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New Hermite-Hadamard Type Inequalities With Applications

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Abstract. In this manuscript, we have established two new integral identities connected with the left hand side of Hermite-Hadamard (H-H) inequality. By using these identities, we have obtained some new bounds for the Hadamard's type inequalities. We have also presented applications for means and for some error estimates of the mid point formula.

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1. INTRODUCTION

A well-known class of functions defined on the interval I in $\mathbb{R}$, is known to be convex on I if the inequality

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v) \tag{1. 1}$$

holds $\forall u, v \in I$ and $\lambda \in [0,1]$. Moreover, If the inequality in (1. 1) holds in the reverse direction, f is said to be concave function. The geometrical interpretation of convexity is that, if there are any three distinct points R, S and T located on the graph of function f with S lies between R and T, then the point S lies on or below the chord joining the points R and T.

For the class of convex functions, many inequalities have been introduced, when this idea was first introduced than a century ago. But among those the most prominent is so called Hermite-Hadamard's inequality (or (H-H)). For the statement of this inequality is (see for example [15]):

Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function defined on I such that $b_1, b_2 \in I$ with $b_1 < b_2$. Then the inequalities

$$f\left(\frac{b_1+b_2}{2}\right) \leq \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx \leq \frac{f(b_1)+f(b_2)}{2} \tag{1. 2}$$

hold. If the function f is concave on I, then both the inequalities in (1. 2) hold in the reverse direction. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f. These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics.

For more related results, generalizations, improvements and refinements to Hermite-Hadamard inequality see [1–14, 16–30, 32] and the references cited therein.

For simplicity we symbolize the function

$$\Delta := \Delta(f; b_1, b_2) = f\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx, \tag{1. 3}$$

where $f : [b_1, b_2] \to \mathbb{R}$ is an integrable function.

Throughout this paper, we assume that I is an interval in $\mathbb{R}$ and I° is interior of I.
In 1998, Dragomir and Agarwal [10] have proved the following important lemma:

Lemma 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If $f' \in L[b_1, b_2]$, then*

$$\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx = \frac{b_2-b_1}{2}\int_0^1 (1-2z)f'(zb_1+(1-z)b_2)dz \tag{1. 4}$$

holds.

The following two results are the ultimate consequences of Lemma 1, which have been presented in [10] .

Theorem 1. *Let $f : I^{\circ} \to \mathbb{R}$ be a differentiable function such that $f' \in L[b_1, b_2]$, where $b_1, b_2 \in I^{\circ}$ with $b_1 < b_2$. If the function $|f'|$ is convex on $[b_1, b_2]$, then the inequality*

$$\left|\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx\right| \le \frac{(b_2-b_1)(|f'(b_1)|+|f'(b_2)|)}{8} \tag{1. 5}$$

holds.

Theorem 2. *Let $f : I^{\circ} \to \mathbb{R}$ be a differentiable function such that $f' \in L[b_1, b_2]$, where $b_1, b_2 \in I^{\circ}$ with $b_1 < b_2$. If the function $|f'|^{\frac{p}{p-1}}$ for $p > 1$ is convex on $[b_1, b_2]$, then the inequality*

$$\left|\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx\right| \le \frac{b_2-b_1}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(b_1)|^{\frac{p}{p-1}}+|f'(b_2)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}} \tag{1. 6}$$

holds.

In 2000, Pearce and Pecaric [31] proved the following theorem by using Lemma 1.

Theorem 3. *Let $f : I^{\circ} \to \mathbb{R}$ be a differentiable function such that $f' \in L[b_1, b_2]$, where $b_1, b_2 \in I^{\circ}$ with $b_1 < b_2$. If for the function $|f'|^q$ is concave on $[b_1, b_2]$ for $q > 1$, then the inequality*

$$\left|\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx\right| \le \frac{b_2-b_1}{4}\left|f'\left(\frac{b_1+b_2}{2}\right)\right| \tag{1. 7}$$

holds.

