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On the Direct and Skew Sums of Γ_1-Non Deranged Permutations

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Abstract. Direct and skew sum operations are invaluable techniques for linking permutations while retaining their original structure in the resulting concatenation. In this work we apply the direct and skew sum operations on the elements of the Γ_1-non deranged permutation group ($\mathcal{G}_p^{\Gamma_1}$), and present relations and schemes on the structures and fixed points of the permutations obtained from these operations. Furthermore, if π is the direct sum of these Γ_1- non deranged permutations, then the collection of permutations in the form of π is an abelian group under composition, denoted as $\mathcal{G}_p^{m\oplus}$. We present an expression relating the direct and skew sum operations, and we establish an isomorphism between $\mathcal{G}_p^{\Gamma_1} \times \mathcal{G}_p^{\Gamma_1}$ and $\mathcal{G}_p^{m\oplus}$.

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Group.

1. Introduction

Researchers in combinatorics have mainly applied the direct and skew sum operations in the study of separable and decomposable permutations, and these permutations are often arbitrary, and are thus of different structures. A permutation is said to be separable if it can be expressed as the direct or skew sum of the trivial permutation 1, and decomposable if it can be expressed as the direct or skew sum of two nonempty permutations. Separable and decomposable permutations are fundamental in the study of pattern avoidance, equipopularity of patterns, the Möbius function and topology of a permutation poset. Albert et al. [1], in their study of the equipopularity classes of separable permutations, showed that the number of equipopularity classes for length n patterns of the separable permutations is equal to the number of partitions of $n-1$. Homberger [6] examined the direct and skew sum operations as examples of inflation operations in his work on patterns in permutations and involutions. Smith [10] gave a formula for the Möbius function of intervals $[1, \pi]$, where π is a permutation with exactly one descent. McNamara and Steingrímsson [9] examined decomposable and indecomposable permutations in the shellability of intervals, and Hoffmann [5] demonstrated the regularity of sets of plus- and minus-(in)decomposable permutations and $\sigma-$decomposable permutations.

It is important to note that researchers are yet to investigate the direct and skew sum operations on a particular permutation group, whose members share the same structure.

Ibrahim et al. [8] extended the work of Garba and Ibrahim [4] by establishing a natural identity permutation for a special class of permutations generated by a modulo p function of Ibrahim [7], and thus constructed a new permutation group called the Γ_1-non deranged permutation group, denoted as $\mathcal{G}_p^{\Gamma_1}$. More recently, Garba et al. [3] investigated the non standard Young tableaux of $\mathcal{G}_p^{\Gamma_1}$, and Aremu et al. [2] studied the fuzzy subgroup of $\mathcal{G}_p^{\Gamma_1}$.

This paper applies the direct and skew sum operations to the Γ_1-non deranged permutation group with the aim of studying the algebraic and combinatorial properties of the group ($\mathcal{G}_p^{\Gamma_1}$) and the operations themselves.

2. Notations and Preliminaries

Definition 2.1. *The* direct sum *of permutations is an operation, denoted by $\oplus$, for concatenating two permutations into a longer one. Given two permutations α of length m and β of length n, the direct sum of α and β is given as*

$$(\alpha \oplus \beta)(i) = \begin{cases} \alpha(i) \;\; \textit{for}\ 1 \leq i \leq m \\ \beta(i-m) + m \;\; \textit{for}\ m+1 \leq i \leq m+n. \end{cases}$$

Definition 2.2. *The* skew sum *of permutations is an operation, denoted by $\ominus$, for concatenating two permutations into a longer one. Given two permutations α of length m and β of length n, the skew sum of α and β is given as*

$$(\alpha \ominus \beta)(i) = \begin{cases} \alpha(i) + n \;\; \textit{for}\ 1 \leq i \leq m \\ \beta(i-m) \;\; \textit{for}\ m+1 \leq i \leq m+n. \end{cases}$$

Remark 2.3. *For any permutations α, β :*

(i) $\oplus$ *and* $\ominus$ *are associative but not commutative; and the permutations* $\alpha \oplus \beta$ *and* $\alpha \ominus \beta$ *are of length* $m+n$.

(ii) $[\alpha \oplus \beta]^{-1} = \alpha^{-1} \oplus \beta^{-1}$ *and* $[\alpha \ominus \beta]^{-1} = \beta^{-1} \ominus \alpha^{-1}$.

