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On Multigrid Solver for Generalized Stokes Equations

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Abstract. A multigrid solver for the generalized Stokes equations with coarsening by a factor of three is presented. Finite difference approximations are used on staggered grids and a distributive Gauss-Seidel smoothing scheme is employed on these grids. Numerical experiments are performed to validate the effectiveness and efficiency of the proposed multigrid staggered grid framework.

AMS (MOS) Subject Classification Codes: 65N55, 65F10
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1. Introduction

The generalized Stokes equations as well as the need of their (numerical) solution usually appear in incompressible fluid dynamics. In particular, in the numerical investigation (simulation) of fluid flow problems such as time-dependent Stokes or Navier-Stokes equations, we also need to solve a generalized Stokes problem at each, respectively, linear or nonlinear iteration step. The Stokes system and its solution procedures has also been studied in some applications (mechanics problems regarding structures) such as plasticity, beam and shell.

For the positive definite linear system (in connection with the elliptic systems) that arises from the discretization (of finite differences or finite elements), multigrid algorithms have been used as one of the most efficient schemes, e.g., see [3, 12, 13, 25]. On the other hand [1, 10, 22, 24, 26], for the saddle point problems that appears in the solution procedures of Stokes or Navier-Stokes equations, the smoothing step is not an easy task, i.e., the resulting matrices after discretization are not positive definite and thus the well known procedure for the smoothing cannot be applied in this case (in a standard way). Solving Stokes equation with multigrid has a long history. In most of the cases, two methods are used. The first method is known as the distributed (Gauss-Seidel) iteration [4, 25], i.e., decoupled smoothing on the transformed system. The second method refers to an approximate decoupling of the system, see [17].

In this paper, we extend the work [4, 6, 7] to the generalized Stokes equations and develop a multigrid solver using coarsening by a factor-of-three strategy. In particular, the scope of this paper is more on the numerical efficiency of the proposed multigrid technique. Adopting a coarsening by the factor of three strategy on staggered grids has the potential advantage of simplifying the intergrid transfer operators, coarsening more quickly and ultimately reducing the number of levels and communication steps in a parallel implementation. The iteration counts we have observed in this paper show robust results in grid size h, $\alpha \geqslant 0$, and $\nu > 0$ (viscosity or size of the diffusion term). For $\alpha = 0$, the generalized Stokes equations becomes the standard Stokes problem. For this case, various multigrid solvers are available in literature, for example, see [2, 5, 26] and the papers cited in [24]. For robust convergence behavior when $\alpha > 0$, the construction of multigrid algorithm is more involved, see [15]. This paper aims at to construct an efficient multigrid scheme on staggered grids for solving the generalized Stokes problems, which appears in time-dependent Stokes flow problems, when $\alpha > 0$ (which correspond to small time-steps). Here, we focus on the development of the coarsening-by-three multigrid algorithm, and leave a rigorous analysis of its convergence for future work.

The paper is organized as follows. The generalized Stokes system in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ is discussed in the next section. Discretization on staggered grids using finite differences is presented in Section 3. In Section 4, the proposed full multigrid scheme is presented with a distributive Gauss-Seidel relaxation scheme. Results of numerical experiments are presented in Section 5, and at the end conclusions are given in the last section.

2. THE GENERALIZED STOKES EQUATIONS

In this paper, we consider the following generalized Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$: Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ (the velocity vector) and $p \in L_0^2(\Omega)$ (the pressure field) satisfying

$$\alpha\mathbf{u} - \nu\triangle\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \tag{2. 1}$$

$$-\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \tag{2. 2}$$

$$\mathbf{u} = 0 \text{ on } \Gamma, \tag{2. 3}$$

where the source term $\mathbf{f} \in \mathbf{L}^2(\Omega)$ represents the force, $\alpha \geq 0$, and viscosity $\nu > 0$ are given. Moreover, when $\alpha = 0$, we have the standard Stokes problem. The parameter $\alpha \geq 0$ is a quantity, proportional to the inverse of the time-step, that appears in an auxiliary problem for implicit time-stepping approaches to solve nonstationary (time-dependent) Stokes flow problems. For existence of a unique solution, we further required $\int_\Omega p\, dx = 0$.