Theorem 4. *Let $f : I^{\circ} \to \mathbb{R}$ be a differentiable function such that $f' \in L[b_1, b_2]$, where $b_1, b_2 \in I^{\circ}$ with $b_1 < b_2$. If the function $|f'|^q$ is concave on $[b_1, b_2]$ for $q > 1$, then the inequality*

$$\left|f\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx\right| \le \frac{b_2-b_1}{4}\left|f'\left(\frac{b_1+b_2}{2}\right)\right| \tag{1. 8}$$

holds.

In this paper, we establish two new integral identities connected with the left hand side of (H-H) inequality. By using these identities, we obtain some new bounds for the Hadamard's type inequalities. Our one new bound is better than the earlier obtained bound (see Remark 1). We also present applications for means and for some error estimates of the mid point formula.

2. MAIN RESULTS

To establish our main results connected with the left-hand side of (H-H) inequality for differentiable convex function, we need the following lemma.

Lemma 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and let $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If $f' \in L[b_1, b_2]$, then*

$$\Delta = \frac{b_2 - b_1}{4}\left[\int_0^1 zf'\left(\frac{zb_2}{2} + \frac{2-z}{2}b_1\right)dz - \int_0^1 (1-z)f'\left(\frac{1-z}{2}b_1 + \frac{1+z}{2}b_2\right)dz\right]$$

holds, where Δ is defined as in (1. 3).

Proof. Integration by parts gives that

$$\begin{aligned}
I_1 &= \int_0^1 zf'\left(\frac{zb_2}{2} + \frac{2-z}{2}b_1\right)dz \\
&= \frac{zf\left(\frac{zb_2}{2} + \frac{2-z}{2}b_1\right)}{\frac{b_2-b_1}{2}}\Bigg|_0^1 - \frac{2}{b_2-b_1}\int_0^1 f\left(\frac{zb_2}{2} + \frac{2-z}{2}b_1\right)dz \\
&= \frac{2}{b_2-b_1}f\left(\frac{b_1+b_2}{2}\right) - \frac{2}{b_2-b_1}\int_0^1 f\left(\frac{zb_2}{2} + \frac{2-z}{2}b_1\right)dz.
\end{aligned}$$

By change of variable we have that

$$\begin{aligned}
I_1 &= \frac{2}{b_2-b_1}f\left(\frac{b_1+b_2}{2}\right) - \frac{2}{b_2-b_1}\int_{b_1}^{\frac{b_1+b_2}{2}} f(x)\frac{2}{b_2-b_1}dx \\
&= \frac{2}{b_2-b_1}f\left(\frac{b_1+b_2}{2}\right) - \frac{4}{(b_2-b_1)^2}\int_{b_1}^{\frac{b_1+b_2}{2}} f(x)dx. \qquad (2.\ 9)
\end{aligned}$$

Similarly, we can write

$$\begin{aligned}
I_2 &= \int_0^1 (1-z)f'\left(\frac{1-z}{2}b_1 + \frac{1+z}{2}b_2\right)dz \\
&= -\frac{2}{b_2-b_1}f\left(\frac{b_1+b_2}{2}\right) + \frac{4}{(b_2-b_1)^2}\int_{\frac{b_1+b_2}{2}}^{b_2} f(x)dx. \qquad (2.\ 10)
\end{aligned}$$

Now by subtraction (2. 10) from (2. 9) and then multiplying by $\frac{b_2-b_1}{4}$, we obtain the required result. □

Theorem 5. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function such that $f' \in L[b_1, b_2]$, where $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If the function $|f'|^q$ is concave on $[b_1, b_2]$ for $q > 1$, then the inequality*

$$|\Delta| \leq \frac{b_2-b_1}{8}\left[\left|f'\left(\frac{b_2+2b_1}{3}\right)\right| + \left|f'\left(\frac{b_1+2b_2}{3}\right)\right|\right]$$

holds, where Δ *is defined as in* (1. 3).