Definition 2.4. *The* Γ_1-non deranged permutation group *(*$\mathcal{G}_p^{\Gamma_1}$*) is a permutation group developed from a special class of permutations generated by a modulo* p *function that exhibits the following properties:*

(i) Each element of the group has the following form

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & \cdots & (1+(p-1)i)_{mp} \end{pmatrix},$$

for $1 \leq i < p$, $p \geq 5$, p *a prime.*

(ii) The length of each $\omega_i \in \mathcal{G}_p^{\Gamma_1}$ *is* p, *and the order of* $\mathcal{G}_p^{\Gamma_1}$ *is* $p-1$, *for* $p \geq 5$, p *a prime.*

(iii) The Γ_1-*non deranged permutation group is abelian.*

3. Group theoretic properties of $\mathcal{G}_p^{\Gamma_1}$ under direct and skew sums

To investigate the group theoretic properties of $\mathcal{G}_p^{\Gamma_1}$ under direct and skew sums, we first provide the structures of the direct and skew sums of Γ_1-non deranged permutations.

Proposition 3.1. *Let* $\pi = \omega_i \oplus \omega_j$, *for* $\omega_i, \omega_j \in \mathcal{G}_p^{\Gamma_1}$. *Then* π *is of the form*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & p & p+1 & p+2 & p+3 & \cdots & 2p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & \cdots & (1+(p-1)i)_{mp} & p+1 & [p+(1+j)_{mp}] & [p+(1+2j)_{mp}] & \cdots & [p+(1+(p-1)j)_{mp}] \end{pmatrix}.$$

Proof. Since $\ell(\omega_i) = \ell(\omega_j) = p$, then the length of π is $2p$. The images of π are the images of ω_i concatenated to the images of ω_j each incremented by p. $\square$

Corollary 3.2. *Let* $\pi = \omega_i \oplus \omega_j \oplus \omega_k \oplus \cdots \oplus \omega_n$, *where* $\omega_i, \omega_j, \omega_k, \ldots, \omega_n \in \mathcal{G}_p^{\Gamma_1}$ *are each of length* p. *Then* π *is of the form*

$$\pi = \begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & 2p & 2p+1 & \cdots \\ 1 & \cdots & [(1+(p-1)i)_{mp}] & p+1 & \cdots & [p+(1+(p-1)j)_{mp}] & 2p+1 & \cdots \end{pmatrix.$$
$$\begin{matrix} 3p & 3p+1 & \cdots & (n-1)p+1 & \cdots & np \\ [2p+(1+(p-1)k)_{mp}] & 3p+1 & \cdots & (n-1)p+1 & \cdots & [(n-1)p+(1+(p-1)n)_{mp}] \end{matrix}\Bigg).$$

Proof. The proof follows from the proof of Proposition 3.1. $\square$

Proposition 3.3. *Let* $\sigma = \omega_i \ominus \omega_j$, *where* $\omega_i, \omega_j \in \mathcal{G}_p^{\Gamma_1}$. *The form of* σ *is given as*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & p & p+1 & p+2 & p+3 & \cdots & 2p \\ p+1 & [p+(1+i)_{mp}] & [p+(1+2i)_{mp}] & \cdots & [p+(1+(p-1)i)_{mp}] & 1 & (1+j)_{mp} & (1+2j)_{mp} & \cdots & (1+(p-1)j)_{mp} \end{pmatrix}.$$

Proof. The proof follows from Proposition 3.1 except that the skew sum operation only increases the images of the left permutation. $\square$

Corollary 3.4. *Let* $\sigma = \omega_i \ominus \omega_j \ominus \omega_k \ominus \cdots \ominus \omega_n$, *where* $\omega_i, \omega_j, \omega_k, \ldots, \omega_n \in \mathcal{G}_p^{\Gamma_1}$ *are each of length* p. *Then* σ *is of the form*

$$\sigma = \Bigg(\begin{matrix} 1 & \cdots & p & p+1 & \cdots \\ (1+(n-1)p) & \cdots & [(1+(p-1)i)_{mp}+(n-1)p] & (1+(n-2)p) & \cdots \end{matrix}$$
$$\begin{matrix} 2p & 2p+1 & \cdots & (n-1)p+1 & \cdots & np \\ [(1+(p-1)j)_{mp}+(n-2)p] & (1+(n-3)p) & \cdots & 1 & \cdots & [(1+(p-1)k)_{mp}] \end{matrix}\Bigg).$$

