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Exact Solution for Some Rotational Motions of Fractional Oldroyd-B Fluids Between Circular Cylinders

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Abstract. In this analysis motion of an incompressible fractional Oldroyd-B fluid [IFOBF] in an annulus is studied. The movement in fluid is created by the motion of the both cylinders. At $(t = 0^+)$, the inner cylinder exerts rotational shear which is time dependent and outer cylinder moves with time depended linear velocity. The analysis of velocity field and shear stress are made with the help of integral transform techniques. For simplicity and better perception, generalized R and G functions are used to represent the obtained results. These solutions are satisfied all the imposed conditions. By imposing favourable limits, general solution is reduced to ordinary Oldroyd-B, Maxwell, second grade and Newtonian fluids for the same motion. From the final expressions of velocity fields and shear stress, we recover the previously obtained results by choosing some limits. Finally, the effect of parameters and their comparison are explained graphically.

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Key Words: Fractional Oldroyd-B fluid; Integral transforms; Exact solution; Velocity field; Shear stress

1. NOTATIONS

μ	dynamic viscosity (Kg/m sec)
ν	kinematic viscosity $(\frac{m^2}{sec})$
v	velocity (m/sec)
S	extra stress tensor $(\frac{N}{m^2})$
τ	shear stress $(\frac{N}{m^2})$
g_1	the dimension of this constant depends on the parameter d
g_2	a constant whose dimension is $(\frac{m}{(sec)^2})$
d	Positive real number
η	fractional parameter(sec^{η})
γ	fractional parameter (sec^{γ})
R_{1c}	inner cylinder(m)
R_{2c}	outer cylinder(m)
s	Laplace transform parameter
Λ	relaxation parameter(sec)
Λ_r	retardation parameter (sec)

2. INTRODUCTION

During the past decades, researchers show considerable concern to the flow problems of Non-Newtonian fluids (blood, cosmetic fluids, paint, polymer, sludge, certain oils and greases, etc) as compare to Newtonian fluids. It is due to broad applications of non-Newtonian fluids in food industry, petroleum industry, chemical engineering, geophysics, biological analysis etc. Fluids have a large number of applications in our practical life[13, 25, 32]. Marshall [24] consider the porous medium into account and studied about the flow of viscoelastic fluids. The Non-Newtonian fluids have nonlinear stress-strain relationship. The behaviour of various viscoelastic fluids can not be studied by using Newtonian fluid model. So, Non-Newtonian fluids [35, 36] are essential to study because they involves in many aspects of life. Since non-Newtonian fluids have complex structure and typical properties therefore a number of fluid models presented in literature. The Dunn et al. [6] studied the fluid of differential type in critical review sense and Rajagopal et al. [28] worked on the rate type fluid models. Among them the rate type fluids model have special concentrated by researchers due to their several technological applications[2, 3, 4, 10, 12, 15, 16, 17, 18, 19, 20, 27, 41, 43] .

Fluid in translating, oscillating or rotating cylindrical system is very important and interesting [33]. The analysis and exact solutions for generalized second grade fluid can be seen in [1, 8, 9, 29, 30, 37, 39] within an annulus and over a flat plate. The study of the mechanism of viscoelastic fluids is very critical. It has a lot of applications in many walks of industry, such as chemical industry, bio engineering and oil exploitation.

Viscoelasticity is the property of fluids which has major effect on the motion of the fluids or movement of the bodies through fluids even at micro level. It is practically verified that differential equations with non-integer order is very reasonable to describe the viscoelasticity and some other properties of the fluids. Thats why fractional calculus approach has