Here and in the following, $L^2(\Omega)$ and $H^1(\Omega)$ denote the standard Lebesque and Sobolev spaces with $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$, respectively, as associated standard norms. The usual inner product associated with $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$. Moreover, we have the space $L_0^2(\Omega)$, which is the space of functions in $L^2(\Omega)$ with mean value 0, i.e.,

$$L_0^2(\Omega) = \left\{ \phi \in L^2(\Omega) : \int_\Omega \phi\, dx = 0 \right\}$$

and $\mathbf{H}_0^1(\Omega)$, the space in $\mathbf{H}^1(\Omega)$ vanishing on the boundary, i.e.,

$$\mathbf{H}_0^1(\Omega) = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = 0 \text{ on } \partial\Omega \right\}.$$

The weak solution $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ of (2. 1)-(2. 3) is the solution of

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \ \ \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ b(\mathbf{u}, q) &= 0 \ \ \forall\, q \in L_0^2(\Omega), \end{aligned}$$

where $a : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$ and $b : \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ are the bilinear forms defined as

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \sum_{j=1}^{2} \int_\Omega \nabla \mathbf{u}_j \cdot \nabla \mathbf{v}_j \\ b(\mathbf{v}, p) &:= -\int_\Omega p \nabla \cdot \mathbf{v} \end{aligned}$$

For $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ the problem has a unique solution [11]. Moreover, if Ω is a convex polygon and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, then $\mathbf{u} \in \mathbf{H}^2(\Omega), p \in H^1(\Omega)$ [14] and there exists $C = C(\Omega) > 0$ satisfying

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

3. Discretization

The chosen scheme (resulting from finite difference or finite elements discretization), the discretization of (2. 1)-(2. 3) ultimately gives a linear system, i.e.,

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}$$

where (the block matrix) A is the (symmetric positive definite) discrete operator $\alpha - \nu\triangle$ acting on each velocity components (u and v respectively). Moreover, B^T and B represents the discrete gradient, respectively, the discrete divergence; the term C is needed to avoid spurious solutions by some discretization schemes. We refer [23] and [8], respectively, for more details about (finite difference and finite element) discretization.

In the following, we present the discretization of the generalized Stokes equations using finite difference approximations on staggered grids. It is well known that the given finite difference scheme (or MAC scheme) is naturally stable with $C = 0$; for example see [19]. Moreover, in our case of finite differences (and later for distributive Gauss-Seidel relaxation scheme, c.f. [4]), we have $C = \alpha\, h^2\, \Delta^h$ a scaled discrete Laplacian acting on pressure unknowns.

First, we consider a sequence of grids $\{\Omega_h\}_{h>0}$, which is defined as follows

$$\Omega_h = \{\mathbf{x} \in \mathbb{R}^2 : x_i = i\, h,\, y_j = j\, h, \quad i, j \in \mathbb{Z}\} \cap \Omega.$$

Here, we consider Ω as a rectangular domain. The values of grid size h are chosen such that the grid lines coincides with the boundaries of Ω. The variables (velocity vector), on the staggered grids, are placed on (horizontal or vertical) centers of cell faces and (the pressure variable) on cell centers. We use Ω_h^s, $s \in \{ev, eh, c\}$ to denote theses grid points, e.g., by Ω_h^{ev} we mean grid points defined on center of cell edge-vertical.

Next, the discrete L^2-scalar product, for grid functions u^h and v^h, defined on Ω_h^s, with associated norm $\|u^h\|_{L_h^2(\Omega_h^s)} = (u^h, u^h)^{1/2}_{L_h^2(\Omega_h^s)}$ is given by

$$(u^h, v^h)_{L_h^2(\Omega_h^s)} = h^2 \sum_{\mathbf{x}\in\Omega_h^s} u^h(\mathbf{x})\, v^h(\mathbf{x}).$$

Moreover, we denote the (discrete) space $L_h^2(\Omega_h^s)$ of grid points u^h defined on Ω_h^s, and endowed with $\|u^h\|_{L_h^2(\Omega_h^s)}$, as norm; by $\mathcal{U}_h$, $\mathcal{V}_h$ and $\mathcal{P}_h$ we mean the space of the grid points u^h, v^h and p^h, respectively.

Here, the (velocity) variable $\mathbf{u}$ is defined at center of faces (horizontal Ω_h^{ev} or vertical Ω_h^{eh}), and the (pressure) variable p is defined at cell centers Ω_h^c; see Figure 1. Therefore, we can write the discrete Stokes equations as follows

$$\begin{aligned} Q^h \mathbf{u}_j^h + \partial_j^h p^h &= \mathbf{f}_j^h, \qquad \text{at centers of } j\text{-faces} \\ \sum_{j=1}^{2} \partial_j^h\, \mathbf{u}_j^h &= 0, \qquad \text{at cell centers,} \qquad (j = 1, 2) \end{aligned}$$

where $Q^h = \alpha - \nu\Delta^h$ is some finite difference approximation of $Q = \alpha - \nu\Delta$, and Δ^h is the usual 5-point difference approximation; $\mathbf{u}_1^h = u^h, \mathbf{u}_2^h = v^h$, $\partial_1^h = \partial_x^h$, and $\partial_2^h = \partial_y^h$. However, near a boundary, $\Delta^h\, \mathbf{u}_j^h(\mathbf{x})$ may involve an exterior (so-called ghost points) value. This value is defined by quadratic extrapolation, see [4].