Remark 3.5. *The length of* π *or* σ *is* np, *for* $n \in \mathbb{N}$ $p \geq 5$, p *is prime.*

Proposition 3.6. *Let π and σ be the direct and skew sums of Γ_1-non deranged permutations respectively. Then the reduction modulo p of every $\pi(i)$ or $\sigma(i)$ decomposes π and σ to their generating Γ_1-non deranged permutations.*

Proof. Let π and σ be the direct and skew sums respectively of any $\omega_i \in \mathcal{G}_p^{\Gamma_1}$. Every element of π and σ is either $\omega_i(j)$ or $[\omega_i(j) + np]$, where np is the length of either ω_i, π, or σ, for $n \in \mathbb{N}$, and $p \geq 5$, p a prime. Since

$$[\omega_i(j) + np] \equiv \omega_i(j)(mod\, p),$$

for $p \geq 5$, p a prime, the result holds. □

Lemma 3.7. *Let π and σ be the direct and skew sums of Γ_1-non deranged permutations respectively. Then*

(i) $\pi_1 \circ \pi_2 = \pi$
(ii) $\sigma_1 \circ \sigma_2 = \pi$
(iii) $\pi \circ \sigma = \sigma$
(iv) $\sigma \circ \pi = \sigma$.

Lemma 3.8. *Let $\omega_i, \omega_j \in \mathcal{G}_p^{\Gamma_1}$. Suppose that $\omega_i = \omega_j^{-1}$, then*

(i) $[\omega_i \oplus \omega_j]^{-1} = \omega_j \oplus \omega_i$,
(ii) $[\omega_i \ominus \omega_j]^{-1} = \omega_i \ominus \omega_j$.

Proof. From Remark 2.3 (ii), the proof follows. □

Lemma 3.9. *Let $\ell(\omega_i)$ denote the length of ω_i. Then*

(i) $[\omega_i \oplus \omega_j] \circ [\omega_k \oplus \omega_l] = [\omega_i \circ \omega_k] \oplus [\omega_j \circ \omega_l]$, *where* $\ell(\omega_i) = \ell(\omega_k)$ *and* $\ell(\omega_j) = \ell(\omega_l)$,
(ii) $[\omega_i \ominus \omega_j] \circ [\omega_k \ominus \omega_l] = [\omega_j \circ \omega_k] \oplus [\omega_i \circ \omega_l]$, *where* $\ell(\omega_j) = \ell(\omega_k)$ *and* $\ell(\omega_i) = \ell(\omega_l)$.

Proof. The left and right hand sides of (i) and (ii) are simply rearrangements of the same logic in each case.

(i) The composition operator acts separately on the pairs ω_i, ω_k and ω_j, ω_l, and the direct sum operator increases each element of ω_j and ω_l by the lengths of ω_i and ω_k respectively.
(ii) Since the skew sum operation increases the elements of ω_i and ω_k, the composition operator acts separately
on the pairs ω_j, ω_k and ω_i, ω_l, since the indices of ω_j and ω_i correspond to the images of ω_k and ω_l respectively.

□

We next investigate the group theoretic properties of $\mathcal{G}_p^{\Gamma_1}$ under direct and skew sums.

Proposition 3.10. *The collection*

$$\mathcal{G}_p^{m\oplus} = \{\pi_{i,j} : \pi_{i,j} = \omega_i \oplus \omega_j,\ \omega_i, \omega_j \in \mathcal{G}_p^{\Gamma_1}\}$$

is a subgroup of the symmetric group S_{np} under composition, for $n \in \mathbb{N}$, $p \geq 5$, p a prime.

Proof. Lemma 3.8(i) could be written as

$$\pi_{i,j}^{-1} = \pi_{j,i},$$

whenever ω_i, ω_j which composed π are inverses of each other.
Since i and j are arbitrary, it implies that every π has its unique inverse, and by Lemma 3.7, this inverse is closed under composition. Thus

$$\pi_1^{-1} \circ \pi_2 = \pi \in \mathcal{G}_p^{m\oplus},$$

and therefore $\mathcal{G}_p^{m\oplus}$ is a group under composition, and thus a subgroup of the symmetric group S_{np}. $\square$

Remark 3.11. *The collection $\mathcal{G}_p^{m\ominus} = \{\sigma_{i,j} : \sigma_{i,j} = \omega_i \ominus \omega_j,\ \omega_i, \omega_j \in \mathcal{G}_p^{\Gamma_1}\}$ is not a group as $e \notin \mathcal{G}_p^{m\ominus}$ and for $\sigma_1, \sigma_2 \in \mathcal{G}_p^{m\ominus}$,*

$$\sigma_1^{-1} \circ \sigma_2 = \pi \notin \mathcal{G}_p^{m\ominus}.$$

Let $\mathcal{G}_p^{m\ominus}$ hence denote the set of skew sums of Γ_1-non deranged permutations.

