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Stability of Bright Solitons in Bose-Einstein Condensate

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Abstract. We consider a system of effectively one dimensional cigar shaped Bose Einstein condensate. We present a numerical study for the existence and stability of travelling bright solitons in the time dependent nonlinear Schrödinger equation that describes the dynamics of Bose-Einstein condensate. Furthermore, the effects of strength of the trap on the stability of bright solitons are investigated when the condensate is placed under the influence of a trapping potential.

AMS (MOS) Subject Classification Codes: 03.75.Mn; 03.75.Lm; 74.50.+r
Key Words: Solitons, Bose-Einstein condensate, Quantum states, Stability.

1. Introduction

Bose-Einstein Condensate (BEC) is a captivating phase of matter in which atoms of bosons are cooled to a very low temperature near absolute zero so that they amalgamate to a lowest energy state and can be narrated by a macroscopic wave function [18, 16]. In 1924, BEC was first predicted theoretically by Albert Einstein and Satyendra Nath Bose and was experimentally created by Cornell and Wieman for rubidium atoms at Joint Institute for Laboratory Astrophysics (JILA) in 1995 [1]. After this, Ketterle formed BEC by using sodium atoms at Massachusetts Institute of Technology (MIT) [5].

The ground state and macroscopic nonlinear excitations of BEC can be accurately described by the nonlinear Schrödinger equation (NLSE) [11, 29]. NLSE is used to describe the dynamical properties of BEC at adequately low temperature [12]. In 1972, Aleksey Shabat and Vladimir Zakharov accomplished solitons as a solution of NLSE [33, 34].

Solitons are formed by nonlinear interaction of wave packets which are caused by dispersion and diffraction [6]. The most well-known and captivating hallmark of this wave packet is that it can cover long distances without any distortion [23]. Some examples relevant to this localized wave packets are plasma waves, sound waves and water waves. The first interpretation of the appearance of soliton was given by an engineer J. S. Russel from scotland while perceiving the motion of a boat at the Union canal, Scotland in 1834 and called it wave of translation [24]. Later, in 1965, Zabusky and Kruskal called these waves of translation as soliton while discussing solitary waves.

Solitary wave solutions are considered to be an important part in physics [26]. In quantum pair ion plasmas that consists of negative and positive ions along with immobile charged dust particles, the small amplitude dust ion acoustic waves can be described by the cylindrical Kadomtsev-Petriashvili (CKP) equation. This equation admits single and multiple soliton solutions which were obtained analytically in [31]. In mathematics, several methods have been employed to obtain the soliton solutions. The Backlund transformation of Ricatii equation, the trial solution method and the generalized Kudryashov methods were used to obtain the analytical solutions of the NLSE having non-kerr law nonlinearity [2]. Using exp-function method, the solitary wave solution of nonlinear fractional Cahn-Allen equation was obtained [30]. The finite difference methods [19] were used to obtain the solitary wave solutions of Korteweg de-Varies (KdV) equation [4, 14]. The concept of artificial boundaries was introduced to control the Tsunami like solitary waves [32].

There exist different types of soliton solutions and the most studied soliton solutions are dark and bright solitons. Both dark and bright solitons are eminent in experiments of BEC and optical fibers. Dark solitons are also known as kink states and emerge in BEC when the interaction between atoms is repulsive. In a BEC, the dynamics and stability dark soliton solutions were discussed in [9, 15] and their concept of creation in [7, 25]. Dark solitons show a behavior of particle like objects in one dimension [9], but in higher dimension they show snake instability [15]. Solitons were forced to circulate in one-dimensional potential created by a beam of laser. The experimental and theoretical studies on vortices in BEC were presented in [10]. The stability and existence of Josephson vortices [20] and their bound states were discussed in [21, 22].

Bright solitons can be characterized as a non-dispersive and non-spreading localized wave packets, which are emerged in BEC with attractive interaction of atoms. From experimental point of view, bright solitons are formed themselves as a condensate [13, 27, 28]. On the other hand dark solitons exist as holes and notches in the condensate [3]. Bright solitons propagate over large distances than dark solitons. The production of bright matter wave solitons were experimentally observed by Khaykovich et al. in rubidium-87 [13] and Strecker et al. in lithium-7 [27]. The transformation from repulsive to attractive interactions in BEC of lithium-7 formed soliton by using a Feshbach resonance.