Next, for the given step size h in the $2D$ space domain Ω, we consider the grid points, in a lexicographic order including the boundaries, i.e., the vertices coordinate $x_i = (i-1)\,h$ and $y_j = (j-1)\,h$; $1 \le i \le N_x, 1 \le j \le N_y$. The equations are discretized on a squared staggered grid, see Fig. 1. Thus p is located at the interior point $p_{i+1/2,j+1/2}$ (i.e., the discrete point $p(x_i + h/2, y_j + h/2)$)); $u_{i,j}$ is located at $((i-1)\,h, (j-1/2)\,h)$ and $v_{i,j}$ at $((i-1/2)\,h, (j-1)\,h)$. Therefore, by using second-order central differences, we get

$$\begin{aligned} &\alpha\, u_{i,j+1/2} - \Big(\frac{u_{i-1,j+1/2} - 2u_{i,j+1/2} + u_{i+1,j+1/2}}{h^2} \\ + \quad &\frac{u_{i,j-1/2} - 2u_{i,j+1/2} + u_{i,j+3/2}}{h^2}\Big) + \frac{p_{i+1/2,j+1/2} - p_{i-1/2,j+1/2}}{h} = f_{i,j+1/2} \qquad (3.\,4) \\ &\alpha\, v_{i+1/2,j} - \Big(\frac{v_{i-1/2,j} - 2v_{i+1/2,j} + v_{i+3/2,j}}{h^2} \\ + \quad &\frac{v_{i+1/2,j-1} - 2v_{i+1/2,j} + v_{i+1/2,j+1}}{h^2}\Big) + \frac{p_{i+1/2,j+1/2} - p_{i+1/2,j-1/2}}{h} = g_{i+1/2,j} \qquad (3.\,5) \\ &\frac{u_{i+1,j+1/2} - u_{i,j+1/2}}{h} + \frac{v_{i+1/2,j+1} - v_{i+1/2,j}}{h} = 0, \qquad (3.\,6) \end{aligned}$$

where the first momentum equation (3. 4) is centered (relaxed) at (internal cell) edge-vertical Ω_h^{ev}, the second momentum equation (3. 5) at (internal cell) edge-horizontal Ω_h^{eh}, and the continuity equation (3. 6) at (internal cell centers) Ω_h^c.

Summarizing, equation (3. 4)-(3. 6) constitute the discrete Stokes system with the following boundary conditions

$$\begin{aligned} u_{i,j+1/2} &= 0, \qquad (i = 1, N_x + 1, j = 1, \cdots, N_y) \\ v_{i+1/2,j} &= 0, \qquad (j = 1, N_y + 1, i = 1, \cdots, N_x). \end{aligned} \qquad (3.\,7)$$