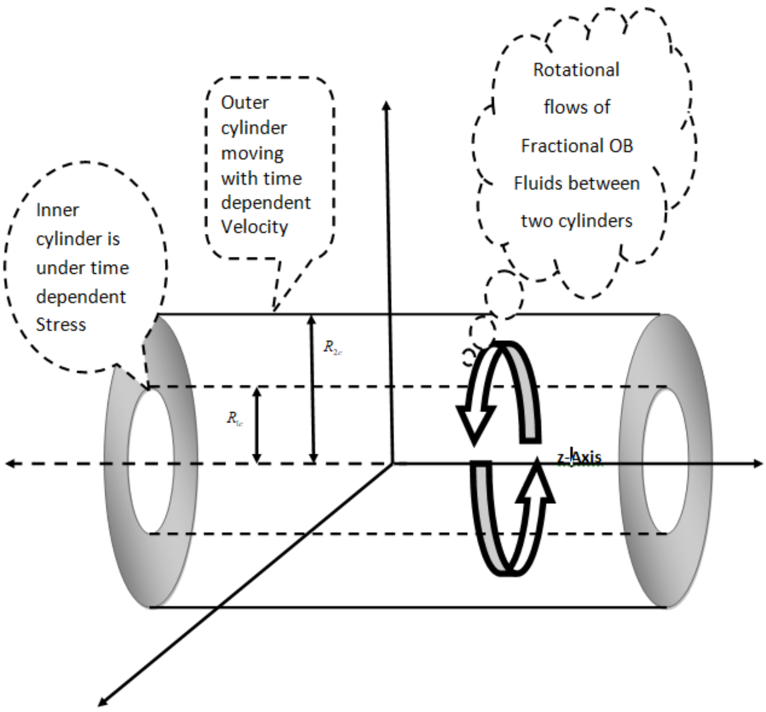

FIGURE 1. Geometry of the flow problem

been used mostly in flow problems for last few years. Some investigations with fractional derivatives can be seen [4, 31, 42]

In (1923) Taylor [38] investigate his famous result related to stability of viscous fluids through annulus of two cylinders. A successful attempt for finding solutions for the helical flow of unsteady fluids was made by Srivastava [34]. Some important material and applications related to this work can be seen [11, 14, 21] as per interest of readers.

In this article our task is to determine the velocity field and shear stress of an [IFOBF] in an annulus of coaxial moving cylinders having infinite length and to recover some previous results. At the moment $(t = 0^+)$, inner cylinder pulled by rotational shear which is time dependent and the outer cylinder moves with some velocity. The well known integral transforms namely Laplace and Hankel are used to solve the above flow model. The final results are expressed in the form of generalized *R* and *G* functions. As limiting cases results for Oldroyd-B, Maxwell, second grade and Newtonian fluids are also established with ordinary and fractional derivatives.

3. GOVERNING EQUATIONS

For the considered flow, let velocity and stress tensor are [12]

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\mathbf{e}_\theta, \quad \mathbf{S} = \mathbf{S}(r,t), \tag{3. 1}$$

where $\mathbf{e}_\theta$ indicates the flow direction in cylindrical coordinate system. Initially an [IFOBF] in an annulus of concentric circular cylinders are at rest. At $(t = 0^+)$, cylinders start moving. Inner cylinder moves by the application of stress on its boundary which is time dependent and outer cylinder moves with time dependent velocity. The constraint of incompressibility for above flow model is satisfied ultimately. In addition, we assume, initially fluid and cylinders are at rest

$$\mathbf{v}(r,0) = \mathbf{0}, \qquad \mathbf{S}(r,0) = \mathbf{0}. \tag{3. 2}$$

The governing equations for the said fluid are [27, 18],

$$\left(1 + \Lambda \frac{\partial}{\partial t}\right) \frac{\partial w(r,t)}{\partial t} = \nu \left(1 + \Lambda_r \frac{\partial}{\partial t}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t); \tag{3. 3}$$

$$\left(1 + \Lambda \frac{\partial}{\partial t}\right) \tau(r,t) = \mu \left(1 + \Lambda_r \frac{\partial}{\partial t}\right) \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t), \tag{3. 4}$$

The system of governing equations for [IFOBF], expressing the similar motion, is obtained by replacing the ordinary derivative from Eqs. (3. 3) and (3. 4), with the fractional differential operator [23, 26]

$$D_t^\gamma g(t) = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t \frac{g(\tau)}{(t-\tau)^\gamma} d\tau, & 0 \le \gamma < 1; \\ \frac{d}{dt} g(t), & \gamma = 1, \end{cases} \tag{3. 5}$$

where $\Gamma(\cdot)$ is the Gamma function.

