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Numerical Analysis of a Modified SIR Epidemic Model with the Effect of Time Delay

Muhammad Asghar Ali
Department of Mathematics and Statistics,
University of Lahore, Lahore, Pakistan.
Email: muhammadasghar.bss@gmail.com

Muhammad Rafiq
Faculty of Engineering,
University of Central Punjab, Lahore, Pakistan.

Muhammad Ozair Ahmad
Department of Mathematics and Statistics,
University of Lahore, Lahore, Pakistan.

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Abstract. In the study of disease dynamics SIR (Susceptible-Infected-Recovered) models have a great importance. In this paper, a modified SIR epidemic model for the transmission dynamics of an infectious disease with the susceptibility effect, in a human population, has been proposed. Its corresponding SIR time delay epidemic model has also been presented and analysed numerically by developing an unconditionally convergent numerical model i.e. Non Standard Finite Difference (NSFD) Scheme. It has been shown that the proposed discrete model (NSFD scheme) must exhibit the same behaviour as the continuous model and preserves all the essential properties like positivity and boundedness of the solution, stability and dynamical consistency, for all time steps h and delay factors τ. A well-known numerical scheme, RK-4, is used to compare the results, which fails at large time step and/or delay factor. Finally, the effect of time delay on the dynamics of the disease and threshold parameter R_0 has also been presented to prove the importance of delay in the dynamics of a disease and its eradication.

AMS (MOS) Subject Classification Codes: 65P40; 65P99; 37M10
Key Words: modified SIR model; delayed SIR model; RK-4 method; NSFD scheme.

1. Introduction

Mathematical modelling is a useful tool to study the mechanism that how an infectious disease can spread into a population. The future course of an outbreak and measures to

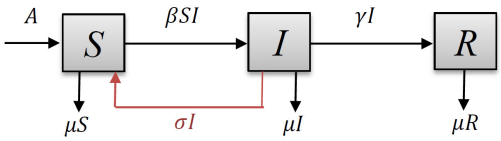


FIGURE 1. Flow diagram of Modified SIR Model with susceptibility effect

control an epidemic can be predicted through modelling [6]. The basic idea of modelling is to study the rapid rise and fall of infected population. This idea was given by W. Kermak and A. McKendrik [11], in 1927. Murray [15], Bailey [2] and Anderson and May [1] have given a detailed history of SIR epidemic models in their respective books.
The transmission dynamics of infectious diseases are governed by non-linear dynamical systems [20, 21, 22, 23, 24]. Due to their non-linear behaviour, it is hard to analyse them analytically. Some semi analytic techniques for epidemic models have been reported in scientific literature [8]. Most of semi analytic and standard numerical methods fail to explain the true behaviour of an infectious disease in a population. These techniques have certain limitations and do not remain consistent with the biological nature of the continuous model. To investigate a precise behaviour of such system, a reliable numerical analysis is needed which preserves all the essential features of the model.
In this paper, a modified SIR epidemic model with the susceptibility effect and delay factor τ has been presented numerically. This model is based on the assumptions proposed by Hethcote [9] in 1976. According to Hethcote;

- the population size is constant and large enough so that we can consider the population of each compartment as a continuous model.
- the birth and death rates are equal.
- the population is homogeneously mixed and uniform.

2. MODIFIED SIR EPIDEMIC MODEL

In our proposed modified SIR epidemic model, susceptibility effect is added. If the immunity is temporary, the infective individuals will return to the susceptible class after an infective period with the rate σ, as used in SIS epidemic model in [7] and delayed SIRS epidemic model in [25]. In the case of getting permanent immunity the infective individuals will be removed from infected compartment with a rate of γI as they will not catch the same infection again as shown in Fig. 1 (given above).

2.1. **Variables and Parameters.** A detailed information about variables and parameters used in above model is as;
S = Susceptible Population
I = Infected Population
R = Recovered Population
A = Rate of Recruitment
μ = Death rate
β = Infection rate for susceptible population

σ = Rate at which infected individuals will become susceptible again
γ = Rate at which infected individuals will become recover

2.2. **System of Differential Equations.** The differential equations obtained from above modified SIR model are;

$$\begin{aligned} \frac{dS}{dt} &= A - \mu S(t) - \beta S(t)I(t) + \sigma I(t) \\ \frac{dI}{dt} &= \beta S(t)I(t) - (\mu + \sigma + \gamma)I(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t) \end{aligned} \tag{2. 1}$$