Corollary 3.12. *The group $\mathcal{G}_p^{m\oplus}$ is abelian.*

Proof. We want to show that

$$\pi_{i,j} \circ \pi_{k,l} = \pi_{k,l} \circ \pi_{i,j},$$

for any $\omega_i, \omega_j, \omega_k, \omega_l \in \mathcal{G}_p^{\Gamma_1}$.
By Lemma 3.9(i),

$$\begin{aligned} \pi_{i,j} \circ \pi_{k,l} &= [\omega_i \oplus \omega_j] \circ [\omega_k \oplus \omega_l] \\ &= [\omega_i \circ \omega_k] \oplus [\omega_j \circ \omega_l]. \end{aligned}$$

Since $\mathcal{G}_p^{\Gamma_1}$ is abelian,

$$\begin{aligned} \Longrightarrow [\omega_i \circ \omega_k] \oplus [\omega_j \circ \omega_l] &= [\omega_k \circ \omega_i] \oplus [\omega_l \circ \omega_j] \\ &= [\omega_k \oplus \omega_l] \circ [\omega_i \oplus \omega_j] \\ &= \pi_{k,l} \circ \pi_{i,j}. \end{aligned}$$

Hence, $\mathcal{G}_p^{m\oplus}$ is abelian. $\square$

Theorem 3.13. *Let np be the length of $\pi \in \mathcal{G}_p^{m\oplus}$. Then the order of $\mathcal{G}_p^{m\oplus}$ is $(p-1)^n$.*

Proof. Since there are $p-1$ permutations in any $\mathcal{G}_p^{\Gamma_1}$, then by a direct sum operation table, there will only be $(p-1)(p-1) = (p-1)^2$ possible unique direct sums ($\omega_i \oplus \omega_j = \pi \in \mathcal{G}_p^{\oplus}$), for $\omega_i, \omega_j \in \mathcal{G}_p^{\Gamma_1}$, $1 \le i, j \le p-1$, $p \ge 5$, p a prime. The non-commutativity of the direct sum operation guarantees uniqueness.

Similarly, there are only $(p-1)^2(p-1) = (p-1)^3$ possible unique results of $[\omega_i \oplus \omega_j] \oplus \omega_k = \pi \in \mathcal{G}_p^{2\oplus}$.

Also, there are only $(p-1)^3(p-1) = (p-1)^4$ possible unique results of $[\omega_i \oplus \omega_j \oplus \omega_k] \oplus \omega_l = \pi \in \mathcal{G}_p^{3\oplus}$.

Therefore, by induction, there are only $(p-1)^n$ possible unique results of $\omega_i \oplus \omega_j \oplus \omega_k \oplus \cdots \oplus \omega_n = \pi \in \mathcal{G}_p^{m\oplus}$ where $m = n-1$. $\square$

4. The fix points and transpositions of $\pi \in \mathcal{G}_p^{m\oplus}$ and $\sigma \in \mathcal{G}_p^{m\ominus}$

First we present some results on the algebra of $\pi \in \mathcal{G}_p^{m\oplus}$ and $\sigma \in \mathcal{G}_p^{m\ominus}$.

Lemma 4.1. *Let $\pi \in \mathcal{G}_p^{m\oplus}$ and $\sigma \in \mathcal{G}_p^{m\ominus}$. Then*

(i) $\pi \oplus \sigma = \pi(i)(\sigma(j) + np)$, *for* $1 \le i \le \frac{1}{2}np$, $\frac{1}{2}np \le j \le np$, *where np is the length of $\pi \oplus \sigma$.*

(ii) $\sigma \oplus \pi = \sigma(i)(\pi(j) + np)$, *for* $1 \le i \le \frac{1}{2}np$, $\frac{1}{2}np \le j \le np$, *where np is the length of $\sigma \oplus \pi$.*

(iii) $\pi \ominus \sigma = \sigma$

(iv) $\sigma \ominus \pi = \sigma$

(v) $\pi \ominus \pi = \sigma$

(vi) $\sigma \oplus \sigma = \sigma$.