Finally, the governing equations for [IFOBF] model are [41]

$$(1 + \Lambda^\eta D_t^\eta) \frac{\partial w(r,t)}{\partial t} = \nu \left(1 + \Lambda_r^\gamma D_t^\gamma\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t); \tag{3. 6}$$

$$(1 + \Lambda^\eta D_t^\eta) \tau(r,t) = \mu \left(1 + \Lambda_r^\gamma D_t^\gamma\right) \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t), \tag{3. 7}$$

When $\eta\,, \gamma \to 1$, then above fractional model Eqs. (3. 6) and (3. 7) reduced to ordinary model Eqs. (3. 3) and (3. 4), because

$$D_t^1 g = \frac{dg}{dt}. \tag{3. 8}$$

4. FLOW THROW AN ANNULUS

We assume, at $(t = 0)$, the fluid between coaxial cylinders of radii R_{1c} and $R_{2c}(> R_{1c})$ are at rest. At $(t = 0^+)$, both cylinders start moving, outer cylinder moves with some velocity $g_2 t$ and the inner one start rotating around axis due to torsional shear stress on its boundary which is time dependent and defined as

$$\tau(R_{1c}, t) = \frac{g_1}{\Lambda^\eta} \Gamma(d+1) R_{\eta,-d-1}\left(-\frac{1}{\Lambda^\eta}, t\right); \qquad 0 < \eta < 1, \quad d \ge 0, \tag{4. 9}$$

the well known generalized R function is defined as [22]

$$R_{\ell,\jmath}(\sigma\,,t)=L^{-1}\left\{\frac{s^{\jmath}}{(s^{\ell}-\sigma)}\right\}$$
$$=\sum_{m=0}^{\infty}\frac{\sigma^{m}t^{(m+1)\ell-\jmath-1}}{\Gamma[(m+1)\ell-\jmath]};\ \mathrm{Re}(\ell-\jmath)>0,\ \ \mathrm{Re}(s)>0,\ \ \left|\frac{\sigma}{s^{\ell}}\right|<1. \quad (4.\ 10)$$

The initial and boundary conditions are

$$w(r,0)=\frac{\partial w(r,0)}{\partial t}=0,\qquad \tau(r,0)=0,\quad r\in[R_{1c},R_{2c}], \quad (4.\ 11)$$

and

$$(1+\Lambda^{\eta}D_t^{\eta})\,\tau(r,t)|_{r=R_{1c}}=\mu\,(1+\Lambda_r^{\gamma}D_t^{\gamma})\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t)|_{r=R_{1c}}$$
$$=g_1t^{d},t>0,\quad d\geq 0,\quad , \quad (4.\ 12)$$

$$w(R_{2c},t)=g_2t\quad,\quad t>0. \quad (4.\ 13)$$

Eq. (4. 9) is the solution of Eq. (4. 12). For the solution of above problem there exits a class of methods in literature but we use most efficient, systematic and powerful integral transform techniques.

4.1. **Calculation of the velocity field.** By the implimentation of integral transformation, Eqs.(3. 6), (4. 12), (4. 13) implies

$$\left(s+\Lambda^{\eta}s^{\eta+1}\right)\overline{w}(r,s)=\nu\,(1+\Lambda_r^{\gamma}s^{\gamma})\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{w}(r,s), \quad (4.\ 14)$$

$$\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{w}(R_{1c},s)=\frac{g_1\Gamma(d+1)}{\mu s^{d+1}\,(1+\Lambda_r^{\gamma}s^{\gamma})},$$

$$\overline{w}(R_{2c},s)=\frac{g_2}{s^2}. \quad (4.\ 15)$$

where $\overline{w}(r,s)$, $\overline{w}(R_{1c},s)$ and $\overline{w}(R_{2c},s)$ represent the Laplace transforms of $w(r,t)$, $w(R_{1c},t)$ and $w(R_{2c},t)$ respectively and s is Laplace parameter. The Hankel transform of $\overline{w}(r,s)$ is $\overline{w}_H(r_\xi,s)$, and defined as [7, 40]

$$\overline{w}_H(r_\xi,s)=\int_{R_{1c}}^{R_{2c}}r\,\overline{w}(r,s)B_{1w}(r,r_\xi)dr\,, \quad (4.\ 16)$$

where

$$B_{1w}(r,r_\xi)=J_1(rr_\xi)Y_2(R_{1c}r_\xi)-J_2(R_{1c}r_\xi)Y_1(rr_\xi), \quad (4.\ 17)$$

