\oplus F\otimes\mathcal{B}_4(F) \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_9(3) & \xrightarrow{d} & G_8(3) & \xrightarrow{h_2^5} & \beta_3^D(F)\otimes\wedge^2F^\times\oplus F\otimes\mathcal{B}_3(F)\otimes F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_8(2) & \xrightarrow{d} & G_7(2) & \xrightarrow{h_1^5} & \beta_2^D(F)\otimes\wedge^3F^\times\oplus F\otimes\mathcal{B}_2(F)\otimes\wedge^2F^\times \\
\downarrow p & & \downarrow p & & \downarrow \partial^D \\
G_7(1) & \xrightarrow{d} & G_6(1) & \xrightarrow{h_0^5} & F\otimes\wedge^4F^\times
\end{array} \tag{K}$$

where

$$h_2^5(q_0,...,q_7) = +\frac{1}{45}\sum_{i=0}^{7}(-1)^i\Big[[\![r(q_0,...,\hat{q}_i,,\hat{q}_j,...,q_7)]\!]_3^D\otimes\prod_{\substack{j\neq m\\ m=0}}^{7}\triangle(q_j,q_m)\wedge\prod_{\substack{i\neq m\\ m=0}}^{7}\triangle(q_i,q_m)-$$

$$D\log\Big(\prod_{\substack{i\neq m\\ m=0}}^{7}\triangle(q_i,q_m)\Big)\otimes[r(q_0,...,\hat{q}_i,,\hat{q}_j,...,q_7)]_3\wedge\prod_{\substack{j\neq m\\ m=0}}^{7}\triangle(q_j,q_m)+$$

$$D\log\Big(\prod_{\substack{j\neq m\\ m=0}}^{7}\triangle(q_j,q_m)\Big)\otimes[r(q_0,...,\hat{q}_i,,\hat{q}_j,...,q_7)]_3\wedge\prod_{\substack{i\neq m\\ m=0}}^{7}\triangle(q_i,q_m)\Big]$$

$$h_3^5(q_0,...,q_8) = -\frac{1}{105}\sum_{i=0}^{8}(-1)^i\Big[[\![r(q_0,...,\hat{q}_i,...,q_8)]\!]_4^D \otimes \prod_{\substack{i\neq m\\ m=0}}^{8}\triangle(q_i,q_m) -$$

$$D\log(\prod_{\substack{i\neq m\\ m=0}}^{8}\triangle(q_i,q_m))\otimes[r(q_0,...,\hat{q}_i,...,q_8)]_4\Big]$$

and

$$h_4^5(q_0,...,q_9) = \frac{1}{190}Alt_{10}[\![r(q_0,...,q_9)]\!]_5^D$$

Lemma 5.4. $h_2^5\circ d = 0$

Proof. Let $(q_0,...,q_8)\in G_9(5)$, then

$$d(q_0,...,q_8) = \sum_{i=0}^{8}(-1)^i(q_0,...,\hat{q}_i,...,q_8)$$

now by applying morphism h_2^5

$$h_2^5\circ d(q_0,...,q_8) = \frac{1}{45}\sum_{j=i+1}^{8}(-1)^j\sum_{i=0}^{8}(-1)^i\Big[[\![r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_8)]\!]_3^D\otimes\prod_{\substack{j\neq m\\ m=j+1}}^{8}\triangle(q_j,q_m) -$$

$$D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{8}\triangle(q_j,q_m))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_8)]_3\Big] \qquad (5.\ 13)$$

In Eq. (5. 13) the summand $\sum[\![r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_8)]\!]_3^D\in\beta_3^D(F)$ and $\sum[r(q_0,\ldots,\hat{q}_i,\hat{q}_j,..., q_8)]_3\in\mathcal{B}_3(F)$ are functional equations of these polylog groups. Therefore $h_2^5\circ d = 0$