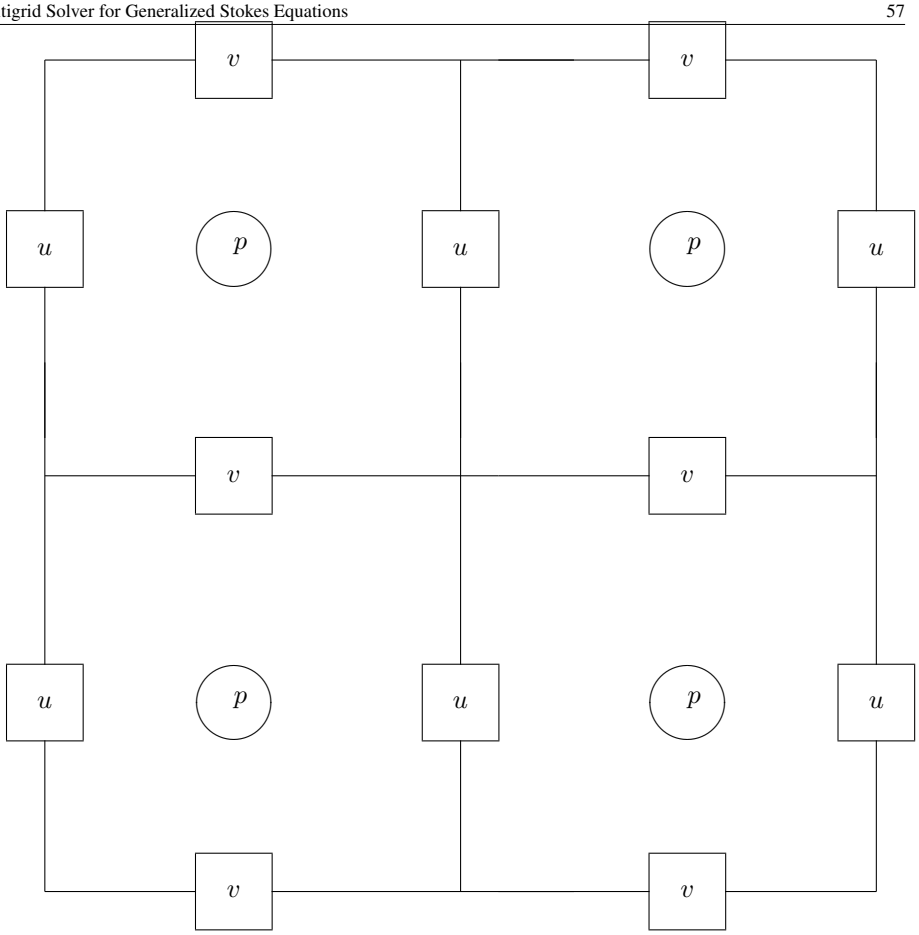

FIGURE 1. Coarsest staggered grid for the Stokes system.

4. A FULL MULTIGRID FRAMEWORK

In the following, a full multigrid method, on staggered grid, with distributive smoothing scheme to solve the discrete Stokes system (3. 4)-(3. 7) is presented. The coarsest staggered grid is shown in Figure 1.

As we know that multigrid scheme uses different discretization grids and such grids are usually obtained from a coarse grid, e.g., by halving the mesh (coarsest grid) [21], which gives hierarchy of grids that are non-nested and therefore additional efforts are required in construction of intergrid transfer operators; see [4, 16]. Keeping this fact in mind, we note that when (tripling) a coarsening by a factor-of-three to the mesh size is used (on the coarsest grid), a nested sequence of grids has been obtained, c.f. [6, 7]. It is also

worth noting that by using factor-of-three coarsening the use of intergrid transfer operators (*Prolongation and Restriction operators*) have been simplified in the (solution procedures) multigrid algorithms, see Fig. 2. Therefor, in the following we use coarsening by a factor-of-three strategy which is explained as follows:

A sequence of nested grids (or levels) Ω_k is considered with mesh size $h_{xk} = h_{x1}/3^{(k-1)}$ and $h_{yk} = h_{y1}/3^{(k-1)}$, where $k = 1, \ldots, L$, and $k = L$ denotes the finest level. Moreover, at the coarsest mesh, we use $h_{x1} = h_{y1} = 1/2$ as the mesh sizes. Here, we also remark that in this way, a variable X_{IJ}^{k-1} at the grid point (I, J) on the coarse grid Ω_{k-1} has the same spatial placement (location) as the variable X_{ij}^{k} at the grid point (i, j) on the fine grid Ω_k as follows; see Figure 2

- $u_{I,J+1/2}^{k-1} \longleftrightarrow u_{i,j+1/2}^{k}$ $\quad (\, i = 3I - 2, j = 3J - 1)$;
- $v_{I+1/2,J}^{k-1} \longleftrightarrow v_{i+1/2,j}^{k}$ $\quad (\, i = 3I - 1, j = 3J - 2)$;
- $p_{I+1/2,J+1/2}^{k-1} \longleftrightarrow p_{i+1/2,j+1/2}^{k}$ $\quad (\, i = 3I - 1, j = 3J - 1)$.

4.1. **A smoothing scheme and intergrid transfer operators .** In the following, we illustrate the (distributed Gauss-Seidel) smoothing scheme as well as the intergrid transfer operators.

Following [4], we note that the momentum equations (2. 1) are elliptic but the continuity equation (2. 2) is not elliptic, i.e., it is only a part of an elliptic (Stokes) system. Therefore, the momentum equations can be relaxed by a classical Gauss-Seidel scheme but for the continuity equation we need to relax it by a *distributive relaxation*, which is explained as follows:

Let (u^h, v^h, p^h) be the current approximation to the discretized system (3. 4)-(3. 6). First we relax the residuals of the momentum equations (3. 4)-(3. 5), by a pointwise Gauss-Seidel scheme, at all the interior points where u^h and v^h is defined. Then, we relax the continuity equation (3. 6) by distributive Gauss-Siedel (DGS) scheme, c.f. [4]. It is done as follows: let $\mathbf{x} = (i + 1/2, j + 1/2)$ be the current cell center and let

$$r_p^h = 0 - \partial_x^h u^h - \partial_y^h v^h$$

be the residual (just before relaxing there). The relaxation step at current cell center is made up of the following nine changes

$$\begin{aligned} u_j^h &\leftarrow u_j^h - \delta_p h \partial_j^h \chi_{\mathbf{x}}^h, \\ p^h &\leftarrow p^h + \delta_p h Q^h \chi_{\mathbf{x}}^h, \end{aligned}$$