and r_ξ are the positive roots of $B(R_{2c},r)=0$, $J_a(.)$ and $Y_a(.)$ represent bessel functions of the first and second kind of order a. $rB_{1w}(r,r_\xi)$ is multiplied with Eq. (4. 14) and then integrate from R_{1c} to R_{2c} with respect to r, using Eq.(4. 15) and the predefined identity [20]

$$\int_{R_{1c}}^{R_{2c}} r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{w}(r,s)B_{1w}(r,r_\xi)dr = -r_\xi^2\overline{w}_H(r_\xi,s)+\frac{2}{\pi r_\xi}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)$$
$$\times\,\overline{w}(r,s)|_{r=R_{1c}}+R_{2c}r_\xi\overline{w}(R_{2c},s)\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$=-r_\xi^2\overline{w}_H(r_\xi,s)+\frac{2}{\pi r_\xi}\frac{g_1\Gamma(d+1)}{\mu s^{d+1}\left(1+\Lambda_r^\gamma s^\gamma\right)}$$
$$+\,R_{2c}r_\xi\frac{g_2}{s^2}\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right], \tag{4. 18}$$

we obtained

$$\overline{w}_H(r_\xi,s)=\frac{2g_1\Gamma(d+1)}{\rho\pi r_\xi}\frac{1}{\left[s^{d+1}(s+\Lambda^\eta s^{\eta+1}+\nu r_\xi^2+\nu r_\xi^2\Lambda_r^\gamma s^\gamma)\right]}$$
$$+\,\nu R_{2c}r_\xi g_2\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$\times\,\frac{1+\Lambda_r^\gamma s^\gamma}{\left[s^2(s+\Lambda^\eta s^{\eta+1}+\nu r_\xi^2+\nu r_\xi^2\Lambda_r^\gamma s^\gamma)\right]}. \tag{4. 19}$$

for more suitable, we can rewrite above equation as

$$\overline{w}_H(r_\xi,s)=\frac{2g_1}{\mu\pi {r_\xi}^3}\frac{\Gamma(d+1)}{s^{d+1}}-\frac{2g_1\Gamma(d+1)}{\mu\pi {r_\xi}^3}\frac{1+\Lambda^\eta s^\eta+\nu\, r_\xi^2\Lambda_r^\gamma s^{\gamma-1}}{s^d(s+\Lambda^\eta s^{\eta+1}+\nu r_\xi^2+\nu r_\xi^2\Lambda_r^\gamma s^\gamma)}$$

$$+\frac{g_2R_{2c}}{r_\xi}\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$\times\left[\frac{1}{s^2}-\frac{1+\Lambda^\eta s^\eta}{s(s+\Lambda^\eta s^{\eta+1}+\nu r_\xi^2+\nu r_\xi^2\Lambda_r^\gamma s^\gamma)}\right]. \tag{4. 20}$$

Since Inverse Hankel transformation [15]

$$\overline{w}(r,s)=\frac{\pi^2}{2}\sum_{\xi=1}^{\infty}\frac{r_\xi^2J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)}\overline{w}_H(r_\xi,s). \tag{4. 21}$$

Applying inverse Hankel transform given by Eq. (4. 21) to Eq. (4. 20) and using well known result

$$\int_{R_{1c}}^{R_{2c}}\left(r^2-{R_{2c}}^2\right)B_{1w}(r,r_\xi)dr=\frac{4}{\pi r_\xi^3}\left(\frac{R_{2c}}{R_{1c}}\right)^2, \tag{4. 22}$$

we get

$$\begin{aligned}\overline{w}(r,s) &= \frac{g_1}{2\mu}\left(r-\frac{R_{2c}^2}{r}\right)\left(\frac{R_{1c}}{R_{2c}}\right)^2\frac{\Gamma(d+1)}{s^{d+1}} - \frac{\pi g_1 \Gamma(d+1)}{\mu}\\ &\times \sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\\ &\times \frac{1+\Lambda^\eta s^\eta+\nu\, r_\xi^2\Lambda_r^\gamma s^{\gamma-1}}{s^d(s+\Lambda^\eta s^{\eta+1}+\nu r_\xi^2+\nu r_\xi^2\Lambda_r^\gamma s^\gamma)} + \frac{\pi^2 g_2 R_{2c}}{2}\sum_{\xi=1}^{\infty}\frac{r_\xi J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\\ &\times\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]\\ &\times\left[\frac{1}{s^2}-\frac{1+\Lambda^\eta s^\eta}{s(s+\Lambda^\eta s^{\eta+1}+\nu r_\xi^2+\nu r_\xi^2\Lambda_r^\gamma s^\gamma)}\right]. \end{aligned} \tag{4. 23}$$