Lemma 5.5. $h_3^5\circ d = 0$

Proof. Let $(q_0,...,q_9)\in G_{10}(5)$, then apply morphism d

$$d(q_0,...,q_9) = \sum_{i=0}^{9}(-1)^i(q_0,...,\hat{q}_i,...,q_9)$$

now by applying morphism h_3^5

$$h_3^5\circ d(q_0,...,q_9) = -\frac{1}{105}\sum_{j=i+1}^{9}(-1)^j\sum_{i=0}^{9}(-1)^i\Big([\![r(q_0,...,\hat{q}_j,\hat{q}_i,...,q_9)]\!]_4^D\otimes\prod_{\substack{i\neq m\\ m=i+1}}^{9}\triangle(q_i,q_m) -$$

$$D\log(\prod_{\substack{i\neq m\\ m=i+1}}^{9}\triangle(q_i,q_m))\otimes[r(q_0,...,\hat{q}_j,\hat{q}_i,...,q_9)]_4\Big) \qquad (5.\ 14)$$

In Eq. (5. 14) the summand $\sum[\![r(q_0,...,\hat{q}_j,\hat{q}_i,...,q_8)]\!]_4^D\in\beta_4^D(F)$ and $\sum[r(q_0,\ldots,\hat{q}_j,\hat{q}_i,...,q_8)]_3\in\mathcal{B}_4(F)$ are functional equations. Therefore $h_3^5\circ d = 0$

Lemma 5.6. $h_4^5\circ d = 0$

Proof. Let $(q_0, ..., q_{10}) \in G_{11}(5)$, then apply morphism d

$$d(q_0, ..., q_{10}) = \sum_{i=0}^{10} (-1)^i (q_0, ..., \hat{q}_i, ..., q_{10})$$

now by applying morphism h_4^5

$$h_4^5 \circ d(q_0, ..., q_{10}) = \frac{1}{190} \sum_{j=0}^{10} (-1)^i Alt_{10} [\![r(q_0, ..., \hat{q}_i, ..., q_{10})]\!]_5^D$$

It is an 11 terms relation functional equation of $\beta_5^D(F)$. Therefore $h_4^5 \circ d = 0$

Theorem 5.7. $h_1^5 \circ p = \partial^D \circ h_2^5$

Proof. Let $(q_0, ..., q_7) \in G_8(3)$, then apply morphism p

$$p(q_0, ..., q_7) = \sum_{i=0}^{7} (-1)^i (q_i | q_0, ..., \hat{q}_i, ..., q_7)$$

now by applying map h_1^5, then

$$h_1^5 \circ p(q_0, ..., q_7) = -\frac{1}{10} \Big[\sum_{\substack{i=0 \\ j=i+1 \\ k=i+2 \\ l=i+3}}^{7} (-1)^i ([\![r(q_i | q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, ..., q_7)]\!]_2^D \otimes \prod_{\substack{l \neq m \\ m=k+1}}^{7} (q_i | q_l, q_m)$$

$$\wedge \prod_{\substack{k \neq m \\ m=k+1}}^{7} \triangle(q_i | q_k, q_m) \wedge \prod_{\substack{j \neq m \\ m=j+1}}^{7} \triangle(q_i | q_j, q_m) -$$

$$D \log(\prod_{\substack{j \neq m \\ m=j+1}}^{7} \triangle(q_i | q_j, q_m)) \otimes [r(q_i | q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_7)]_2$$

$$\wedge \prod_{\substack{k \neq m \\ m=k+1}}^{7} \triangle(q_i | q_k, q_m) +$$

$$D \log(\prod_{\substack{k \neq m \\ m=k+1}}^{7} \triangle(q_i | q_k, q_m) \otimes [q_i | q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_7]_2$$

$$\wedge \prod_{\substack{i \neq m \\ m=i+1}}^{7} \triangle(q_i | q_j, q_m)]$$

$$D \log(\prod_{\substack{l \neq m \\ m=l+1}}^{7} \triangle(q_i | q_l, q_m) \otimes [r(q_i | q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_7)]_2$$

$$\wedge \prod_{\substack{i\neq m\\ m=i+1}}^{7} \triangle(q_i|q_j,q_m))\Big] \tag{5. 15}$$