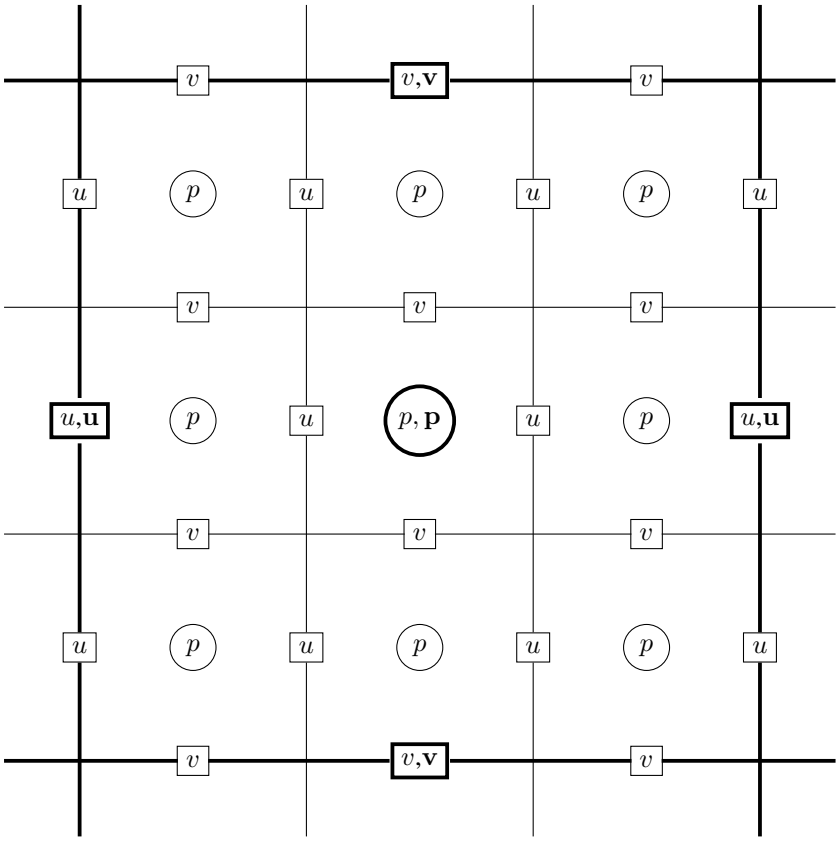

FIGURE 2. Illustration of coarsening by a factor-of-three strategy on staggered grids for the state variables. Bold lines represent the coarse- and the finer lines the fine-grid, respectively.

where $\chi^h_{\mathbf{x}}$ is the characteristic function of the cell center $\mathbf{x}$, i.e.,

$$
\begin{aligned}
u_{i,j+1/2} &\leftarrow u_{i,j+1/2} + \delta_p,\\
v_{i+1/2,j} &\leftarrow v_{i+1/2,j} + \delta_p,\\
u_{i,j-1/2} &\leftarrow u_{i,j-1/2} - \delta_p,\\
v_{i-1/2,j} &\leftarrow v_{i-1/2,j} - \delta_p,\\
p_{i+1/2,j+1/2} &\leftarrow p_{i+1/2,j+1/2} + \frac{4\nu}{h}\delta_p + \alpha h \delta_p,\\
p_{i+3/2,j+1/2} &\leftarrow p_{i+3/2,j+1/2} - \frac{\nu}{h}\delta_p,\\
p_{i+1/2,j+3/2} &\leftarrow p_{i+1/2,j+3/2} - \frac{\nu}{h}\delta_p,\\
p_{i-1/2,j+1/2} &\leftarrow p_{i-1/2,j+1/2} - \frac{\nu}{h}\delta_p,\\
p_{i+1/2,j-1/2} &\leftarrow p_{i+1/2,j-1/2} - \frac{\nu}{h}\delta_p,
\end{aligned}
$$

where

$$\delta_p = \frac{4}{h} r_p^h.$$

Above changes and δ_p are such that, after changing, r_p^h vanishes. The pressure changes are such that the momentum equations residuals

$$r_j^h \quad = \quad f_j^h - \alpha \mathbf{u}_j + \nu \Delta^h \mathbf{u}_j^h - \partial_j^h p^h, \qquad (j = 1, 2) \tag{4. 8}$$

at all points are preserved, at least in the approximate sense.

Near the boundary it is not possible to precisely preserve r_j^h while relaxing the continuity equation. It is enough to relax r_p^h so that the changes introduced to r_j^h do not cause later significant feed back (changes), in r_p^h, when the momentum equations are relaxed. Due to the boundary conditions, boundary feed-back changes, near the boundary, are partly absorbed. Therefore, such feed-back schemes are easy to design. For example, because of boundary conditions, in the case where one of the updates at the boundaries are not allowed, we need to modify δ_p as

$$\delta_p = \frac{h}{4 - d} r_p^h,$$

where $d = 1$, when one of the u or v is not updated, and (at corner) $d = 2$, i.e., where u as well as v (simultaneously) are not updated during the iterative step, see [4].