By using identity [17]

$$\begin{aligned}\frac{1}{(s+\Lambda^\eta s^{\eta+1}+\nu r_\xi^2+\nu r_\xi^2\Lambda_r^\gamma s^\gamma)} &= \tfrac{1}{\Lambda^\eta}\textstyle\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\tfrac{\ell!}{n!(\ell-n)!}\left(\tfrac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\\ &\times\Lambda_r^{\gamma n}\tfrac{s^{\gamma n-\ell-1}}{(s^\eta+\Lambda^{-\eta})^{\ell+1}}, \end{aligned}\tag{4. 24}$$

Equation (4. 23) becomes

$$\begin{aligned}\overline{w}(r,s) &= \frac{g_1}{2\mu}\left(r-\frac{R_{2c}^2}{r}\right)\left(\frac{R_{1c}}{R_{2c}}\right)^2\frac{\Gamma(d+1)}{s^{d+1}} - \frac{\pi g_1\Gamma(d+1)}{\mu\Lambda^\eta}\\ &\times\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\frac{\ell!}{n!(\ell-n)!}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\\ &\times\Lambda_r^{\gamma n}\frac{s^{\gamma n-\ell-d-1}+\Lambda^\eta s^{\eta+\gamma n-\ell-d-1}+\nu\Lambda_r^\gamma r_\xi^2 s^{\gamma(n+1)-\ell-d-2}}{(s^\eta+\Lambda^{-\eta})^{\ell+1}}\\ &+\frac{\pi^2 g_2 R_{2c}}{2}\sum_{\xi=1}^{\infty}\frac{r_\xi J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\\ &\times\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]\left[\frac{1}{s^2}\right.\\ &\left.-\frac{1}{\Lambda^\eta}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\frac{\ell!}{n!(\ell-n)!}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\Lambda_r^{\gamma n}\frac{s^{\gamma n-\ell-2}+\Lambda^\eta s^{\eta+\gamma n-\ell-2}}{(s^\eta+\Lambda^{-\eta})^{\ell+1}}\right]. \end{aligned}\tag{4. 25}$$

To find the required velocity field, we apply discreet inverse Laplace transform instead of lengthy calculations of residues and contour integrations, using definition of generalized $G_{\ell,\jmath,\zeta}(\sigma\,,\,t)$ function which is defined as [22]

$$G_{\ell,\jmath,\zeta}(\sigma\,,t) = L^{-1}\left\{\frac{s^\jmath}{(s^\ell-\sigma)^\zeta}\right\} =$$

$$\sum_{i=0}^{\infty}\frac{\sigma^i\,\Gamma(\zeta+i)}{\Gamma(\zeta)\Gamma(i+1)}\frac{t^{(\zeta+i)\ell-\jmath-1}}{\Gamma[(\zeta+i)\ell-\jmath]}\,;\ \mathrm{Re}(\ell\zeta-\jmath)>0,\ \mathrm{Re}(s)>0,\quad \left|\frac{\sigma}{s^\ell}\right|<1. \tag{4. 26}$$

Finally we find

$$w(r,t)=\frac{1}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r-\frac{R_{2c}^2}{r}\right)g_1t^d-\frac{\pi g_1\Gamma(d+1)}{\mu\Lambda^\eta}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\frac{\ell!}{n!(\ell-n)!}\Lambda_r^{\gamma n}\times\Big[G_{\eta,\gamma n-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)+\Lambda^\eta$$
$$\times G_{\eta,\eta+\gamma n-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)+\nu\, r_\xi^2\Lambda_r^\gamma G_{\eta,\gamma(n+1)-\ell-d-2,\ell+1}\left(-\Lambda^{-\eta},t\right)\Big]+\frac{\pi^2g_2R_{2c}}{2}$$
$$\times\sum_{\xi=1}^{\infty}\frac{r_\xi J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$\times\Bigg[t-\frac{1}{\Lambda^\eta}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\frac{\ell!}{n!(\ell-n)!}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\Lambda_r^{\gamma n}$$
$$\times\left(G_{\eta,\gamma n-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)+\Lambda^\eta G_{\eta,\eta+\gamma n-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)\right)\Bigg]. \tag{4. 27}$$