In this paper, we numerically investigate the existence and stability of bright soliton solutions in a one dimensional BEC. The property of translational invariance motivated us to study the existence and stability of travelling solitons in the same context. Finally, we study the trapping strength variation and its effects over the stability of bright solitons.

2. Mathematical Model and Methods

BEC is a physical phenomenon that occurs at a very low temperature near absolute zero. At this temperature, the particles of BEC are at the lowest quantum state and these same state particles can be represented by a wave function. The dynamics of these particles of BEC can be described by NLSE. The dimensionless form of NLSE is given as

$$i\varphi_t = -\frac{1}{2}\varphi_{xx} - \varkappa|\varphi|^2\varphi + \c{E}\varphi, \tag{2. 1}$$

where t and x represent, respectively, the temporal and spatial variables. φ represents the wave function of atoms of BEC. The parameter $\varkappa$ represents the nonlinearity coefficient and the interaction among the atoms of BEC is considered to be attractive i.e. without loss of generality, we can consider $\varkappa = 1$ [8]. $\c{E}$ is the external magnetic potential which is produced due to the intrinsic moment of particles. In a magnetic potential, magnetic fields are used to trap the neutral particles with electric current. Although, such magnetic traps have been used in physics to trap microchip atoms and cold atoms, they have been best recognized as the last phase in cooling atoms to accomplish BEC. $\c{E}$ can be given as

$$\c{E} = \frac{1}{2}\breve{e}^2x^2, \tag{2. 2}$$

where $\breve{e}$ shows the strength of external magnetic potential. We substitute $\varphi = \tilde{\varphi}e^{i\omega t}$ (where ω is the chemical potential) in equation (2. 1) to obtain

$$i\tilde{\varphi}_t = -\frac{1}{2}\tilde{\varphi}_{xx} - \varkappa|\tilde{\varphi}|^2\tilde{\varphi} + \omega\tilde{\varphi} + \c{E}\tilde{\varphi}. \tag{2. 3}$$

We first obtain the bright soliton solution when there is no external magnetic trap, i.e. $\c{E} = 0$, so that equation (2. 3) becomes

$$i\tilde{\varphi}_t = -\frac{1}{2}\tilde{\varphi}_{xx} - \varkappa|\tilde{\varphi}|^2\tilde{\varphi} + \omega\tilde{\varphi}. \tag{2. 4}$$

For time-independent solution, we substitute $\tilde{\varphi}_t = 0$ in equation (2. 4) to obtain

$$\frac{1}{2}\tilde{\varphi}_{xx} + \varkappa|\tilde{\varphi}|^2\tilde{\varphi} - \omega\tilde{\varphi} = 0. \tag{2. 5}$$

As $\tilde{\varphi}$ is complex, we substitute $\tilde{\varphi} = \zeta + i\gamma$ in the above equation. Then equating the real and imaginary parts on both sides and discretizing the resulting equations by applying central difference formula, we get

$$\frac{1}{2}\left(\frac{\zeta_{i+1} - 2\zeta_i + \zeta_{i-1}}{h^2}\right) + \varkappa(\zeta_i^3 + \zeta_i\gamma_i^2) - \omega\zeta_i = 0, \tag{2. 6}$$

$$\frac{1}{2}\left(\frac{\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}}{h^2}\right) + \varkappa(\gamma_i^3 + \zeta_i^2\gamma_i) - \omega\gamma_i = 0, \tag{2. 7}$$

where $i = 1, 2, 3, ..., N$. Equations (2. 6) and (2. 7) represent a nonlinear algebraic system. This system can be solved using Newton's method and obtains a bright soliton solution which is depicted in Fig. 1. It is observed that the density of atoms in the center of the condensate decreases with ω as shown in Fig. 2