Proof. By concavity of $|f'|^q$ and the power mean inequality, we may write

$$|f'(zb_1+(1-z)b_2)|^q \geq z|f'(b_1)|^q+(1-z)|f'(b_2)|^q \geq (z|f'(b_1)|+(1-z)|f'(b_2)|)^q.$$

Since we have

$$|f'(zb_1+(1-z)b_2)| \geq z|f'(b_1)|+(1-z)|f'(b_2)|,$$

$|f'|$ is also concave function. By using triangle inequality and Lemma 2, we have:

$$\begin{aligned}
&\left| f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx\right| \\
&\leq \frac{b_2-b_1}{4}\left[\int_0^1 z\left|f'\left(\frac{zb_2}{2}+\frac{2-z}{2}b_1\right)\right|dz+\int_0^1(1-z)\left|f'\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)\right|dz\right].
\end{aligned} \tag{2. 11}$$

Now by Jensen's integral inequality, we have

$$\begin{aligned}
&\int_0^1 (1-z)|f'\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)|dz \\
&\leq \int_0^1(1-z)dz\left|f'\left(\frac{\int_0^1(1-z)\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)dz}{\int_0^1(1-z)dz}\right)\right|.
\end{aligned} \tag{2. 12}$$

Since $\int_0^1(1-z)dz=\frac{1}{2}$ and $\int_0^1\left(\frac{(1-z)^2}{2}b_1+\frac{1-z^2}{2}b_2\right)dz=\frac{1}{6}b_1+\frac{1}{3}b_2$, (2. 12) turns out to

$$\int_0^1(1-z)\left|f'\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)\right|dz \leq \frac{1}{2}\left|f'\left(\frac{2b_2+b_1}{3}\right)\right|. \tag{2. 13}$$

Similarly, we have

$$\int_0^1 z\left|f'\left(\frac{zb_2}{2}+\frac{2-z}{2}b_1\right)\right|dz \leq \frac{1}{2}\left|f'\left(\frac{b_2+2b_1}{3}\right)\right|. \tag{2. 14}$$

□

By substituting (2. 13) and (2. 14) in (2. 11) we get the required result.

Remark 1. Note that the bound in (2. 11) is better than that in (1. 8). Since $|f'|$ is concave on $[b_1, b_2]$, we have

$$
\begin{aligned}
& \frac{b_2-b_1}{8}\left[\left|f'\left(\frac{b_2+2b_1}{3}\right)\right|+\left|f'\left(\frac{b_1+2b_2}{3}\right)\right|\right] \\
= & \frac{b_2-b_1}{4}\left[\frac{1}{2}\left|f'\left(\frac{b_2+2b_1}{3}\right)\right|+\frac{1}{2}\left|f'\left(\frac{b_1+2b_2}{3}\right)\right|\right] \leq \frac{b_2-b_1}{4}\left|f'\left(\frac{b_1+b_2}{2}\right)\right|.
\end{aligned}
$$

Lemma 3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and let $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If $f'' \in L[b_1, b_2]$, then*

$$
\begin{aligned}
\Delta = \frac{(b_2-b_1)^2}{16}\Bigg[& \int_0^1 (1-z)^2 f''\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)dz \\
& + \int_0^1 z^2 f''\left(\frac{zb_2}{2}+\frac{2-z}{2}b_1\right)dz\Bigg]
\end{aligned}
$$

holds, where Δ is defined as in (1. 3).

Proof. Integration by parts gives that

$$
\begin{aligned}
I_1 &= \int_0^1 (1-z)^2 f''\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)dz \\
&= \left.\frac{(1-z)^2 f'\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)}{\frac{b_2-b_1}{2}}\right|_0^1 + \frac{4}{b_2-b_1}\int_0^1 (1-z) f'\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)dz \\
&= -\frac{2}{b_2-b_1} f'\left(\frac{b_1+b_2}{2}\right) - \frac{8}{(b_2-b_1)^2} f\left(\frac{b_1+b_2}{2}\right) \\
&\quad + \frac{8}{(b_2-b_1)^2}\int_0^1 f\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)dz.
\end{aligned}
$$

By change of variables, we have

$$
I_1 = -\frac{2}{b_2-b_1} f'\left(\frac{b_1+b_2}{2}\right) - \frac{8}{(b_2-b_1)^2} f\left(\frac{b_1+b_2}{2}\right) + \frac{16}{(b_2-b_1)^3}\int_{\frac{b_1+b_2}{2}}^{b_2} f(x)dx. \tag{2. 15}
$$