Alternating form of velocity field is

$$w(r,t)=\frac{1}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r-\frac{R_{2c}^2}{r}\right)g_1t^d-\frac{\pi g_1\Gamma(d+1)}{\mu\Lambda^\eta}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\frac{\ell!}{n!(\ell-n)!}\Lambda_r^{\gamma n}\times\Big[G_{\eta,\gamma n-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)$$
$$+\Lambda^\eta G_{\eta,\eta+\gamma n-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)+\nu\, r_\xi^2\Lambda_r^\gamma G_{\eta,\gamma(n+1)-\ell-d-2,\ell+1}\left(-\Lambda^{-\eta},t\right)\Big]$$
$$+\frac{g_2R_{2c}}{{R_{2c}}^2-{R_{1c}}^2}\left(r-\frac{R_{1c}^2}{r}\right)t+\frac{1}{\Lambda^\eta}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\frac{\ell!}{n!(\ell-n)!}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\Lambda_r^{\gamma n}\Big[(\Phi_1-\Phi_2\nu r_\xi^2)G_{\eta,\gamma n-\ell-3,\ell+1}$$
$$\left(-\Lambda^{-\eta},t\right)+(\Lambda^\eta\Phi_1-\Phi_2\nu\Lambda_r^\gamma r_\xi^2)G_{\eta,\gamma n-\ell-3+\gamma,\ell+1}\left(-\Lambda^{-\eta},t\right)-\Phi_2$$
$$\times G_{\eta,\gamma n-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)-\Phi_2\Lambda^\eta G_{\eta,\eta+\gamma n-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)\Big], \tag{4. 28}$$

where $\Phi_1=\pi r_\xi^2g_2\nu\frac{J_2(R_{1c}r_\xi)}{J_1(R_{2c}r_\xi)}$ and $\Phi_2=\frac{2g_2R_{2c}}{{R_{2c}}^2-{R_{1c}}^2}\left(\frac{R_{1c}}{R_{2c}}\right)^2$.

4.2. **Calculation of the shear stress.**

From Eq.(3. 7), we get

$$\tau(r,t)=\frac{\mu\left(1+\Lambda_r^\gamma D_t^\gamma\right)}{\left(1+\Lambda^\eta D_t^\eta\right)}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t) \tag{4. 29}$$

using the Laplace transformation, Eq. (4. 29) implies

$$\overline{\tau}(r,s)=\frac{\mu\left(1+\Lambda_r^{\gamma}s^{\gamma}\right)}{(1+\Lambda^{\eta}s^{\eta})}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{w}(r,s) \tag{4. 30}$$

using Eq. (4. 23) in Eq. (4. 30), we obtain

$$\begin{aligned}\overline{\tau}(r,s)&=g_1\Gamma(d+1)\left(\frac{R_{1c}}{r}\right)^2\frac{1}{s^{d+1}(1+\Lambda^{\eta}s^{\eta})}+\pi g_1\Gamma(d+1)\\&\times\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_{\xi})B_{2w}(r,r_{\xi})}{[J_2^2(R_{1c}r_{\xi})-J_1^2(R_{2c}r_{\xi})]}\frac{1}{s^d(s+\Lambda^{\eta}s^{\eta+1}+\nu r_{\xi}^2+\nu r_{\xi}^2\Lambda_r^{\gamma}s^{\gamma})}\\&-\frac{\mu\pi^2g_2R_{2c}}{2}\sum_{\xi=1}^{\infty}\frac{r_{\xi}^2J_1^2(R_{2c}r_{\xi})B_{2w}(r,r_{\xi})}{}\\&\times[J_2^2(R_{1c}r_{\xi})-J_1^2(R_{2c}r_{\xi})]\left[Y_2(R_{1c}r_{\xi})J_2(R_{2c}r_{\xi})-J_2(R_{1c}r_{\xi})Y_2(R_{2c}r_{\xi})\right]\\&\times\left[\frac{1+\Lambda_r^{\gamma}s^{\gamma}}{s^2(1+\Lambda^{\eta}s^{\eta})}-\frac{1+\Lambda_r^{\gamma}s^{\gamma}}{s(s+\Lambda^{\eta}s^{\eta+1}+\nu r_{\xi}^2+\nu r_{\xi}^2\Lambda_r^{\gamma}s^{\gamma})}\right].\end{aligned} \tag{4. 31}$$

where

$$B_{2w}(r,r_{\xi})=J_2(rr_{\xi})Y_2(R_{1c}r_{\xi})-J_2(R_{1c}r_{\xi})Y_2(rr_{\xi}). \tag{4. 32}$$