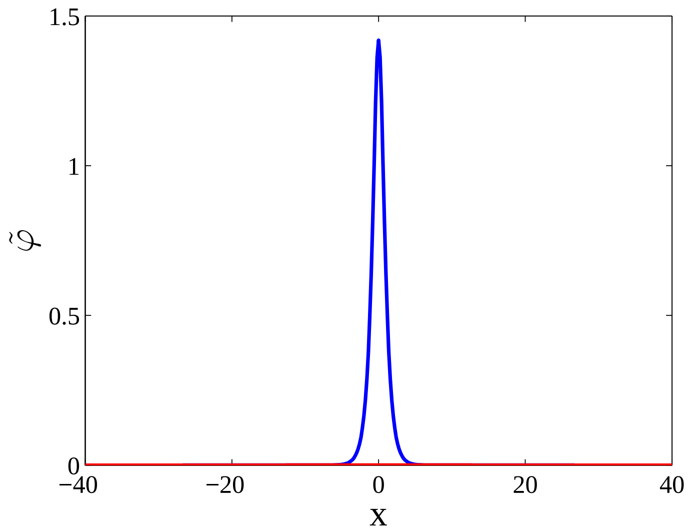

FIGURE 1. The density profile of the bright soliton for $\varkappa = 1$, $\omega = 1$. The curve displays the real part and the red horizontal line shows the imaginary part of the solution.

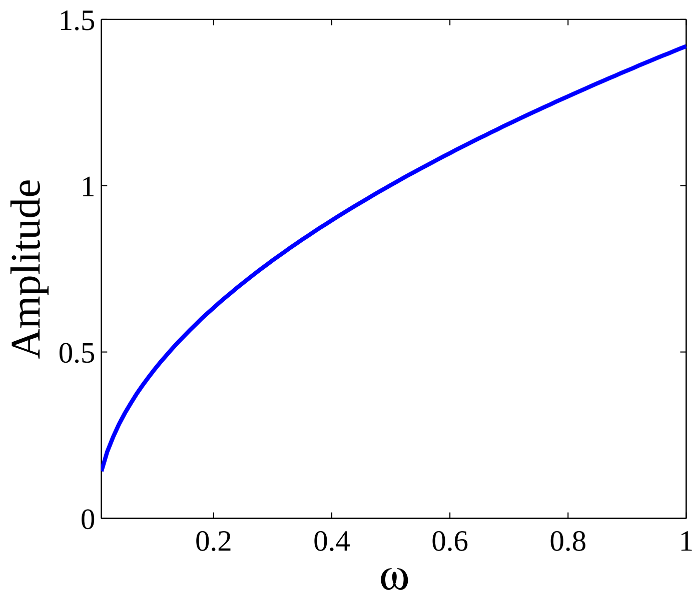

FIGURE 2. Plot of ω versus the amplitude of the bright soliton solution.

3. STABILITY OF BRIGHT SOLITONS

Now we investigate the stability of the bright soliton. We assume that $\hat{\varphi}$ be the time independent solution of equation (2. 4) and $p(x,t)$ be the perturbation added to that steady state solution. This perturbation is supposed to be too small that its squares, cubes and higher powers can be excluded,and we can write

$$\tilde{\varphi} = \hat{\varphi} + p. \tag{3. 8}$$

We insert this value in equation (2. 4) and linearized the resulting equation to obtain

$$-ip_t = \frac{1}{2}p_{xx} + \varkappa\bar{p}\hat{\varphi}^2 + 2\varkappa p|\hat{\varphi}|^2 - \omega p, \tag{3. 9}$$

where bar represents the complex conjugate. Taking complex conjugate of equation (3. 9) and substituting $p = ç$, $\bar{p} = ḑ$, we obtain

$$iç_t = -\frac{1}{2}ç_{xx} - \varkappa ḑ\hat{\varphi}^2 - 2\varkappa ç|\hat{\varphi}|^2 + \omega ç = \lambda ç, \tag{3. 10}$$

$$iḑ_t = \frac{1}{2}ḑ_{xx} + \varkappa ç(\bar{\hat{\varphi}})^2 + 2\varkappa ḑ|\hat{\varphi}|^2 - \omega ḑ = \lambda ḑ. \tag{3. 11}$$