Without any loss of generality, third equation of above system can be excluded because the first two equations are not dependent on it, therefor system (2. 1) becomes;

$$\begin{aligned} \frac{dS}{dt} &= A - \mu S(t) - \beta S(t)I(t) + \sigma I(t) \\ \frac{dI}{dt} &= \beta S(t)I(t) - (\mu + \sigma + \gamma)I(t) \end{aligned} \tag{2. 2}$$

2.3. **Condition for Epidemic.** Epidemic will exist in the population only if an infected individual appears in it. To analyze the transmission dynamics of any disease, the basic reproductive number or threshold parameter R_0 is very important, because if;
i) $R_0 < 1$, there will be no epidemic or disease will die out.
ii) $R_0 > 1$, epidemic will occur or disease will persist.
For proposed modified SIR model R_0 is;

$$R_0 = \frac{\beta A}{\mu(\mu + \sigma + \gamma)}$$

2.4. **Equilibrium Points of the Model.** There are two equilibrium states of proposed modified SIR epidemic model.

2.4.1. *Disease Free Equilibrium (DFE).* System (2. 2) has a DFE if $R_0 < 1$. This is a stage when there does not exist any infection in the population i.e. $I = 0$. Therefore $E_1 = (S_0, I_0) = (\frac{A}{\mu}, 0)$ is the DFE point.

2.4.2. *Endemic Equilibrium (EE).* System (2. 2) has an EE, if $R_0 > 1$. This is a stage when some disease will persist into the population i.e. $I \neq 0$. Hence $E_2 = (S^*, I^*) = (\frac{A}{\mu R_0}, \frac{A\mu(1-R_0)}{\mu\sigma R_0 - \beta A})$ is the EE point.

2.5. **Stability Analysis of modified SIR Model.** The local stability of the modified SIR model is performed at DFE point i.e. $E_1(\frac{A}{\mu}, 0)$.
The eigenvalues of system (2. 2) are;

$$\lambda_1 = -\mu < 0$$

$$\lambda_2 = \frac{\beta A}{\mu} - (\sigma + \mu + \gamma) < 0, for R_0 < 1$$

As both the eigenalues are less than zero for R_0, therefore $E_1(\frac{A}{\mu}, 0)$ is locally asymptotically stable.

3. Modified SIR Model with the Effect of Time Delay

3.1. **Delay Differential Equation Model.** By introducing the effect of time delay in infected population, the model (2. 1) takes the following form;

$$\begin{aligned}\frac{dS}{dt} &= A - \mu S(t) - \beta S(t)I(t) + \sigma I(t)\\ \frac{dI}{dt} &= \beta S(t-\tau)I(t-\tau)e^{-\mu\tau} - (\mu+\sigma+\gamma)I(t)\\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t)\end{aligned} \tag{3. 3}$$

Here τ is the delay factor or incubation period. It is the time during which an infected individual will become infectious i.e. it can spread the infection further [28, 5]. The incidence rate $\beta S(t)I(t)e^{-\mu\tau}$ appearing in the second equation of system (3. 3) represents the rate at time $t-\tau$ at which susceptible individuals are leaving the susceptible class and entering the infectious class at time *t*. Therefore, the fraction follows from the assumption that the death of individuals is following a linear law given by the term $-\mu\tau$ [10].

3.2. **Condition for Epidemic.** R_{d0} is the basic reproductive number or threshold parameter of the delayed SIR model, and its value is;

$$R_{d0} = \frac{\beta A e^{-\mu\tau}}{\mu(\sigma+\mu+\gamma)}$$

3.2.1. *Equilibrium Points.* Two equilibrium points are;

- DFE at $R_{d0} < 1$ is $E_{d1}(\frac{A}{\mu}, 0)$
- EE at $R_{d0} > 1$ is $E_{d2}(\frac{A}{\mu R_{d0}}, \frac{A\mu(1-R_{d0})}{\mu\sigma R_{d0}-\beta A})$

3.3. **Stability Analysis of Delayed SIR Model.** The local stability of the delayed SIR model is performed at $E_{d1}(\frac{A}{\mu}, 0)$ by considering the first two equations of the system (3. 3). The two eigenvalues are;

$$\begin{aligned}\lambda_{d1} &= -\mu < 0\\ \lambda_{d2} &= \frac{\beta A e^{-\mu\tau}}{\mu} - (\sigma+\mu+\gamma) < 0, for R_{d0} < 1\end{aligned}$$

Since both eigenvalues are less than zero, $E_{d1}(\frac{A}{\mu}, 0)$ is locally asymptotically stable.