Proposition 4.2. *Let $\pi_1 \in \mathcal{G}_p^{m_1\oplus}$ and $\pi_2 \in \mathcal{G}_p^{m_2\oplus}$. Then*

$$[\pi_1 \oplus \pi_2] \circ [\pi_1^{-1} \oplus \pi_2^{-1}] = e \in \mathcal{G}_p^{m\oplus},$$

where $m = m_1 + m_2$.

Proof. By Lemma 3.9(i), the result follows. □

Remark 4.3. *Proposition 4.2 could be rewritten as*

$$[\pi_1 \oplus \pi_2] \circ [\pi_1 \oplus \pi_2]^{-1} = e,$$

by Remark 2.3 (ii).

Proposition 4.4. *Let $\pi \in \mathcal{G}_p^{m\oplus}$ and $\sigma \in \mathcal{G}_p^{m\ominus}$. Then*

(i) $[\pi \oplus \sigma] \circ [\sigma \oplus \pi] = \sigma$

(ii) $[\pi \ominus \sigma] \circ [\sigma \ominus \pi] = \pi$.

Proof. (i) Follows from Lemma 3.9(i) and Lemma 4.1(vi).

(ii) Follows from Lemma 3.9(ii). Also follows from Lemma 4.1(iii) and (iv), and Lemma 3.7(ii).

□

Corollary 4.5. *Suppose that e is the identity of $\mathcal{G}_p^{\Gamma_1}$ or $\mathcal{G}_p^{m\oplus}$. Then*

$$e^{m\oplus} = e \in \mathcal{G}_p^{m\oplus},$$

where $e^{m\oplus} = e_1 \oplus e_2 \oplus \cdots \oplus e_{m+1}$.

We next give some results on the fix points of $\pi \in \mathcal{G}_p^{m\oplus}$.

Theorem 4.6. *Let $\pi \in \mathcal{G}_p^{m\oplus}$. Then π has fix points at $np+1$, for $n \ge 0, n \in \mathbb{Z}$, and $p \ge 5$, p a prime.*

Proof. Let

$$\pi_1 = \omega_i \oplus \omega_j \text{ and } \pi_2 = \pi_k \oplus \pi_l.$$

Then by the direct sum operation, the indices of π_1 and π_2 proceeding np is $np+1$, and since every ω_i and π has a fix point at 1, the image of $np+1$ is also $np+1$.

Hence, π has a fix point at $np+1$, for $n \ge 0$, $n \in \mathbb{Z}$, and $p \ge 5$, p a prime. □

Remark 4.7. *The fix points of $\pi \in \mathcal{G}_p^{m\oplus}$ are elements of the sequence $\{np+1\}_{n=0}^{\infty}, p \geq 5$, p a prime.*

Corollary 4.8. *Let $\pi \in \mathcal{G}_p^{m\oplus}$ be of length np, for $n \in \mathbb{N}, p \geq 5$, p a prime. Then there are at least n fix points in any π.*

Proof. By Theorem 4.6 the fix points of π occur at $np+1$. Since π is of length np its fix points must precede np, and are, in descending order,

$$(n-1)p+1, (n-2)p+1, \cdots, (n-n)p+1 = 1,$$

the first fix point. Hence the result follows. □

Corollary 4.9. *Let n be a non-negative integer with $p \geq 5$, p a prime. Then*

(i) For every $\pi \in \mathcal{G}_p^{m\oplus}$, p is the difference of successive fix points.
(ii) In any $\pi \in \mathcal{G}_p^{m\oplus}$ or $\sigma \in \mathcal{G}_p^{m\ominus}$, p is the difference of successive $np+1$ elements.
(iii) The fix points of $\pi \in \mathcal{G}_p^{m\oplus}$ (and equivalently, the $np+1$ elements of $\sigma \in \mathcal{G}_p^{m\ominus}$) are the positive elements of the residue class $[1]$ modulo p.