where λ represents the eigenvalue corresponding to eigenvector $X = [ç, ḑ]^t$. Equations (3. 10) and (3. 11) after discretization can be written as an eigenvalue problem $MX = \lambda X$ where

$$M = \begin{bmatrix} M_1 & M_2 \\ \\ M_3 & M_4 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} \frac{1}{2h^2} - 2\varkappa|\hat{\varphi}|^2 + \omega & -\frac{1}{2h^2} & 0 & \cdots & 0 \\ -\frac{1}{2h^2} & \frac{1}{h^2} - 2\varkappa|\hat{\varphi}|^2 + \omega & -\frac{1}{2h^2} & \cdots & 0 \\ 0 & -\frac{1}{2h^2} & \frac{1}{h^2} - 2\varkappa|\hat{\varphi}|^2 + \omega \cdots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2h^2} - 2\varkappa|\hat{\varphi}|^2 + \omega \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -\varkappa(\hat{\varphi})^2 & 0 & 0 & \cdots & 0 \\ 0 & -\varkappa(\hat{\varphi})^2 & 0 & \cdots & 0 \\ 0 & 0 & -\varkappa(\hat{\varphi})^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\varkappa(\hat{\varphi})^2 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} \varkappa(\bar{\hat{\varphi}})^2 & 0 & 0 & \cdot & 0 \\ 0 & \varkappa(\bar{\hat{\varphi}})^2 & 0 & \cdots & 0 \\ 0 & 0 & \varkappa(\bar{\hat{\varphi}})^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varkappa(\bar{\hat{\varphi}})^2 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} -\frac{1}{2h^2} + 2\varkappa|\varphi|^2 - \omega & \frac{1}{2h^2} & 0 & \cdots & 0 \\ \frac{1}{2h^2} & -\frac{1}{h^2} + 2\varkappa|\varphi|^2 - \omega & \frac{1}{2h^2} & \cdots & 0 \\ 0 & \frac{1}{2h^2} & -\frac{1}{h^2} + 2\varkappa|\hat{\varphi}|^2 - \omega \cdots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots - \frac{1}{2h^2} + 2\varkappa|\hat{\varphi}|^2 - \omega & \end{bmatrix}.$$

When all the eigenvalues are real, the solution will be stable. But, the solution will be unstable, when at least a single eigenvalue is imaginary. The eigenvalues of matrix M are evaluated and are shown in Fig. 3. It is easy to notice that all the eigenvalues are laying horizontally, which shows the stability of bright soliton solution.

4. TRAVELLING BRIGHT SOLITON

The bright soliton solution of equation (2. 4) is translationally invariant. However, the property of translational invariance for stationary bright soliton solution is not guaranteed by the existence of travelling bright solitons, this is a prerequisite for bright soliton mobility [17, 8]. A suitable parametrization for the travelling solution is represented by the conversion of variables as

$$\eta = x - \upsilon\tau \tag{4. 12}$$

where υ represents the velocity of travelling bright soliton solution. Substituting this value into equation (2. 4), the time dependent NLSE takes the form

$$i\tilde{\varphi}_\tau = -\frac{1}{2}\tilde{\varphi}_{\eta\eta} - \varkappa|\tilde{\varphi}|^2\tilde{\varphi} + \omega\tilde{\varphi} + i\upsilon\tilde{\varphi}_\eta. \tag{4. 13}$$

For the time independent solution, we have $\tilde{\varphi}_\tau = 0$. As $\tilde{\varphi}$ is complex, we substitute $\tilde{\varphi} = \zeta + i\gamma$ and separate the real and imaginary parts to obtain

$$\frac{1}{2}\zeta_{\eta\eta} + \varkappa(\zeta^3 + \zeta\gamma^2) - \omega\zeta + \upsilon\gamma_\eta = 0, \tag{4. 14}$$

$$\frac{1}{2}\gamma_{\eta\eta} + \varkappa(\zeta^2\gamma + \gamma^3) - \omega\gamma - \upsilon\zeta_\eta = 0. \tag{4. 15}$$

Solving the above equations numerically to obtain the travelling bright soliton solution as shown in Fig. 4.