Assume again $(q_0,...,q_7) \in G_8(3)$ and apply morphism h_2^5

$$h_2^5(q_0,...,q_7) = \frac{1}{45}\Big[\sum_{\substack{i=0\\ j=i+1}}^{7}(-1)^i([\![r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_7)]\!]_3^D \otimes \prod_{\substack{j\neq m\\ m=j+1}}^{7}\triangle(q_j,q_m)\wedge\prod_{\substack{i\neq m\\ m=i+1}}^{7}\triangle(q_i,q_m)-$$

$$D\log(\prod_{\substack{i\neq m\\ m=i+1}}^{7}\triangle(q_i,q_m))\otimes[r(q_0,...,\hat{q}_i,\hat{q}_j,...,q_7)]_3\wedge\prod_{\substack{j\neq m\\ m=j+1}}^{7}\triangle(q_j,q_m)+$$

$$D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{7}\triangle(q_j,q_m)\otimes[r(q_0,...,\hat{q}_j,...,q_7)]_3\wedge\prod_{\substack{i\neq m\\ m=i+1}}^{7}\triangle(q_i,q_m))\Big]$$

after applying morphism ∂^D and all properties of tensor, wedge, odd cycle and Siegel cross ratio, it becomes

$$\partial^D\circ h_2^5(q_0,...,q_7) = -\frac{1}{10}\Big[\sum_{\substack{i=0\\ j=i+1\\ k=i+2\\ l=i+3}}^{7}(-1)^i([\![r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,\hat{q}_l,...,q_7)]\!]_2^D\otimes\prod_{\substack{l\neq m\\ m=l+1}}^{7}(q_i|q_l,q_m)$$

$$\wedge\prod_{\substack{k\neq m\\ m=k+1}}^{7}\triangle(q_i|q_k,q_m)\wedge\prod_{\substack{j\neq m\\ m=j+1}}^{7}\triangle(q_i|q_j,q_m)-$$

$$D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{7}\triangle(q_i|q_j,q_m))\otimes[r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_7)]_2$$

$$\wedge\prod_{\substack{k\neq m\\ m=k+1}}^{7}\triangle(q_i|q_k,q_m)+$$

$$D\log(\prod_{\substack{k\neq m\\ m=k+1}}^{7}\triangle(q_i|q_k,q_m)\otimes[q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_7]_2$$

$$\wedge\prod_{\substack{i\neq m\\ m=i+1}}^{7}\triangle(q_i|q_j,q_m)]$$

$$D\log(\prod_{\substack{l\neq m\\ m=l+1}}^{7}\triangle(q_i|q_l,q_m)\otimes[r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_7)]_2$$

$$\wedge \prod_{\substack{i\neq m\\ m=i+1}}^{7} \triangle(q_i|q_j, q_m))\Big] \tag{5. 16}$$

Using Eqs. (5. 15) and (5. 16) it is concluded that $h_1^5 \circ p = \partial^D \circ h_2^5$

Theorem 5.8. $h_2^5 \circ p = \partial^D \circ h_3^5$

Proof. Let $(q_0, ..., q_8) \in G_9(4)$, then apply morphism p

$$p(q_0, ..., q_8) = \sum_{i=0}^{8} (-1)^i (q_i|q_0, ..., \hat{q}_i, ..., q_8)$$

now by applying map h_2^5, then

$$h_2^5 \circ p(q_0, ...,q_8) = \frac{1}{45}\Big[\sum_{\substack{i=0\\ j=i+1\\ k=i+2}}^{8} (-1)^i ([\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_8)]\!]_3^D \otimes \prod_{\substack{k\neq m\\ m=k+1}}^{8} \triangle(q_i|q_k, q_m)$$

$$\wedge \prod_{\substack{j\neq m\\ m=j+1}}^{8} \triangle(q_i|q_j, q_m) -$$

$$D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{8} \triangle(q_i|q_j, q_m)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_8)]_3 \wedge \prod_{\substack{k\neq m\\ m=k+1}}^{8} \triangle(q_i|q_k, q_m) +$$

$$D\log(\prod_{\substack{k\neq m\\ m=k+1}}^{8} \triangle(q_i|q_k, q_m)) \otimes [r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_8)]_3 \wedge \prod_{\substack{j\neq m\\ m=j+1}}^{8} \triangle(q_i|q_j, q_m))\Big] \tag{5. 17}$$