Next, we discuss the prolongation and restriction (intergird transfer) operators. As we discussed that in coarsening by a factor of a three we obtain a nested hierarchy of grids. Moreover, this strategy also makes the implementation of the bilinear interpolation easy. Therefore, we use bilinear interpolation, e.g., consider $\mathcal{U}_k$ of $u^k : \Omega_k^{ev} \to \mathbb{R}$, for $k = 1, \ldots, L$. We define a prolongation operator (among two grids Ω_k and Ω_{k-1}), $I_{k-1}^k : \mathcal{U}_{k-1} \to \mathcal{U}_k$ such that on each discretized rectangular partition, I_{k-1}^k is consistent with the bilinear finite elements.

Note that in coarsening by factor of three, the spatial placement of the coarse-grid points are on the same placement as of the fine-grid points; see Figure 2. Therefore, we use straight injection operator $I_k^{k-1} : \mathcal{U}_k \to \mathcal{U}_{k-1}$ (as the restriction operator) for transfer of variables (residuals and functions) form fine grids to coarse grids. Here, we remark that it is not necessary to use the straight injection operator as a *restriction operator*, one can use full or half weighting. We use straight injection because it is natural choice in the coarsening by a factor-of-three strategy.

4.2. **The multigrid algorithms .** To introduce our approach, we discuss the full approximation scheme (FAS) and the full multigrid method (FMG). The FAS scheme is a natural choice in the treatment of the boundary value problems (BVPs), whereas FMG scheme allows to improve the computational complexity of the FAS scheme. The FMG scheme is obtained combining a nested iteration strategy with the FAS scheme that we discuss next.

Next, we consider the discretized generalized Stokes system (3. 4)-(3. 7) and write this in compact form given by

$$A_k(X_k) = F_k. \tag{4. 9}$$

at the discretization level k for $X_k = (u_k, v_k, p_k)$.

Furthermore, we denote $X_k^{(l)} = S_k\left(X_k^{(l-1)}, F_k\right)$, the smoothing scheme given in Section 4.1. For the purpose of completeness, the FAS cycle for solving $A_k(X_k) = F_k$ (FMG algorithm for solving $A_L(X_L) = F_L$) is summarized as follows:

Algorithm 1**: FAS**(m_1, m_2).

(1) Solve $A_k(X_k) = F_k$ exactly, if $k = 1$.
(2) Pre-smoothing: $X_k^{(l)} = S_k(X_k^{(l-1)}, F_k)$, $(l = 1, \ldots, m_1)$;
(3) Compute the residual: $r_k = F_k - A_k(X_k^{(m_1)})$;
(4) Restriction of residuals: $r_{k-1} = I_k^{k-1} r_k$;
(5) Set $X_{k-1} = I_k^{k-1} X_k^{(m_1)}$;
(6) Set $F_{k-1} = r_{k-1} + A_{k-1}(X_{k-1})$
(7) Call FAS (m times) to solve $A_{k-1}(X_{k-1}) = F_{k-1}$;
(8) Correction: $X_k^{(m_1+1)} = X_k^{(m_1)} + I_{k-1}^k(X_{k-1} - I_k^{k-1} X_k^{(m_1)})$;
(9) Post-smoothing: $X_k^{(l)} = S_k(X_k^{(l-1)}, F_k)$, $(l = m_1 + 2, \ldots, m_1 + m_2 + 1)$;

Note that at each working level, we can perform m two-grid iterations. We have a V-cycle if $m = 1$ and for $m = 2$ we have a W-cycle; for more details see [21].

In the full multigrid method, first we solve the problem on the coarsest grid iteratively or by direct method until the reduction in algebraic error is bellow the discretization error, and then the problem is interpolated to the next finer level. Afterwords some multigrid cycles (V- or W-cycle) are performed, and the resulting solution is interpolated to the next finer level. This process is repeated until the solution has been obtained on the desired finest level.

A description of the full multigrid (FMG) method is given in the following algorithm.

Algorithm 2**: FMG for solving** $A_L(X_L) = F_L$.

(1) For $l = K < L$ set initial approximation u_l.
(2) If $l < L$ then interpolate to the next finer working level: $\tilde{X}_{l+1} = I_l^{l+1} X_l$;
(3) Apply FAS to solve $A_{l+1}(X_{l+1}) = F_{l+1}$, starting with $\tilde{X}_{l+1}$;
(4) Set $l := l + 1$; If $l < L$ go to step 2; else stop.

5. NUMERICAL EXPERIMENTS

In the following, we present the numerical investigation of the proposed full multigrid scheme for solving the generalized Stokes problem (with finite difference approximations on staggered grids). We consider a unit square domain $\Omega = (0, 1)^2$ with uniform grid size h (of each rectangular partition of the domain Ω) for all the presented numerical examples. Our implementation is executed in Matlab 8.5.0 (R2015a) on a laptop i7 with 1.86GHz and 4GB RAM.