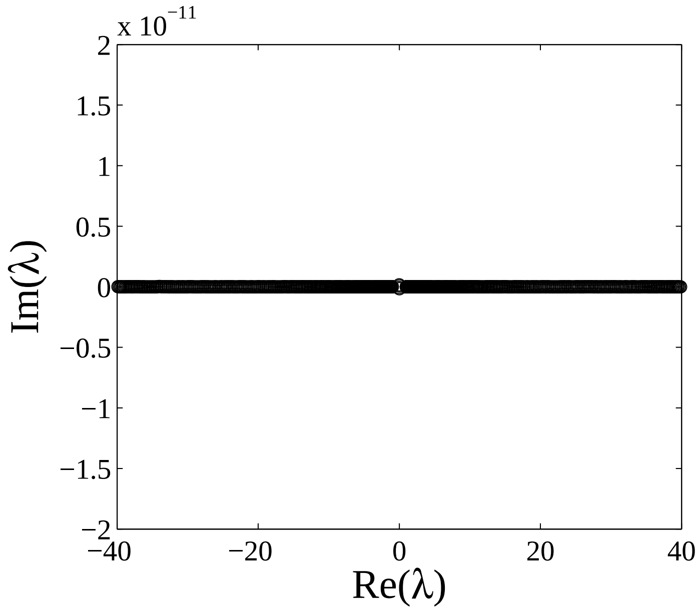

FIGURE 3. The layout of eigenvalues for the bright soliton solution displayed in Fig. 1. All the eigenvalues are structured horizontally and show that the solution is stable.

5. STABILITY OF TRAVELLING BRIGHT SOLITON SOLIUTION

Suppose that $\varphi_\circ$ be the travelling bright soliton solution and $\tilde{\varphi} = \varphi_\circ + p(\eta, \tau)$ be the perturb solution of equation (4. 13), then we obtain

$$-ip_\tau = \frac{1}{2}p_{\eta\eta} + \varkappa\bar{p}\varphi_\circ^2 + 2\varkappa p|\varphi_\circ|^2 - \omega p - i\upsilon p_\eta. \tag{5. 16}$$

We take the complex conjugate of equation (5. 16) and then substitute $p = ç$ and $\bar{p} = ḑ$ to obtain the following equations

$$iç_\tau = -\frac{1}{2}ç_{\eta\eta} - \varkappa ḑ\varphi_\circ^2 - 2\varkappa ç|\varphi_\circ|^2 + \omega ç + i\upsilon ç_\eta = \lambda ç, \tag{5. 17}$$

$$iḑ_\tau = \frac{1}{2}ḑ_{\eta\eta} + \varkappa ç(\bar{\varphi}_\circ)^2 + 2\varkappa ḑ|\varphi_\circ|^2 - \omega ḑ + i\upsilon ḑ_\eta = \lambda ḑ. \tag{5. 18}$$

The above equations after discretization represent an eigenvalue problem that can be expressed in matrix form $LX = \lambda X$, where

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix},$$

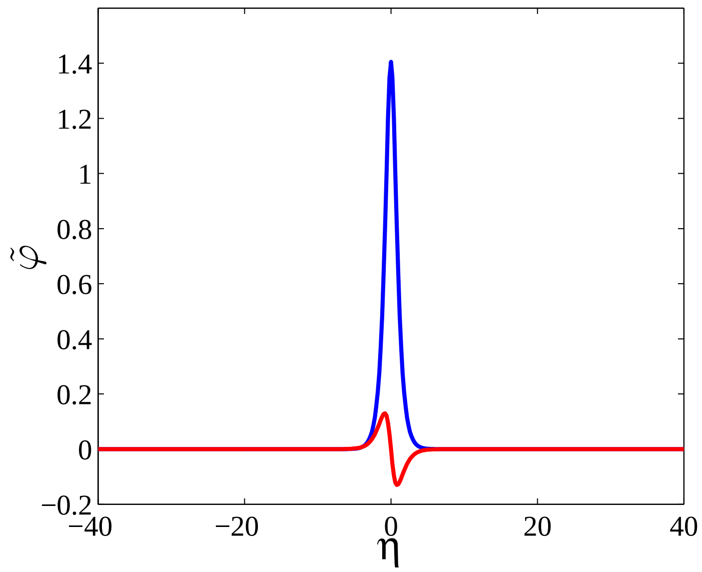

FIGURE 4. Travelling bright soliton for $\upsilon = 0.2$. The blue curve shows the real part and the red curve shows the imaginary part of the travelling solution.