By using Eq. (4. 24) in Eq. (4. 31) and then applying discrete inverse Laplace transform and keeping in mind Eqs. (4. 26) and (4. 10), we obtain tangential stress of the form

$$\begin{aligned}\tau(r,t)&=g_1\Gamma(d+1)\left(\frac{R_{1c}}{r}\right)^2\left(\frac{1}{\Lambda^{\eta}}\right)R_{\eta,-1-d}\left(-\Lambda^{-\eta},t\right)+\frac{\pi g_1\Gamma(d+1)}{\Lambda^{\eta}}\\&\times\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_{\xi})B_{2w}(r,r_{\xi})}{[J_2^2(R_{1c}r_{\xi})-J_1^2(R_{2c}r_{\xi})]}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\left(\frac{-\nu r_{\xi}^2}{\Lambda^{\eta}}\right)^{\ell}\frac{\ell!}{n!(\ell-n)!}\Lambda_r^{\gamma n}\\&\times G_{\eta,\gamma n-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)-\frac{\mu\pi^2g_2R_{2c}}{2\Lambda^{\eta}}\sum_{\xi=1}^{\infty}\frac{r_{\xi}^2J_1^2(R_{2c}r_{\xi})B_{2w}(r,r_{\xi})}{[J_2^2(R_{1c}r_{\xi})-J_1^2(R_{2c}r_{\xi})]}\\&\times\left[Y_2(R_{1c}r_{\xi})J_2(R_{2c}r_{\xi})-J_2(R_{1c}r_{\xi})Y_2(R_{2c}r_{\xi})\right]\\&\times\left[R_{\eta,-2}\left(-\Lambda^{-\eta},t\right)+\Lambda_r^{\gamma}R_{\eta,\gamma-2}\left(-\Lambda^{-\eta},t\right)-\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\left(\frac{-\nu r_{\xi}^2}{\Lambda^{\eta}}\right)^{\ell}\frac{\ell!}{n!(\ell-n)!}\right.\\&\left.\times\Lambda_r^{\gamma n}\left(G_{\eta,\gamma n-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)+\Lambda_r^{\gamma}G_{\eta,\gamma(1+n)-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)\right)\right].\end{aligned} \tag{4. 33}$$

5. THE SPECIAL CASE

5.1. **One cylinder at rest other one is moves.** If we substitute $d=0=g_2$ into Eqs. (4. 28) and (4. 33), we recover the solutions obtained by Kamran et al. [17] represented by Eqs. (4. 23) and (4. 32).

6. THE LIMITING CASES

6.1. **Ordinary Oldroyed-B fluid.** Impose $\eta, \gamma \to 1$ to (4. 28) and (4. 33), then we get velocity fields and shear stresses for ordinary Oldroyed-B fluid

$$\begin{aligned}
w_{OB}(r,t) &= \frac{g_1 t^d}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r - \frac{R_{2c}^2}{r}\right) - \frac{\pi g_1 \Gamma(d+1)}{\mu\Lambda}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \\
&\times \sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^{\ell}\frac{\ell!}{n!(\ell-n)!}\Lambda_r^n \times \Big[G_{1,n-\ell-d-1,\ell+1}\left(-\Lambda^{-1},t\right) \\
&+ \Lambda G_{1,1+n-\ell-d-1,\ell+1}\left(-\Lambda^{-1},t\right) + \nu\, r_\xi^2 \Lambda_r G_{1,n+1-\ell-d-2,\ell+1}\left(-\Lambda^{-1},t\right)\Big] \\
&+ \frac{g_2 R_{2c}}{{R_{2c}}^2 - {R_{1c}}^2}\left(r - \frac{R_{1c}^2}{r}\right)t + \frac{1}{\Lambda}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \\
&\times \sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\frac{\ell!}{n!(\ell-n)!}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^{\ell}{\Lambda_r}^n\Big[(\Phi_1 - \Phi_2\nu r_\xi^2)G_{1,n-\ell-3,\ell+1} \\
&\left(-\Lambda^{-1},t\right) + (\Lambda\Phi_1 - \Phi_2\nu\Lambda_r r_\xi^2)G_{1,n-\ell-2,\ell+1}\left(-\Lambda^{-1},t\right) - \Phi_2 \\
&\times G_{1,n-\ell-2,\ell+1}\left(-\Lambda^{-1},t\right) - \Phi_2\Lambda G_{1,n-\ell-1,\ell+1}\left(-\Lambda^{-1},t\right)\Big], \qquad (6.\ 34)
\end{aligned}$$