Proof. The proofs follow from Remark 4.7. □

Theorem 4.10. *Suppose that $\omega_i \in \mathcal{G}_{p_1}^{\Gamma_1}$ and $\omega_j \in \mathcal{G}_{p_2}^{\Gamma_1}$ such that $p_1 \neq p_2$. Then there exist at least two fix points in $\omega_i \oplus \omega_j$.*

Proof. This follows from the proof of Theorem 4.6 except that the fix points would not occur at $np+1$ (for $n \geq 0, n \in \mathbb{Z}$) since ω_i and ω_j are of different lengths. □

We conclude this section by presenting some results on the transpositions of $\sigma \in \mathcal{G}_p^{m\ominus}$.

Corollary 4.11. *For every $\sigma \in \mathcal{G}_p^{m\ominus}$, the $np+1$ elements are transpositions whenever n is even.*

Proof. Since the $np+1$ elements of $\sigma \in \mathcal{G}_p^{m\ominus}$ do not fix themselves, and from Corollary 4.9 (ii), since the difference of successive $np+1$ elements equals p, then the $np+1$ elements are in the same residue class modulo p, they interchange themselves on the index and image rows, and are therefore decomposable into transpositions. □

Theorem 4.12. *Let $\sigma \in \mathcal{G}_p^{m\ominus}$ be a permutation of odd length. Then σ has at least one fix point.*

Proof. Let $\sigma_r \in \mathcal{G}_p^{m\ominus}$ be an odd-length permutation. Thus, for $\sigma_r = \sigma_s \ominus \sigma_w$, σ_s is either even and σ_w is odd or vice versa. Let ℓ_s and ℓ_w denote the lengths of σ_s and σ_w respectively, and let $\ell_w > \ell_s$.

Case I: For $\sigma_s \ominus \sigma_w = \sigma_r$, there exists an $np+1$ image of σ_w, say $\sigma_w(i)$, such that $\sigma_w(i) = \ell_s + i$, where i is the index of $\sigma_w(i)$. The skew sum operation increases each index of σ_w by ℓ_s, and hence there exists a fix point in σ_r.

Case II: For $\sigma_w \ominus \sigma_s = \sigma_r$, there exists an $np+1$ image of σ_w, say $\sigma_w(i)$, such that $\sigma_w(i) + \ell_s = i$, where i is the index of $\sigma_w(i)$. Since the skew sum operation increases each image of σ_w by ℓ_s, this also guarantees the existence of a fix point in σ_r. □

Theorem 4.13. *Let $\omega \in \mathcal{G}_p^{\Gamma_1}$. Then the permutation $\omega \ominus \omega^{-1}$ decomposes into p disjoint transpositions.*

Proof. Let $\omega \in \mathcal{G}_p^{\Gamma_1}$. For every $1 \le i \le p$,

$$\ominus : \begin{cases} i \longmapsto [\omega(i)+p] & (1) \\ [\omega(i)+p] \longmapsto [\omega(i)+p]^{-1} = [\omega(i)]^{-1} = i. & (2) \end{cases}$$

The preimage of (1) is equal to the image of (2), and the image of (1) is equal to the preimage of (2), and thus $\omega \ominus \omega^{-1}$ can be decomposed into disjoint transpositions. Since the length of $\omega \ominus \omega^{-1}$ is $2p$ then there are $\frac{1}{2} \cdot 2p$ possible transpositions in $\omega \ominus \omega^{-1}$. □

5. The morphism between $\mathcal{G}_p^{\Gamma_1} \times \mathcal{G}_p^{\Gamma_1}$ and $\mathcal{G}_p^{m\oplus}$

Proposition 5.1. *(i) Given the mapping*

$$\varphi : \mathcal{G}_{p_r}^{\Gamma_1} \times \mathcal{G}_{p_s}^{\Gamma_1} \longrightarrow \mathcal{G}_p^{\oplus},$$

define

$$\varphi(\omega_i, \omega_j) = \omega_i \oplus \omega_j$$

for $\omega_i \in \mathcal{G}_{p_r}^{\Gamma_1}$ and $\omega_j \in \mathcal{G}_{p_s}^{\Gamma_1}$. Thus, φ is the direct sum of Γ_1-non deranged permutations.

(ii) Given the mapping

$$\varphi : \prod_{r=1}^{n} \mathcal{G}_{p_r}^{\Gamma_1} \longrightarrow \mathcal{G}_p^{m\oplus},$$

define

$$\varphi(\omega_i, \dots, \omega_n) = \omega_i \oplus \cdots \oplus \omega_n,$$

for $\omega_i \in \mathcal{G}_{p_r}^{\Gamma_1}$. Then, φ is the direct sum of n Γ_1-non deranged permutations.

