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# Radial Basis Function Solution for the LIBOR Market Model PDE

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**Abstract.** This research paper is intended at analyzing the interpolation of LIBOR (London Inter Bank Offer Rate) Model PDE (Partial Differential Equation) in one and two dimensions using Radial Basis Functions (RBF) on full grids. The LIBOR Market model is considered an effective and standard approach for pricing the derivatives which is based on interest rates. In recent times, Monte Carlo methods are often used in practice to price derivatives instruments because of the high dimensionality of the model. This research paper highlights the applicability of the RBF method rather than Finite Difference Method (FDM) for solving the LMM PDE, LIBOR Market Model, with the Bermudan Swaption or Chooser Option as a boundary condition. The results have suggested faster convergence to reference value than FDM in one dimension. Also, the calculation of price is similar to FDM in two dimension.

**AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09**
**Key Words:** Radial Basis Function(RBF), LIBOR Market Model(LMM), Convergence, Bermudan Swaption, Chooser Option,Full grid, Sparse grid.

## 1. Introduction

LIBOR Market model is considered as the best model from the family of interest rate models. LMM was developed by Brace and Musiela (1997) and models the LIBOR rates in the arbitrage-free way. The other main model that also belongs to the interest rate models family is the Swap Market Model. The SMM considers forward interest rates mechanisms. LMM prices the caps according to the Black Caps formula while SMM prices the caps according to Blacks swaption formula [10].

LMM is based on derivatives in the LIBOR market; derivatives are thought of as the basis of financial markets. LIBOR (London Inter-Bank Offer Rate) is a standard interest rate which is considered by the most of the worlds banks for their short term loans. LIBOR rates are determined by the British Bankers Association. Hedging of risk is one of the most crucial considerations for the investor, and that is why they always watch the interest rate derivatives. Here a question arises: what should be the fair price of forward contracts that an investor makes for hedging its future risk. The determination and modelling of the fair price is one of the most challenging tasks for financial industry in todays world. LMM is based on the pay-off of an interest rate derivative and LIBOR rates (forward). These forward LIBOR rate are log-normally distributed. Under a suitable forward measure, forward rates are anticipated to be log-normally distributed. The forward LIBOR rate leads to more stochastic elements in the model, which is why the dimensionality of the LMM can be very high. Presently, the most commonly used method for pricing the derivatives is the Monte Carlo method. This method has some drawbacks; in the Monte Carlo pricing of American derivatives we have to compute the optimal maturity exercise time, also the calculation of Greeks in Monte Carlo method is hard. Moreover, MC methods have low order of convergence in low dimensions, but they are harder to beat for higher dimensions as they become more and more competitive with respect to grid based methods. There are two popular derivatives existing in the interest rate markets, one is the Bermudan Swaption and other is the Chooser Option.The Bermudan Swaption is a type of financial instrument which gives the right but not the obligation for entering in an interest rate swapping contract and the expiry of these contracts are either on maturity dates or other exercise dates within maturity dates. The Bermudan Swap is widely used derivatives in markets; nonetheless no consensus exists about a suitable model for valuing these contracts. The Chooser Option is a contract that gives the holder to right to determine at a specified time whether the contract should correspond to a Put or Call option, thus gives right to the holder for a fixed time period. Modelling of contract prices becomes a challenging factor, even in this computer era where computer aided programs are available to solve many problems. In spite of this they do not carry the capacity to solve this problem in high dimensions,because it requires more memory, high dimension characteristics and timely and fast computation for gaining significant advantage [3]. At present, innovations and discoveries are growing rapidly and most of the problems were solved on the basis of these innovations. In this regard the field of mathematics gives some significant contributions by developing a newly technique involving Radial Basis Functions, to solve numerical simulations of partial differential equation. A Radial Basis Function is a function whose values depend only on the distance from its center. Radial Basis Function is alike to the finite difference method (FDM) and finite element method (FEM). RBF are used globally in projections by approximating partial differential methods while finite difference and element methods utilize the combination of two approaches for solving the solution such as, finite quotients and low order piece-wise continuous polynomials.