4. Numerical Analysis of Delayed SIR Model

For last few decades, a great interest has been observed in treating the delay differential equations numerically. The reason of this interest is the flexibility in the procedures of mathematical modelling and applications in various fields [4, 3, 16]. Numerical modelling includes the study of methods in which the solution of differential equations is approximated. In several cases, numerical modelling yields a series of approximations by iterating the procedure time and again. In a few cases, by using regular methods, it may not be possible or the method can be so long and tedious to work out the precise solution. In many cases the solution of ODE's by using standard numerical methods may not be possible. Some of the methods are Picard's Method, Predictor-Corrector Method, Taylor Series

Method, RK-4 Method, Euler Method etc., but in this work we will only compare our results with standard RK-4 method.

4.1. **Runge-Kutta Method of order four (RK-4 Method).** To develop an explicit RK-4 method it is enough to consider the first two equations of system (3. 3). To construct a numerical scheme of RK-4 method we need to make the following assumptions;
Assumptions: $S(t) \approx S_n, S(t-\tau) \approx S_{n-m}, I(t) \approx I_n, I(t-\tau) \approx I_{n-m}$

$$l_1 = h\{A - \mu S_n - \beta S_n I_n + \sigma I_n\}$$
$$m_1 = h\{\beta S_{n-m} I_{n-m} e^{-\mu\tau} - (\mu + \sigma + \gamma) I_n\}$$

$$l_2 = h\{A - \mu(S_n + \frac{l_1}{2}) - \beta(S_n + \frac{l_1}{2})(I_n + \frac{m_1}{2}) + \sigma(I_n + \frac{m_1}{2})\}$$
$$m_2 = h\{\beta(S_{n-m} + \frac{l_1}{2})(I_{n-m} + \frac{m_1}{2})e^{-\mu\tau} - (\mu + \sigma + \gamma)(I_n + \frac{m_1}{2})\}$$

$$l_3 = h\{A - \mu(S_n + \frac{l_2}{2}) - \beta(S_n + \frac{l_2}{2})(I_n + \frac{m_2}{2}) + \sigma(I_n + \frac{m_2}{2})\}$$
$$m_3 = h\{\beta(S_{n-m} + \frac{l_2}{2})(I_{n-m} + \frac{m_2}{2})e^{-\mu\tau} - (\mu + \sigma + \gamma)(I_n + \frac{m_2}{2})\}$$

$$l_4 = h\{A - \mu(S_n + l_3) - \beta(S_n + l_3)(I_n + m_3) + \sigma(I_n + m_3)\}$$
$$m_4 = h\{\beta(S_{n-m} + l_3)(I_{n-m} + m_3)e^{-\mu\tau} - (\mu + \sigma + \gamma)(I_n + m_3)\}$$

Hence

$$\begin{aligned} l_{n+1} &= S_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\ m_{m+1} &= I_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \end{aligned} \tag{4. 4}$$

4.2. **Non Standard Finite Difference (NSFD) Scheme for Modified SIR Model with Time Delay.** In numerical analysis, NSFD scheme is used to produce a general set of methods to find the numerical solution of ODE's by constructing a discrete model [13]. It was first introduced by R. E. Mickens in 1989. In this scheme the first order time derivative $\dot{f}(t)$ is described as $\frac{df}{dt} = \frac{f(t+h)-f(t)}{\phi(h)} + O(h)$ as $h \to 0$, by using forward difference approximation as in [17, 19, 1, 11] and h is the time step size and should be non-negative [2]. The developed numerical model to solve the system (3) of ODE's must also hold the R.E. Mickens' Law of Conservation [12, 13, 14]. To construct the NSFD scheme there is a need to make the following assumptions in system (3. 3) [19, 18, 26, 27].
(1) For first equation; let

$$\frac{dS}{dt} = \frac{S^{n+1} - S^n}{\phi(h)}, S(t) \approx S^{n+1}, S(t)I(t) \approx S^{n+1}I^n, I(t) \approx I^n$$

(2) For second equation; let

$$\frac{dI}{dt} = \frac{I^{n+1} - I^n}{\phi(h)}, S(t-\tau)I(t-\tau) \approx S^{n-m+1}I^{n-m}, I(t) \approx I^{n+1}$$