$$L_1 = \begin{bmatrix} \tilde{a_1} - \frac{i\upsilon}{2h} & -\frac{1}{2h^2} + \frac{i\upsilon}{2h} & 0 & \cdots & 0 \\ -\frac{1}{2h^2} - \frac{i\upsilon}{2h} & \frac{1}{h^2} - 2\varkappa|\varphi_\circ|^2 + \omega & -\frac{1}{2h^2} + \frac{i\upsilon}{2h} & \cdots & 0 \\ 0 & -\frac{1}{2h^2} - \frac{i\upsilon}{2h} & \frac{1}{h^2} - 2\varkappa|\varphi_\circ|^2 + \omega & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{a_1} + \frac{i\upsilon}{2h} \end{bmatrix},$$

$$L_4 = \begin{bmatrix} \tilde{a_2} - \frac{i\upsilon}{2h} & \frac{1}{2h^2} + \frac{i\upsilon}{2h} & 0 & \cdots & 0 \\ \frac{1}{2h^2} - \frac{i\upsilon}{2h} & -\frac{1}{h^2} + 2\varkappa|\varphi_\circ|^2 - \omega & \frac{1}{2h^2} + \frac{i\upsilon}{2h} & \cdots & 0 \\ 0 & \frac{1}{2h^2} - \frac{i\upsilon}{2h} & -\frac{1}{h^2} + 2\varkappa|\varphi_\circ|^2 - \omega & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{a_2} + \frac{i\upsilon}{2h} \end{bmatrix},$$

L_2 and L_3 can be obtained from M_2 and M_3 by replacing $\hat{\varphi}$ by $\varphi_\circ$, respectively, and $\tilde{a_1} = \frac{1}{2h^2} - 2\varkappa|\varphi_\circ|^2 + \omega$,

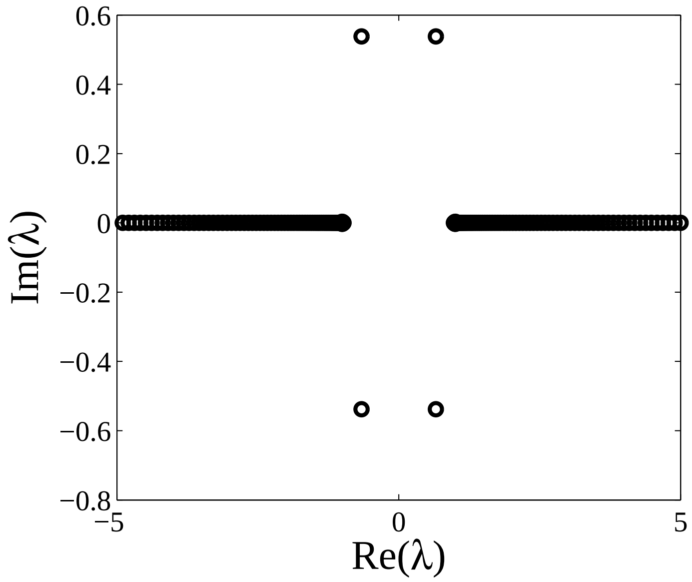

FIGURE 5. The eigenvalues for the travelling bright soliton solution shown in Fig. 4. Some of the eigenvalues lie vertically and depict that the travelling solution is unstable.