Similarly, we have:

$$
I_2 = \frac{2}{b_2-b_1} f'\left(\frac{b_1+b_2}{2}\right) - \frac{8}{(b_2-b_1)^2} f\left(\frac{b_1+b_2}{2}\right) + \frac{16}{(b_2-b_1)^3}\int_{b_1}^{\frac{b_1+b_2}{2}} f(x)dx. \tag{2. 16}
$$

Now by adding (2. 15) and (2. 16) and then multiplying by $\frac{(b_2-b_1)^2}{16}$, we obtain the required result.

□

Theorem 6. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function such that $f'' \in L[b_1, b_2]$, where $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If the function $|f''|$ is concave on $[b_1, b_2]$, then the inequality*

$$|\Delta| \leq \frac{(b_2-b_1)^2}{16}\left|f''\left(\frac{7b_2+5b_1}{12}\right)\right|$$

holds, where Δ is defined as in (1. 3).

Proof. By using Lemma 3 and triangle inequality, we have that

$$|\Delta| \leq \frac{(b_2-b_1)^2}{16}\left[\int_0^1 z^2\left|f''\left(\frac{zb_2}{2}+\frac{2-z}{2}b_1\right)\right|dz + \int_0^1 (1-z)^2\left|f''\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)\right|dz\right].$$

As $(1-z)^2 \leq 1-z^2$ for $z \in [0,1]$, we have

$$|\Delta| \leq \frac{(b_2-b_1)^2}{16}\left[\int_0^1 z^2\left|f''\left(\frac{zb_2}{2}+\frac{2-z}{2}b_1\right)\right|dz + \int_0^1 (1-z^2)\left|f''\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)\right|dz\right]. \quad (2.\ 17)$$

Since $|f''|$ is concave, inequality (2. 17) becomes

$$|\Delta| \leq \frac{(b_2-b_1)^2}{16}\left[\int_0^1 \left|f''\Big(\frac{z^3b_2}{2}+\frac{2z^2-z^3}{2}b_1 + \frac{(1-z^2)(1-z)}{2}b_1+\frac{(1-z^2)(1+z)}{2}b_2\Big)\right|dz\right].$$

Now by applying Jensen's inequality, we get

$$|\Delta| \leq \frac{(b_2-b_1)^2}{16}\left[\left|f''\left(\int_0^1 \Big(\frac{z^3b_2}{2}+\frac{2z^2-z^3}{2}b_1 + \frac{(1-z^2)(1-z)}{2}b_1+\frac{(1-z^2)(1+z)}{2}b_2\Big)dz\right)\right|\right]$$

$$= \frac{(b_2-b_1)^2}{16}\left|f''\left(\frac{11}{24}b_2+\frac{5}{24}b_1+\frac{1}{8}b_2+\frac{5}{24}b_1\right)\right| = \frac{(b_2-b_1)^2}{16}\left|f''\left(\frac{7b_2+5b_1}{12}\right)\right|.$$

□

Theorem 7. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function such that $f'' \in L[b_1, b_2]$, where $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If the function $|f''|^q$ is concave on $[b_1, b_2]$ for $q > 1$, then the inequality*

$$|\Delta| \leq \frac{(b_2-b_1)^2}{48}\left[\left|f''\left(\frac{5b_2+3b_1}{8}\right)\right| + \left|f''\left(\frac{3b_2+5b_1}{8}\right)\right|\right]$$

holds, where Δ is defined as in (1. 3).