## 2. RADIAL BASIS FUNCTIONS

The RBF method will work in $n$ dimensional Euclidean space. The points where we are interested to compute the solution will be $x_1, x_2, ...x_m$ which are in $\mathbf{R}^n$ [4].
As in [8] the RBF approximation is

$$s(x) = \sum_{j=1}^{m} \lambda_j \Phi(\|x - x_j\|), \ \ x \in \mathbf{R}^n.$$

There are different kind of RBFs such as Multiquadric, Inverse Quadric, Gaussian, Inverse Multiquadric etc. In this paper the Multiquadric RBF is used, which is given by $\Phi(r) = \sqrt{r^2 + c^2}$.

RBFs have generally symmetric shape. Thus interpolation of the data on these symmetric shape graphs requires superior thought. Anisotropic radial basis function (ARBFs) can be defined as following [5]:

let $A \in \mathbf{R}^{d*d}$ be an invertible matrix and $\Phi(\|. - x_j\|)$ be the typical RBF which is centred at $x_j \in \mathbf{R}^d$. The ARBFs is defined by $\Phi_A(\|. - x_j\|) = \Phi(\|A(. - x_j)\|)$. Given a positive definite function $\Phi$ , then the the ARBF approximant $S_A$ will be defined as :

$$S_A(x) = \sum_{j=1}^{N} c_j \Phi_A(\|x - x_j\|), \ \ x \in \Omega.$$

## 3. SPARSE KERNEL-BASED INTERPOLATION

In this section we describe sparse kernel-based approximation which we will apply for the LMM PDE [6], which is described in the next section. Let $\Omega := [0, 1]^d$ and $\mathbf{l} = (l_1, ..., l_d) \in \mathbf{N}^d$, then the uniform grids $\mathbf{X_l} : \mathbf{l} \in N^d$ in $\Omega$ is defined by the meshsize $h_\mathbf{l} = 2^{-l} := (2^{-l_1}, ..., 2^{-l_d})$. The number of nodes in $\mathbf{X_l}$ is given by

$$N^\mathbf{l} = \prod_{i=1}^{d} (2^{l_i} + 1).$$

Till here, the transformation matrix corresponding multi-index $\mathbf{l} = (l_1, ..., l_d)$ is defined by

$$\mathbf{A_l} := \operatorname{diag}(2^{-l_1}, ..., 2^{-l_d}).$$

Therefore ARBF solution will be given by

$$S_{A_\mathbf{l}}(x) = \sum_{j=1}^{N^l} c_j \Phi_{A_\mathbf{l}}(\|x - x_j\|).$$

The sparse kernel-based approximant on the sparse grid $X^{n,d}$ will be calculated through the formula

$$\boldsymbol{S}_n(x) = \sum_{q=0}^{d-1} (-1)^q \binom{d-1}{q} \sum_{|\mathbf{l}|=n+(d-1)} S_{A_\mathbf{l}}(x).$$

## 4. LIBOR MARKET MODEL (LMM)

In financial markets, there are several kinds of financial assets that are traded. The most common instruments among these are the risk-free zero coupon bonds. These bonds pay a defined rate of return to its holders till its expiration/maturity date. let $D(t, T)$ be the price of a bond any given time $t$. The LIBOR (London Inter Bank Offer Rate) Market Model is based on the idea of an arbitrage free economy with $N$ numbers of zero coupon bonds, where each bond is determined by a $d$-dimensional Wiener process. The forward LIBOR rate is defined as the given interest rate on which a borrower can generate debt financing from a predetermined future time and a specified maturity date. For our market, there are a number $N$ of given rates. Lets assume that an investor establishes a contract at time $t$ with starting time $T_0$ and ending time $T_N$. The given contract is determined by $L_i(t)$.