5.1. **Example** 1**.** We consider the generalized Stokes problem (2. 1)-(2. 3) and take

$$\begin{aligned} f(x,y) &:= -\nu\left(4y(y-1)(2y-1)\left((x-1)^2 + 4x(x-1) + x^2\right)\right) \\ &- 12x^2(x-1)^2(2y-1) + \alpha\left(2x^2(x-1)^2 y(y-1)(2y-1)\right), \\ g(x,y) &:= \nu\left(4x(x-1)(2x-1)\left((y-1)^2 + 4y(y-1)\right)\right) + 12y^2(y-1)^2(2x-1) \\ &- \alpha\left(2y^2(y-1)^2 x(x-1)(2x-1)\right) + 1, \end{aligned}$$

that gives

$$\begin{aligned} u(x,y) &= 2x^2(x-1)^2y(y-1)(2y-1), \\ v(x,y) &= -2y^2(y-1)^2x(x-1)(2x-1), \\ p(x,y) &= y + C, \end{aligned}$$

where $C = -\frac{1}{2}$ is a constant (satisfying the condition $\int_\Omega p\,dx = 0$ to make the problem well-posed).

We apply the proposed FMG scheme to solve this problem numerically, where the smoothing consist of relaxing the momentum equations by Gauss-Seidel and the continuity equation by a *distributive relaxation* [4], respectively; see Section 4.1. We employ W-cycles with 2-pre and 2-post smoothing steps (unless stated otherwise) and stop the iterations (relaxation process) when the discrete L^2-norm of the residuals is less then the given tolerance, i.e., when

$$\max\left\{\|r_i^h\|_{L^2} : i \in \{u,v,p\}\right\} < tol = 10^{-6}.$$

TABLE 1. L^2-norm of errors and convergence history for Example 1; $\nu = 1$, $tol = 10^{-6}$

$N_x \times N_y$	$\|\mathbf{u}^h - \mathbf{u_e}^h\|_{\mathbf{L}^2}$	$\|p^h - p_e^h\|_{L^2}$	Itr	CPU
$\alpha = 0$				
18×18	$2.1556e-5$	$6.6252e-5$	3	0.48
54×54	$2.0638e-6$	$6.3650e-6$	3	3.82
162×162	$2.1771e-7$	$6.6452e-7$	3	31.96
$\alpha = 10$				
18×18	$1.8658e-5$	$8.5632e-5$	3	0.48
54×54	$1.7895e-6$	$8.3279e-6$	3	3.85
162×162	$1.8899e-7$	$8.7485e-7$	3	36.49
$\alpha = 10^2$				
18×18	$1.0437e-5$	$2.1000e-4$	4	0.58
54×54	$1.0289e-6$	$2.1354e-5$	3	3.55
162×162	$1.1004e-7$	$2.2868e-6$	3	33.44
$\alpha = 10^5$				
18×18	$6.3172e-6$	$8.9327e-2$	3	0.40
54×54	$7.0418e-7$	$9.8894e-3$	3	3.52
162×162	$7.8355e-8$	$1.1032e-3$	3	31.51

The number of iterations (W-cycles) and the CPU time, in seconds, taken for different mesh sizes is given in Table 1. The discrete L^2-norm of the absolute errors are also reported in Table 1. This demonstrate the second-order accuracy for the generalized Stokes equations and efficiency of the proposed multigrid staggered grid framework. That is, when the mesh is refined by a factor of 3, the norm of the errors have been reduced by a factor 3^2.

TABLE 2. Iteration history for Example 1, for different values of α and ν

$N_x \times N_y$	Itr	CPU
$\alpha = 10, \nu = 10^{-1}$		
18×18	3	0.71
54×54	2	3.12
162×162	2	24.26
$\alpha = 10^5, \nu = 10^{-1}$		
18×18	2	0.47
54×54	2	2.97
162×162	1	13.60
$\alpha = 10, \nu = 10^{-3}$		
18×18	2	0.53
54×54	2	3.09
162×162	2	23.67
$\alpha = 10^5, \nu = 10^{-3}$		
18×18	2	0.48
54×54	2	2.91
162×162	2	13.50
$\alpha = 10, \nu = 10^{-5}$		
18×18	2	0.54
54×54	2	2.87
162×162	2	13.78
$\alpha = 10^5, \nu = 10^{-5}$		
18×18	2	0.48
54×54	2	2.98
162×162	1	13.50

Next, we set $tol = 10^{-10}$ to make a comparison with the results given by Table $2-3$ in [9], where a multigrid solution with (equationwise) Gauss-Seidel smoother for the discrete Stokes equations is presented. With this tol, we report iteration count (W-cycles), average convergence factor ($\rho_i = \|r_i^h\|_{L^2}^{new} / \|r_i^h\|_{L^2}^{old} : i \in \{u, v, p\}$) with $\nu = 1$ in Table 3. Although we cannot make an exact comparison between the numerical results of our proposed multigrid scheme with [9] (as it uses coarsening by a factor of 2 and $h^{-1} = 256$, while we use coarsening by a factor of 3 and $h^{-1} = 128$ etc). We can at least see that the proposed solver performs better in respect of iteration count and show robust results in grid size h and $\alpha \geq 0$.