Proof. By Theorem 5, we have $|f''|$ is concave function. By applying triangle inequality in Lemma 3, we have

$$\begin{aligned} |\Delta| \quad \leq \quad & \frac{(b_2-b_1)^2}{16}\left[\left|\int_0^1 z^2 f''\left(\frac{zb_2}{2}+\frac{2-z}{2}b_1\right)dz\right|\right. \\ & \left.+\left|\int_0^1 (1-z)^2 f''\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)dz\right|\right]. \end{aligned} \tag{2. 18}$$

Now by using Jensen's integral inequality, we have:

$$\begin{aligned} & \int_0^1 (1-z)^2 \left|f''\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)\right|dz \\ \leq \quad & \int_0^1 (1-z)^2 dz \left|\int_0^1 f''\left(\frac{(1-z)^2\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)}{\int_0^1 (1-z)^2 dz}\right)\right|dz. \end{aligned} \tag{2. 19}$$

Since $\int_0^1 (1-z)^2 dz = \frac{1}{3}$ and $\int_0^1 \left(\frac{(1-z)^3}{2}b_1 + \frac{(1-z)^2(1+z)}{2}b_2\right) = \frac{1}{8}b_1 + \frac{5}{24}b_2$, (2. 19) turns out to

$$\int_0^1 (1-z)^2 \left|f''\left(\frac{1-z}{2}b_1+\frac{1+z}{2}b_2\right)\right|dz \leq \frac{1}{3}\left|f''\left(\frac{5b_2+3b_1}{8}\right)\right|. \tag{2. 20}$$

Similarly, we have

$$\int_0^1 z^2 f''\left(\frac{zb_2}{2}+\frac{2-z}{2}b_1\right)dz \leq \frac{1}{3}\left|f''\left(\frac{3b_2+5b_1}{8}\right)\right|. \tag{2. 21}$$

□

By substituting (2. 20) and (2. 21) in (2. 18), we get the required result.

Remark 2. It can be noted that if we use the concavity of $|f''|$, then from (2. 18), we can write as

$$\begin{aligned}|\Delta| &\leq \frac{(b_2-b_1)^2}{24}\left[\frac{1}{2}\left|f''\left(\frac{5b_2+3b_1}{8}\right)\right|+\frac{1}{2}\left|f''\left(\frac{3b_2+5b_1}{8}\right)\right|\right]\\ &\leq \frac{(b_2-b_1)^2}{24}\left|f''\left(\frac{b_1+b_2}{2}\right)\right|,\end{aligned}$$

where Δ is defined as in (1. 3).

3. APPLICATION TO MEANS AND TO MID POINT FORMULA

We will consider the following particular means for any $b_1, b_2 \in \mathbb{R}, b_1 \neq b_2$ which are well-known in the literature, see [10]:

$$\begin{aligned}A(b_1,b_2) &= \frac{b_1+b_2}{2}, \qquad b_1,b_2>0,\\ \bar{L}(b_1,b_2) &= \frac{b_2-b_1}{\ln b_2-\ln b_1}, \qquad b_1,b_2>0,\\ L_n(b_1,b_2) &= \left[\frac{b_2^{n+1}-b_1^{n+1}}{(n+1)(b_2-b_1)}\right]^{\frac{1}{n}}, \qquad b_1<b_2, n\in\mathbb{R}.\end{aligned}$$

Proposition 1. *Let* $0 < b_1 < b_2$, $n \in \mathbb{R}$, *and* $1 < n < 2$. *Then the inequality*

$$|A^n(b_1,b_2)-L_n(b_1,b_2)^n| \leq \frac{|n|(b_2-b_1)}{8}\left(\left|\frac{b_2+2b_1}{3}\right|^{n-1}+\left|\frac{2b_2+b_1}{3}\right|^{n-1}\right)$$

holds.

Proof. Choosing the function $f(s) = s^n, s > 0, 1 < n < 2$ in Theorem 5, the proof can be completed. □

Proposition 2. *Let* $0 < b_1 < b_2$, $n \in \mathbb{R}$, *and* $2 < n < 3$. *Then the inequality*

$$|A^n(b_1,b_2)-L_n(b_1,b_2)^n| \leq \frac{|n(n-1)|(b_2-b_1)^2}{16}\left|\frac{7b_2+5b_1}{12}\right|^{n-2}$$

holds.