If we observe $D(t, T)$ and $D(t, T_{i+1})$,then the forward interest rate $L_i(t)$ will be found on no arbitrage condition. If the holder enter into forward contract then the contract will give holder pay-off as $\delta_i L_i(t)$ ans as a result the the contract will worth $\delta_i L_i(t) D(t, T_{i+1})$ in maturity time.
To obtain zero initial value, we must follow the condition as

$$D(t, T_i) - D(t, T_{i+1}) = \delta_i L_i(t) D(t, T_{i+1}),$$

given $D(T_j, T_j) = 1$ the following formula will be obtained for $D(T_j, T_N)$

$$D(T_j, T_N) = \prod_{k=j}^{N-1} \frac{D(T_j, T_{k+1})}{D(T_j, T_k)} = \prod_{k=j}^{N-1} \frac{1}{1 + \delta_k L_k(T_j)}.$$

More detailed formulation on this concept can be found in [2].

According to [9] the model in general form is following the PDE as:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{N-1} \rho_{ij} \sigma_j(t) \sigma_i(t) L_i L_j \frac{\partial^2 u}{\partial L_i \partial L_j} + \Sigma_{j=1}^{N-1} \mu_j(t) L_j \frac{\partial u}{\partial L_j} = 0,$$

where

$$\mu_j(t) = \begin{cases} -\sigma_j(t) \Sigma_{k,j=1}^{N-1} \frac{\delta_k L_k(t)}{1+\delta_k L_k(t)} \rho_{jk} \sigma_k(t) & j < N-1, \\ 0, & j = N-1. \end{cases}$$

## 5. RBF FOR LMM, N=2

In this part we will consider first forward rate as given and will approximate the second one with the help of RBF. The first forward rate, $L_0$, can be assumed as given because we are always in LMM PDE interested to predict the next forward rate,$u(L_1, t)$, based on the previous forward rates. Moreover, it is the way how boundary condition can be applied. Also, the Bermudan Swaption considered as the boundary condition [9] with general definition

$$VBS_w(T_i, ..., T_N) = max(VBS_w(T_{i+1}, ...T_N; T_i, VS_w(T_i, ..., T_N); T_i)),$$

where

$$VBS_w(T_N; T_N) = 0$$

and

$$VS_w((T_i, ...T_N); T_i) = \sum_{j=i}^{N-1} \delta_i (L_j(T_i) - K) D(T_i, T_{i+1})$$

given $D(t,T)$ is the price of bond at any time $t$ and $\delta_i = T_{i+1} - T_i$. The value of swaption depends on $L_1(t)$. To specify the boundary condition to be used for PDE we will use

$$\alpha(t) = VBS_w(T_0, T_1, T_2) = (VS_w(T_0, T_1, T_2), 0)_+ \quad if \quad L_1 = 0,$$

$$\beta(t) = VBS_w(T_0, T_1, T_2) = (VS_w(T_0, T_1, T_2), L_1 - K)_+ \quad if \quad L_1 \to infinity$$

If $N = 2$ the PDE will change into

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 L_1^2 \frac{\partial^2 u}{\partial L_1^2} = 0.$$

Following the method of lines [1] and with the help of MQ RBF, using $T = 1, \sigma = 0.2473, E = 0.055, \theta = 0.5, c = 0.2$, the following results have been achieved, which are presented in Table 1.

TABLE 1. RBF results for LMM N=2

| Number of nodes | Estimated Price(RBF) | Estimated Price(FDM) | Error(RBF) | Error(FDM) |
|---|---|---|---|---|
| 17 | 0.6737 | 0.8866 | 0.0146 | 0.22750 |
| 33 | 0.6685 | 0.7709 | 0.0094 | 0.11181 |
| 65 | 0.6587 | 0.6786 | -0.00042 | 0.01956 |
| 129 | 0.6597 | 0.6659 | 0.000564 | 0.00688 |
| 257 | 0.6591 | 0.6611 | 0.00003 | 0.00209 |
| 513 | 0.6592 | 0.6593 | 0.00006 | 0.00025 |
| 1025 | 0.6591 | 0.6591 | 0.00002 | 0.00005 |

Table 1 shows the RBF function in one dimension; here the speed of convergence is faster than the other method such as finite difference method. Although the convergence rates in two methods has not big difference, 2 in FDM and 1.7 for RBF approach, it worth studying the case in higher dimensions to get better understanding of advantages and possible disadvantages of RBF.