TABLE 3. Iteration counts and convergence factors for Example 1; $\nu = 1$, $tol = 10^{-10}$

$N_x \times N_y$	$Cycle(m_1, m_2)$	Itr	ρ
$\alpha = 0$			
128×128	$W(1,1)$	8	0.051
384×384	$W(1,1)$	7	0.050
$\alpha = 0$			
128×128	$W(2,2)$	5	0.019
384×384	$W(2,2)$	5	0.018
$\alpha = 10^5$			
128×128	$W(1,1)$	11	0.154
384×384	$W(1,1)$	10	0.101
$\alpha = 10^5$			
128×128	$W(2,2)$	6	0.043
384×384	$W(2,2)$	6	0.040

5.2. **Example** 2**.** Next, we consider the generalized Stokes equations (2. 1)-(2. 3) and take an exact solution as follows

$$\begin{aligned} u(x,y) &= (1 - cos(2\pi x))sin(2\pi y), \\ v(x,y) &= (cos(2\pi y) - 1)sin(2\pi x), \\ p(x,y) &= \frac{1}{3}x^3 - \frac{1}{12}. \end{aligned}$$

The values of the force term, $f(x,y)$ and $g(x,y)$, can be computed accordingly. We apply the proposed multigrid method (FMG) to solve this problem numerically, and employ W-cycles with 2-pre and 2-post smoothing steps. We use the same stopping criterion, i.e., we stop the iterations when the discrete L^2-norm of the residuals satisfy

$$\max\left\{\|r_i^h\|_{L^2} : i \in \{u, v, p\}\right\} < tol = 10^{-6}.$$

The discrete L^2-norm of errors and convergence history for $\alpha = 0, 10^1, 10^2, 10^5$ with $\nu = 1$ are reported in Table 4. This demonstrate the second-order accuracy of the proposed multigrid solver, i.e., the norm of the errors reduces by of a factor 3^2 whenever the mesh is refined by a factor of 3. Furthermore, iteration counts with CPU time (seconds) are also given in Table 4.

6. CONCLUSIONS

A multigrid scheme for generalized Stokes equations with coarsening by a factor of three was investigated. On staggered grids, finite differences were used to discretize the Stokes system and a distributive Gauss-Seidel smoothing scheme was employed. One advantage of the proposed multigrid solver is the fact that a nested hierarchy of staggered grids is obtained when coarsening by a factor of three is used. Moreover, this strategy simplify the

TABLE 4. L^2-norm of errors and convergence history for Example 2; $\nu = 1, tol = 10^{-6}$

$N_x \times N_y$	$\|\mathbf{u}^h - \mathbf{u_e}^h\|_{\mathbf{L}^2}$	$\|p^h - p_e^h\|_{L^2}$	Itr	CPU
$\alpha = 0$				
18×18	$8.8866e-3$	$6.1028e-3$	5	0.83
54×54	$9.3299e-4$	$2.3406e-4$	4	4.80
162×162	$1.0689e-4$	$8.8626e-6$	4	42.32
$\alpha = 10$				
18×18	$6.4841e-3$	$3.5608e-3$	5	0.73
54×54	$7.8397e-4$	$1.2295e-4$	4	4.84
162×162	$8.9779e-5$	$3.2755e-5$	4	46.89
$\alpha = 10^2$				
18×18	$2.7774e-3$	$2.5421e-3$	5	0.70
54×54	$3.2379e-4$	$1.0898e-3$	4	4.75
162×162	$3.6875e-5$	$1.5243e-4$	4	41.44
$\alpha = 10^5$				
18×18	$9.9099e-6$	$1.5524e-2$	5	0.60
54×54	$1.6923e-6$	$1.5534e-3$	4	4.52
162×162	$1.8732e-7$	$4.8356e-5$	4	40.41

intergrid transfer operators, i.e., the use of bilinear interpolation and restriction operator becomes easier. The iteration counts we have observed in this paper show better and robust results in grid size h, $\alpha \geqslant 0$, as compared to the recent work [9]. Promising numerical results indicate the feasibility and effectiveness of our proposed solution strategies to solve the time-dependent Stokes and/or Navier-Stokes equations [20] (and other related problems e.g., see [18]), which is our ongoing research work.

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