Proof. Choosing the function $f(s) = s^n, s > 0, 2 < n < 3$ in Theorem 6, the proof can be completed. □

Proposition 3. *Let* $0 < b_1 < b_2$, $n \in \mathbb{R}$, *and* $2 < n < 3$. *Then the inequality*

$$|A^n(b_1,b_2)-L_n(b_1,b_2)^n| \leq \frac{|n(n-1)|(b_2-b_1)^2}{48}\left(\left|\frac{5b_2+3b_1}{8}\right|^{n-2}+\left|\frac{3b_2+5b_1}{8}\right|^{n-2}\right)$$

holds.

Proof. Choosing the function $f(s) = s^n, s > 0, 2 < n < 3$ in Theorem 7 the proof can be completed. □

Now we give applications to mid point formula as in [10]. We consider that σ as partition of the interval $[b_1, b_2]$ in $\mathbb{R}$, i.e. $\sigma : b_1 = s_0 < s_1 < \cdots < s_{n-1} < s_n = b_2$, and the trapezoidal formula defined as

$$T(f,\sigma) = \sum_{i=0}^{n-1} f\left(\frac{s_i + s_{i+1}}{2}\right)(s_{i+1} - s_i).$$

Let f be a twice differentiable mapping on (b_1, b_2) with

$$M = \max_{t \in (b_1, b_2)} |f''(x)| < \infty,$$

then we have

$$\int_{b_1}^{b_2} f(x)dx = T(f,\sigma) + E(f,\sigma),$$

where $E(f,\sigma)$ is the approximation error of the integral $\int_{b_1}^{b_2} f(x)dx$ and by the trapezoidal formula and $T(f,\sigma)$ satisfies

$$|E(f,\sigma)| \leq \frac{M}{12} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3.$$

Proposition 4. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function such that $f'' \in L[b_1, b_2]$, where $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If the function $|f''|$ is concave on $[b_1, b_2]$, then for every division σ of $[b_1, b_2]$, the inequality*

$$|E(f,\sigma)| \leq \frac{1}{16} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left| f''\left(\frac{5s_i + 7s_{i+1}}{12}\right)\right|$$

holds.

Proof. By applying Theorem 6 to the sub intervals $[s_i, s_{i+1}]$, $i = 0, ..., n-1$, of the division σ, we get

$$\left| f\left(\frac{s_i + s_{i+1}}{2}\right) - \int_{s_i}^{s_{i+1}} f(x)dx \right| \leq \frac{1}{16}(s_{i+1} - s_i)^2 \left| f''\left(\frac{5s_i + 7s_{i+1}}{12}\right)\right|. \quad (3.\ 22)$$

Summing up (3. 22) from 0 to $n-1$ and taking into consideration that $|f'|$ is convex, we have

$$\left| T(f,\sigma) - \int_{b_1}^{b_2} f(x)dx \right| \leq \frac{1}{16} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left| f''\left(\frac{5s_i + 7s_{i+1}}{12}\right)\right|$$

by triangle inequality. □

Proposition 5. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function such that $f' \in L[b_1, b_2]$, where $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If the function $|f'|^q$ is concave on $[b_1, b_2]$ for $q > 1$, then for every division σ of $[b_1, b_2]$, the inequality*

$$|E(f,\sigma)| \leq \frac{1}{8}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^2\left[\left|f'\left(\frac{2s_i+s_{i+1}}{2}\right)\right| + \left|f'\left(\frac{s_i+2s_{i+1}}{2}\right)\right|\right]$$

holds.

Proof. The proof is similar to that of Proposition 4. □

Proposition 6. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function such that $f'' \in L[b_1, b_2]$, where $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If the function $|f''|^q$ is concave on $[b_1, b_2]$ for $q > 1$, then for every division σ of $[b_1, b_2]$, the inequality*

$$|E(f,\sigma)| \leq \frac{1}{48}\sum_{i=0}^{n-1}(s_{i+1}-s_i)^3\left[\left|f''\left(\frac{3s_i+5s_{i+1}}{2}\right)\right| + \left|f''\left(\frac{5s_i+3s_{i+1}}{2}\right)\right|\right]$$

holds.

Proof. The proof is similar to that of Proposition 4. □

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