Here briefly MOL, method of lines, which have been used in this part will be presented, more details in [9].

Applying MQ, multi quadric, RBF for space axis $L = (L_0, L_1)$ where $L_1$ is our objective and would be discrete into $L_1 = (m_1, m_2, ...m_N)$, and $L_0$ is the given rate which will come from previous data, the target function $u(L_1, t)$ can be written as

$$u(L_1, t) = \sum_{k=1}^{N} \lambda_k \Phi(||L_1 - m_k||),$$

where $\lambda_k$ are functions of time. Substituting $u$ and all its needed derivatives respect to $L_1$ into the PDE we will get

$$\sum_{k=1}^{N} \frac{\partial \lambda_k}{\partial t} \Phi(||L_1 - m_k||_2) + \frac{1}{2}\sigma^2 L_1^2 \sum_{k=1}^{N} \lambda_k \frac{\partial^2 \Phi(||L_1 - m_k||_2)}{\partial L_1^2} = 0.$$

Rewriting the above equation into matrix notation the following equation will be obtained

$$[\Phi][\frac{\partial \lambda}{\partial t}] + \sigma^2 L_1^2 [\Phi^{''}][\lambda] = 0.$$

Then using $\theta$ method to approximate time derivative and applying boundary condition with the help of Boundary Update Procedure,BUP, and also considering initial condition, European call option pay-off, we can obtain the result has been given in Table 1.

## 6. RBF FOR LMM N=3

In this case, putting $N = 3$, the PDE changed into:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left[\sigma_1^2 L_1^2 \frac{\partial^2 u}{\partial L_1 \partial L_2} + 2\rho\sigma_1\sigma_2 L_1 L_2 \frac{\partial^2 u}{\partial L_1 \partial L_2} + \sigma_2^2 L_2^2 \frac{\partial^2 u}{\partial L_2^2}\right] - \rho\sigma_1\sigma_2 L_1 \frac{\delta_2 L_2}{1+\delta_2 L_2} \frac{\partial u}{\partial L_1} = 0,$$

where $\rho$ is correlation between $L_1$ and $L_2$. The Chooser option here is used as a boundary condition [9], which is kind of options will give right to holder to borrow money for time $\delta_{N-1}$. The expiry time is given here as $T_{N-1}$. To modify the definition in mathematics format in order to use as boundary condition for the PDE we will apply Chooser option as follow

$$V(T_N) = \delta_{N-1}(max(L_i(T_i)) - K, 0)).$$

The result obtained with $T = 2, \rho = \exp(-0.1), \theta = 0.5, E = 0.055 \; and \; \sigma = [0.2245, 0.2473]$ is shown in Table 2:

TABLE 2. RBF results for LMM N=3

| Number of nodes | Sparse grid solution(RBF) | Full grid solution(FDM) |
|---|---|---|
| 9 | 13.2008 | 9.8945 |
| 17 | 10.4583 | 9.8829 |
| 33 | 10.5568 | 10.0700 |

the FDM results are from [2].

As it is shown in the Table 2, the RBF and FDM approach the same number.

## 7. CONCLUSION AND FURTHER STUDY

This study aims to predict LIBOR Market Model forward interest rate with the help of Radial Basis Function in 1 and 2 dimension and compare its result with other methods such FDM, which have previously been used. We were able to show that in 1 dimension the RBF method has faster convergence than FDM, and in dimension 2 RBF and FDM converge to the same number. We are ambitious to apply the sparse grid collocation and compare its results with full grid collocation which was used in this paper. It is expected that sparse grid method will have faster convergence (for the same number of nodes used) than the full grid method.

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