Assume again $(q_0, ..., q_8) \in G_9(4)$ and apply morphism h_3^5

$$h_3^5(q_0, ..., q_8) = -\frac{1}{105}\Big[\sum_{i=0}^{8} (-1)^i ([\![r(q_i|q_0, ..., \hat{q}_i, ..., q_8)]\!]_4^D \otimes \prod_{\substack{i\neq m\\ m=i+1}}^{8} \triangle(q_i, q_m) -$$

$$D\log(\prod_{\substack{i\neq m\\ m=i+1}}^{8} \triangle(q_i, q_m)) \otimes [r(q_i|q_0, ..., \hat{q}_i, ..., q_8)]_4)\Big]$$

after applying morphism ∂^D and all properties of tensor, wedge, odd cycle and Siegel cross ratio, it becomes

$$\partial^D \circ h_3^5(q_0, ...,q_8) = \frac{1}{45}\Big[\sum_{\substack{i=0\\ j=i+1\\ k=i+2}}^{8} (-1)^i ([\![r(q_i|q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_8)]\!]_3^D \otimes \prod_{\substack{k\neq m\\ m=k+1}}^{8} \triangle(q_i|q_k, q_m) \wedge$$

$$\prod_{\substack{j\neq m\\ m=j+1}}^{8} \triangle(q_i|q_j,q_m)-$$

$$D\log(\prod_{\substack{j\neq m\\ m=j+1}}^{8} \triangle(q_i|q_j,q_m))\otimes[r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\wedge\prod_{\substack{k\neq m\\ m=k+1}}^{8}(q_i|q_k,q_m)+$$

$$D\log(\prod_{\substack{k\neq m\\ m=k+1}}^{8} \triangle(q_i|q_k,q_m))\otimes[r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_8)]_3\wedge\prod_{\substack{j\neq m\\ m=j+1}}^{8}\triangle(q_i|q_j,q_m))\Big] \quad (5.\ 18)$$

Using Eqs. (5. 17) and (5. 18), it is concluded that $h^5_2\circ p=\partial^D\circ h^5_3$

Theorem 5.9. $h^5_3\circ p=\partial^D\circ h^5_4$

Proof. Let $(q_0,...,q_9)\in G_{10}(5)$, then apply morphism p

$$p(q_0,.....,q_9)=\sum_{i=0}^{9}(-1)^i(q_i|q_0,...,\hat{q}_i,...,q_9)$$

now by applying map h^5_3, then

$$h^5_3\circ p(q_0,...,q_9)=-\frac{1}{105}\Big[\sum_{i=0}^{8}(-1)^i([\![q_i|q_0,...,\hat{q}_i,...,q_8]\!]^D_4\otimes\prod_{\substack{i\neq m\\ m=i+1}}^{8}(q_i,q_m)-$$
$$D\log(\prod_{\substack{i\neq m\\ m=i+1}}^{8}(q_i,q_m))\otimes[q_i|q_0,...,\hat{q}_i,...,q_8]_4)\Big] \quad (5.\ 19)$$

Assume again $(q_0,...,q_9)\in G_{10}(5)$ and apply morphism h^5_4

$$h^5_4(q_0,...,q_9)=\frac{1}{190}Alt_{10}[\![r(q_0,...,q_9)]\!]^D_5 \quad (5.\ 20)$$

after applying morphism ∂^D and all properties of tensor, wedge, odd cycle, Siegel cross ratio and simplifying, it becomes

$$\partial^D\circ h^5_4=-\frac{1}{105}\Big[\sum_{i=0}^{8}(-1)^i([\![q_i|q_0,...,\hat{q}_i,...,q_8]\!]^D_4\otimes\prod_{\substack{i\neq m\\ m=i+1}}^{8}(q_i,q_m)-$$
$$D\log(\prod_{\substack{i\neq m\\ m=i+1}}^{8}(q_i,q_m))\otimes[q_i|q_0,...,\hat{q}_i,...,q_8]_4)\Big] \quad (5.\ 21)$$

From Eqs. (5. 19) and (5. 21), it is observed that $h^5_3\circ p=\partial^D\circ h^5_4$

6. Conclusion

This work represents the first time that morphisms are defined to connect Polylogarithmic groups for weight $n > 3$ with free abelian groups of Grassmannian complex. As other researchers have only connected these two complexes up to polylog group of weight $n = 3$ ($\beta_3^D(F)$ and $\mathcal{B}_3(F)$). Since Grassmannian is naturally associated with Polylog groups, this work will have significant value in the geometry of Grassmannian with other form of Polylog groups and their complexes.
The above results will prove to be useful in the future by providing researchers a way to define generalized geometry between the Grassmannian complex and variant of Cathelineau complex. The functional equations and discovered properties of infinitesimal groups will be useful for mathematicians in the research field. These results will surely play a significant role in the fields of Algebraic geometry, K- theory, Homological algebra, and other forms of polylogarithmic group theory.

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