$\tilde{a_2} = -\frac{1}{2h^2} + 2\varkappa|\varphi_\circ|^2 - \omega.$

The eigenvalues of matrix L are evaluated and are displayed in Fig. 5. This is easy to note that all eigenvalues are laying horizontally except some of the eigenvalues which are lying vertically. The eigenvalues that are laying vertically illustrate that the travelling bright soliton is unstable. To investigate the stability of travelling bright solitons for distinct values of υ, we plot a graph of velocity υ and the maximum value of imaginary parts of eigenvalues as shown in Fig. 6. It is observed that the travelling bright soliton solution remains unstable for all values of υ.

6. Bright Solitons under the influence of magnetic trap

Let us now study the variation of strength of the trapping potential on the bright soliton solution. For this, we find the steady state solution in the presence of external harmonic trap, so that, equation (2. 2) reduces to

$$\frac{1}{2}\zeta_{xx} + \varkappa(\zeta^3 + \zeta\gamma^2) - \omega\zeta - \text{Ȩ}\zeta = 0, \tag{6. 19}$$

$$\frac{1}{2}\gamma_{xx} + \varkappa(\zeta^2\gamma + \gamma^3) - \omega\gamma - \text{Ȩ}\gamma = 0. \tag{6. 20}$$

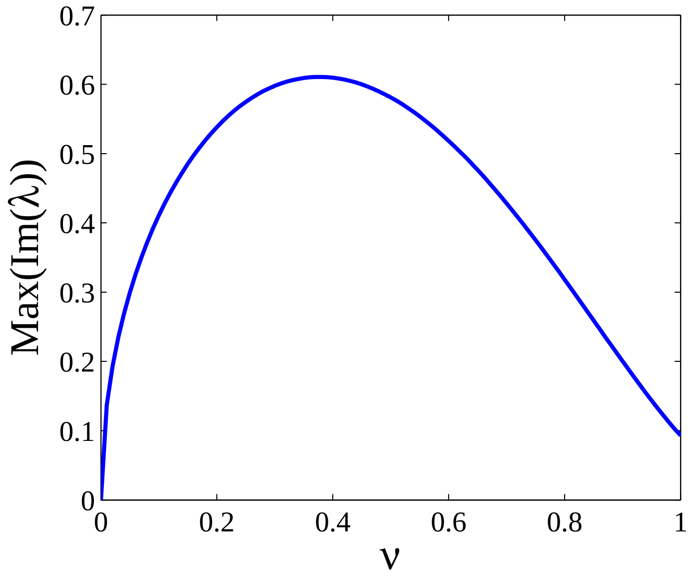

FIGURE 6. The graph of velocity versus $Max(Im(\lambda))$ showing that the solution remains unstable for all values of υ.

The above equations are solved numerically to obtain the trapped bright soliton solutions for distinct values of trapping strength $\breve{e}$. We noticed that the shape of bright soliton remains same as in Fig. 1. This means that the trapping strength does not alter the profile of the solution.

7. STABILITY OF BRIGHT SOLITONS IN THE TRAP

Now, we investigate the stability of bright solitons under the influence of magnetic trap. For this, we substitute $\tilde{\varphi} = \grave{\varphi} + p$ into equation (2. 3) and obtains the following equations

$$i\c{c}_t = -\frac{1}{2}\c{c}_{xx} - \varkappa \c{d}\grave{\varphi}^2 - 2\varkappa \c{c}|\grave{\varphi}|^2 + (\omega + \c{E})\c{c} = \lambda \c{c}, \tag{7. 21}$$

$$i\c{d}_t = \frac{1}{2}\c{d}_{xx} + \varkappa \c{c}(\bar{\grave{\varphi}})^2 + 2\varkappa \c{d}|\grave{\varphi}|^2 - (\omega + \c{E})\c{d} = \lambda \c{d}. \tag{7. 22}$$

These equations define an eigenvalue problem. We calculate the eigenvalues of the coefficient matrix corresponding to a particular value of trapping strength and found that the trapped bright soliton solution is stable. We, then, examine the stability of bright soliton under the influence of magnetic trap for different values of $\breve{e}$. We plot the maximum value of the imaginary parts versus $\breve{e}$ in Fig. 7. It is easy to see that the trapped solution remains stable while increasing the trapping strength. This shows that the magnetic trap does not influence the stability of the bright soliton.