By using above assumptions, the first two equations of system (3) can be simplified as follows;

$$\begin{aligned}
\frac{S^{n+1}-S^n}{\phi(h)} &= A-\mu S^{n+1}-\beta S^{n+1}I^n+\sigma I^n \\
S^{n+1}-S^n &= \phi(h)\{A-\mu S^{n+1}-\beta S^{n+1}I^n+\sigma I^n\} \\
S^{n+1}+\phi(h)\{\mu S^{n+1}+\beta S^{n+1}I^n\} &= S^n+\phi(h)(A+\sigma I^n) \\
S^{n+1}\{1+\phi(h)(\mu+\beta I^n)\} &= S^n+\phi(h)(A+\sigma I^n)
\end{aligned}$$

$$S^{n+1}=\frac{S^n+\phi(h)(A+\sigma I^n)}{1+\phi(h)(\mu-\beta I^n)} \tag{4. 5}$$

$$\begin{aligned}
\frac{I^{n+1}-I^n}{\phi(h)} &= \beta S^{n-m+1}I^{n-m}e^{-\mu\tau}-(\mu+\sigma+\gamma)I^{n+1} \\
I^{n+1}-I^n &= \phi(h)\{\beta S^{n-m+1}I^{n-m}e^{-\mu\tau}-(\mu+\sigma+\gamma)I^{n+1}\} \\
I^{n+1}+\phi(h)(\mu+\sigma+\gamma)I^{n+1} &= I^n+\phi(h)\beta S^{n-m+1}I^{n-m}e^{-\mu\tau} \\
I^{n+1}\{1+\phi(h)(\mu+\sigma+\gamma)\} &= I^n+\phi(h)\beta S^{n-m+1}I^{n-m}e^{-\mu\tau}
\end{aligned}$$

$$I^{n+1}=\frac{I^n+\phi(h)\beta S^{n-m+1}I^{n-m}e^{-\mu\tau}}{1+\phi(h)(\mu+\sigma+\gamma)} \tag{4. 6}$$

4.2.1. *Stability Analysis of NSFD Scheme.* Stability analysis of the proposed NSFD scheme of the delayed SIR model is performed at disease free equilibrium point (DFE), $E_{d1}(\frac{A}{\mu},0)$. By using equation (4. 5) and (4. 6), the Jacobian matrix is;

$$J=\begin{pmatrix} f_S & f_I \\ g_S & g_I \end{pmatrix}$$

$$J=\begin{pmatrix} \frac{1}{1+\mu\phi(h)} & \frac{\sigma\phi(h)(1+\mu\phi(h))-\beta A(1/\mu+\phi(h))\phi(h)}{(1+\mu\phi(h))^2} \\ 0 & \frac{\mu+\phi(h)\beta Ae^{-\mu\tau}}{\mu(1+\phi(h)(\mu+\sigma+\gamma))} \end{pmatrix}$$

From above Jacobian matrix we obtain the following eigenvalues,

$$\chi_1=\frac{1}{1+\mu\phi(h)}$$

and

$$\chi_2=\frac{\mu+\phi(h)\beta Ae^{-\mu\tau}}{\mu(1+\phi(h)(\mu+\sigma+\gamma))}$$

TABLE 1. Table of parameter values

Parameters	Values
σ	0.1
A	0.95
μ	0.05
β	0.01(DFE), 0.1(EE)
γ	0.5
τ	≥ 0

Clearly $\chi_1 < 1$, but we need to prove it for χ_2. Let

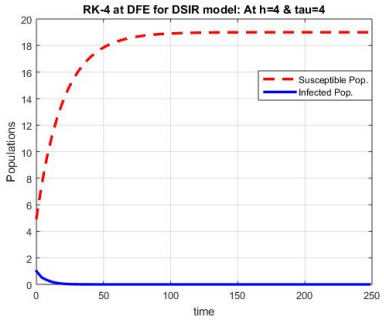
$$\begin{aligned} \chi_2 &< 1 \\ \frac{\mu + \phi(h)\beta A e^{-\mu\tau}}{\mu(1 + \phi(h)(\mu + \sigma + \gamma))} &< 1 \\ \mu + \phi(h)\beta A e^{-\mu\tau} &< \mu(1 + \phi(h)(\mu + \sigma + \gamma)) \\ \beta A e^{-\mu\tau} &< \mu(\mu + \sigma + \gamma) \\ \frac{\beta A e^{-\mu\tau}}{\mu(\mu + \sigma + \gamma)} &< 1 \\ \Rightarrow R_{d0} &< 1 \end{aligned}$$

Whereas, $R_{d0} = \frac{\beta A e^{-\mu\tau}}{\mu(\mu+\sigma+\gamma)}$. Hence the proposed NSFD scheme converges to DFE for any arbitrary value of time step size h whenever $R_{d0} < 1$.