Theorem 5.2. *The function φ of Proposition 5.1 is an isomorphism.*

Proof. We prove Proposition 5.1 (i).

(i) *Homomorphism*: Since the binary operation of $\mathcal{G}_{p_r}^{\Gamma_1}$, $\mathcal{G}_{p_s}^{\Gamma_1}$, and $\mathcal{G}_p^{\oplus}$ is composition, we define the homomorphism of the groups as

$$\begin{aligned} \varphi(\omega_i \circ \omega_j, \omega_k \circ \omega_l) &= \varphi(\omega_i, \omega_k) \circ \varphi(\omega_j, \omega_l) \\ &= \varphi(\omega_i, \omega_l) \circ \varphi(\omega_j, \omega_k), \end{aligned}$$

for $\omega_i, \omega_j \in \mathcal{G}_{p_r}^{\Gamma_1}$ and $\omega_k, \omega_l \in \mathcal{G}_{p_s}^{\Gamma_1}$, and the proof of this homomorphism follows from Lemma 3.9. We note that $\circ$ is commutative.

(ii) *Monomorphism*: For $e_i \in \mathcal{G}_{p_r}^{\Gamma_1}$ and $e_j \in \mathcal{G}_{p_s}^{\Gamma_1}$, by Corollary 4.5 we have that

$$e_i \times e_j \longmapsto e_{i,j} \in \mathcal{G}_p^{m\oplus}.$$

(iii) *Epimorphism*: For $\pi_{i,j} \in \mathcal{G}_p^{\oplus}$,

$$\pi_{i,j} = \varphi(\omega_i, \omega_j),$$

and thus $\pi_{i,j}$ is unique since the direct sum operation is not commutative.

Hence, φ is an isomorphism. □

6. Conclusion

The structure $\mathcal{G}_p^{m\oplus}$ is a permutation group and its respective elements have multiple fixed points. The structure $\mathcal{G}_p^{m\ominus}$ is not closed under composition; its respective elements are derangements except when they are of odd length.

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References

[1] M. Albert, C. Homberger and J. Pantone, *Equipopularity classes in the separable permutations*, arXiv:1410.7312v1[math.CO] (2014).

[2] K. O. Aremu, O. Ejima and M. S. Abdullahi, *On the fuzzy Γ_1-non deranged permutation group $\mathcal{G}_p^{\Gamma_1}$*, Asian Journal of Mathematics and Computer Research **18**, No. 4 (2017) 152-157.

[3] A. I. Garba, O. Ejima, K. O. Aremu and H. Usman, *Non standard young tableaux of Γ_1-non deranged permutation group $\mathcal{G}_p^{\Gamma_1}$*, Global Journal of Mathematical Analysis **5**, No. 1 (2017) 21-23.

[4] A. I. Garba and A. A. Ibrahim, *A new method of constructing a variety of finite group based on some succession scheme*, International Journal of Physical Science **2**, No. 3 (2010) 23-26.

[5] R. Hoffmann, *On dots in boxes, or permutation pattern classes and regular languages*, (Doctoral thesis, University of St Andrews, Fife, Scotland, UK), (2015) Retrieved from `https://personal.research-repository.st-andrews.ac.uk/bitstream/handle/10023/7034/RuthHoffmanPhDThesis.pdf`

[6] C. Homberger, *Patterns in permutations and involutions: A structural and enumerative approach*, arXiv:1410.2657v1 [math.CO] (2014).

[7] A. A. Ibrahim, *Group theoretical interpretation of Bara'at model*, Proceedings of Annual National Conference, Mathematical Association of Nigeria, (2004) 35-46.

[8] A. A. Ibrahim, O. Ejima and K. O. Aremu, *On the representations of Γ_1-non deranged permutation group $\mathcal{G}_p^{\Gamma_1}$*, Advances in Pure Mathematics **6**, No. 9 (2016) 608-614.

[9] P. R. McNamara and E. Steingrímsson, *On the topology of the permutation pattern poset*, arXiv:1305.5569v2 [math.CO] (2015).

[10] J. P. Smith, *On the Möbius function and topology of the permutation poset*, (Doctoral thesis, University of Strathclyde, Glasgow, UK), (2015) Retrieved from `http://personal.strath.ac.uk/jason.p.smith/Thesis.pdf`