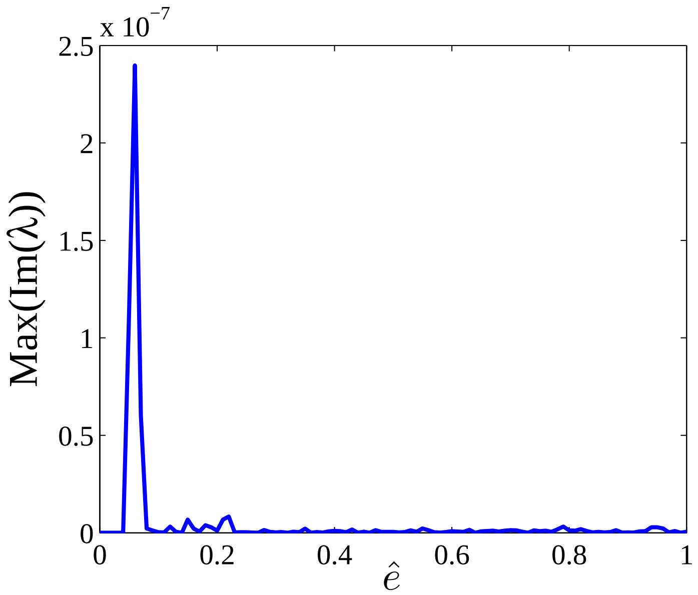

FIGURE 7. The graph of trapping strength versus $Max(Im(\lambda))$ showing the stability of the bright soliton at each value of the strength of trapping potential.

For the justification of the results obtained, we solved equation (2. 3) numerically by employing fourth order Runge-Kutta method. The time evolution of the trapped bright soliton solution for the trapping strength $\breve{e} = 0.1$ is shown in Fig. 8. The contour plot depicts that the solution remains stable under the influence of the magnetic trap and agrees with the results obtained experimentally in [28].

8. Conclusions

In this work, a fascinating state of matter that exists at extremely low temperatures near absolute zero in dilute gases of alkali metals known as BEC was taken into account. This state admits different topological structures such as vortices and solitons. According to the application point of view, atomic soliton lasers could be used in precision measurement such as interferometry [28]. Here, the existence and stability of static as well as travelling bright solitons in nonlinear Schrödinger equation that describes the dynamics of BEC was investigated. It was found that the static bright soliton solution is stable. However, the travelling bright soliton solution remains unstable for different values of velocity υ. This means that motion with certain velocity causes the instability of the bright soliton. We also analyzed the existence and stability of bright soliton solution under the influence of a magnetic trap while changing the strength of trapping potential. It was found that the trapping strength does not affect the stability of bright soliton.

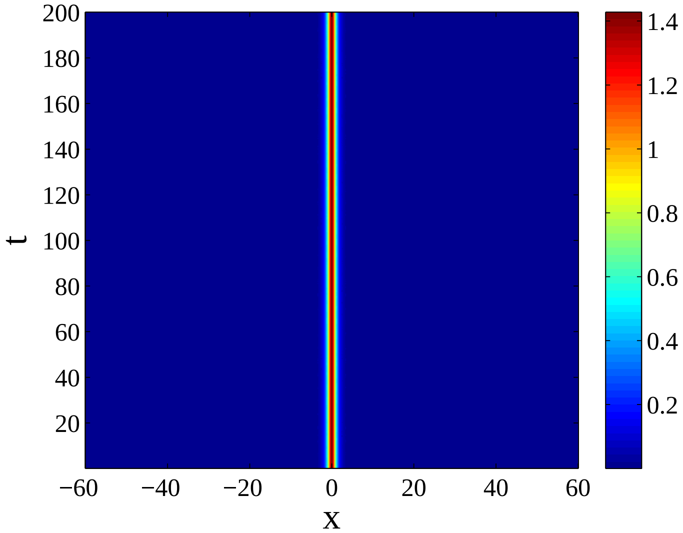

FIGURE 8. The contour plot showing the time dynamics of the trapped bright soliton solution with the trapping strength $\breve{e} = 0.1$.

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