5. NUMERICAL RESULTS AND DISCUSSION

The parameter values given in the following table (1) is used to execute the numerical experiments [10];

5.1. **RK-4 Method.** In this section RK-4 method is used to analyze the dynamics of susceptible and infected populations of delayed SIR model, The experiments are performed for both DFE point and EE point by selecting the different values of time step h and delay parameter τ. In figure 2 and 3, the graphs of susceptible and infected populations for DFE point are plotted by taking $h = \tau = 4$ and $h = \tau = 5.2$ respectively. In both graphs the method is converging to respective true steady states of the continuous model for small time step sizes like $h = 4$ and $h = 5.2$. In figure 3 it is observed that there is a significant increase in the infected population before the disease dies out. This behavior of infected population is not consistent with the biological nature of the continuous model and is due to scheme dependent instability of RK-4 method. In figure 4, $h = \tau = 5.3$ is taken to plot the graphs at DFE point for both the populations. The method is failed as it is showing a diverging behaviour. In figure 5 and 6, susceptible and infected populations are plotted for EE point at $h = 1.5$, $\tau = 6$ and $h = 1.7$, $\tau = 6.8$ respectively. In figure 5, the method is converging to the true steady state of EE point for a small step size as $h = 1.5$ but in figure 6, the method is losing its positivity and showing non-physical oscillations about its true steady state. In figure 7, for EE point at $h = 1.8$ and $\tau = 7.2$, the method is failed



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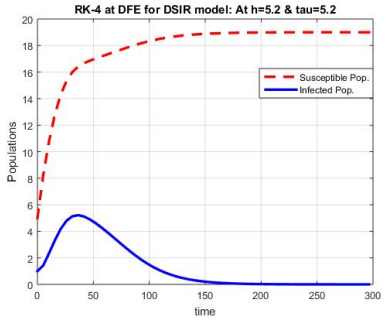

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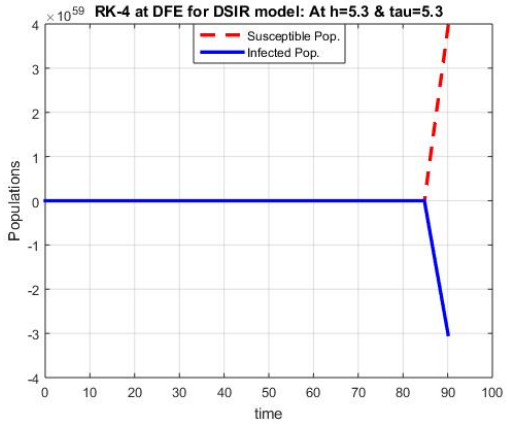

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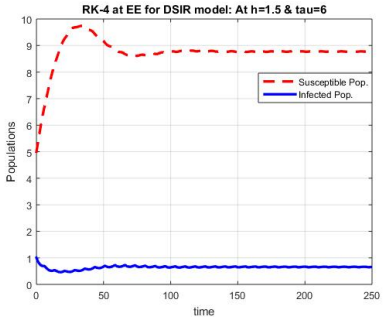

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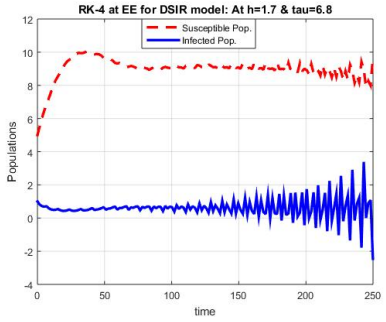

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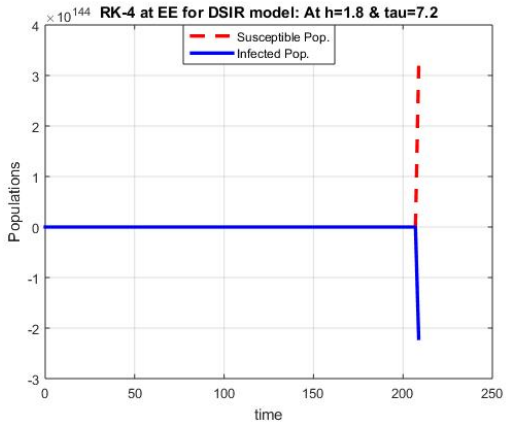

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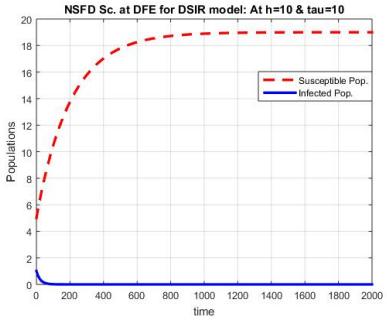

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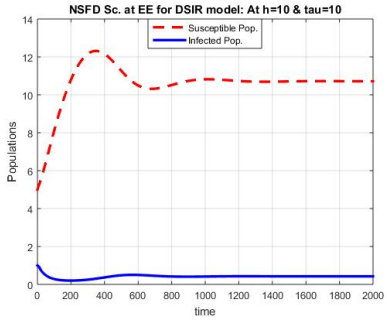
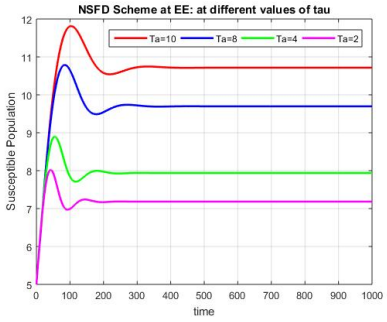


FIGURE 9

to converge the true steady state. Hence, figures 4 and 7 have proved that RK-4 method is highly dependent on time step size h and may diverge after a certain value of time step size h.

5.2. **NSFD Scheme.** In figure 8 and 9, susceptible and infected populations are plotted for both equilibria at a large step size $h = 10$ and $\tau = 10$. It can be clearly seen that the proposed NSFD method is converging to the true steady states. It is independent of time step size and allows us to make a better study of the dynamics of any disease at any time step. So, the NSFD method is more reliable and accurate in producing results and following the biological nature of the epidemic model.

5.3. **Effect of Delay (τ) by using NSFD Scheme.** In figure 10 and 11, the graphs of susceptible and infected populations is plotted for EE point at $h = 2$ and $\tau = 2, 4, 8, 10$. During these numerical experiments, it has been observed that the increase in the delay factor is causing an increase in the susceptible individuals by decreasing the infected individuals.



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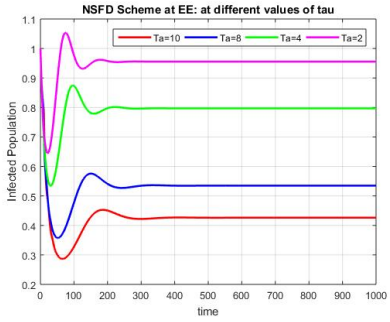

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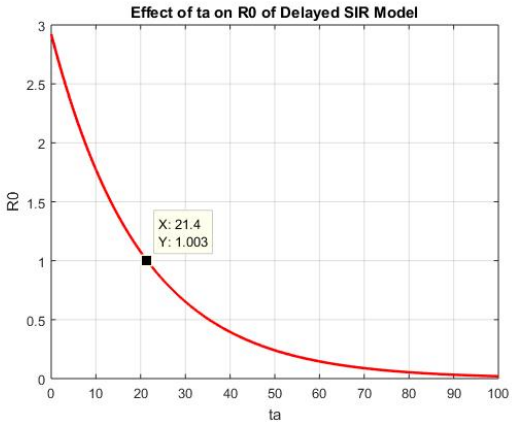

FIGURE 12

5.4. **Effect of τ on R_{d0} of Delayed SIR Model.** In the end, in figure 12, the comparison of R_{d0} and τ reveals that an increase in the value of τ can change an endemic equilibrium state into disease free-state, which is quite realistic that a delay in spread of any infectious disease can help in its cure or eradication. This delay can be made through awareness, medication or by taking some precautionary measures.

6. CONCLUSION

From above discussion, it can be revealed that RK-4 is conditionally convergent and may diverge for certain values of discretization parameter h. So we can conclude that the proposed NSFD scheme remains convergent for all step sizes and preserves all the essential properties of the continuous model.

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