$$\begin{aligned}
\tau_{OB}(r,t) &= g_1\Gamma(d+1)\left(\frac{R_{1c}}{r}\right)^2\left(\frac{1}{\Lambda}\right)R_{1,-1-d}\left(-\Lambda^{-1},t\right) + \frac{\pi g_1\Gamma(d+1)}{\Lambda} \\
&\times \sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^{\ell}\frac{\ell!}{n!(\ell-n)!}\Lambda_r^n \\
&G_{1,n-\ell-d-1,\ell+1}\left(-\Lambda^{-1},t\right) - \frac{\mu\pi^2 g_2 R_{2c}}{2\Lambda}\sum_{\xi=1}^{\infty}\frac{r_\xi^2 J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \\
&\times \left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi) - J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]\Big[R_{1,-2}\left(-\Lambda^{-1},t\right) \\
&+ \Lambda_r R_{1,-1}\left(-\Lambda^{-1},t\right) - \sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^{\ell}\frac{\ell!}{n!(\ell-n)!}\Lambda_r^n \\
&\times \left(G_{1,n-\ell-2,\ell+1}\left(-\Lambda^{-1},t\right) + \Lambda_r G_{1,n-\ell-1,\ell+1}\left(-\Lambda^{-1},t\right)\right)\Big]. \qquad (6.\ 35)
\end{aligned}$$

If we substitute $d = 0$ and $g_2 = 0$ into Eqs. (6. 34) and (6. 35), we recover the solutions obtained by Fetecau et al. [10, Eqs. (17) and (19)].

6.2. **Fractional Maxwell fluid.** Let $\Lambda_r^\gamma \to 0$ in Eqs. (4. 28) and (4. 33), similar solution for fractional Maxwell fluid is

$$w_{FM}(r,t)=\frac{g_1t^d}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r-\frac{R_{2c}^2}{r}\right)-\frac{\pi g_1\Gamma(d+1)}{\mu\Lambda^\eta}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\left[G_{\eta,-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)+\Lambda^\eta G_{\eta,\eta-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)\right]$$
$$+\frac{g_2R_{2c}}{{R_{2c}}^2-{R_{1c}}^2}\left(r-\frac{R_{1c}^2}{r}\right)t+\frac{1}{\Lambda^\eta}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell\Big[(\Phi_1-\Phi_2\nu r_\xi^2)G_{\eta,-\ell-3,\ell+1}$$
$$\left(-\Lambda^{-\eta},t\right)+\Lambda^\eta\Phi_1G_{\eta,\gamma-\ell-3,\ell+1}\left(-\Lambda^{-\eta},t\right)-\Phi_2$$
$$\times G_{\eta,-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)-\Phi_2\Lambda^\eta G_{\eta,\eta-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)\Big],\tag{6. 36}$$
$$\tau_{FM}(r,t)=g_1\Gamma(d+1)\left(\frac{R_{1c}}{r}\right)^2\left(\frac{1}{\Lambda^\eta}\right)R_{\eta,-d-1}\left(-\Lambda^{-\eta},t\right)+\frac{\pi g_1\Gamma(d+1)}{\Lambda^\eta}$$
$$\times\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell G_{\eta,-\ell-d-1,\ell+1}\left(-\Lambda^{-\eta},t\right)-\frac{\pi^2\mu g_2R_{2c}}{2\Lambda^\eta}$$
$$\times\sum_{\xi=1}^{\infty}\frac{r_\xi^2J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$\times\left[R_{\eta,-2}\left(-\Lambda^{-\eta},t\right)-\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda^\eta}\right)^\ell G_{\eta,-\ell-2,\ell+1}\left(-\Lambda^{-\eta},t\right)\right].\tag{6. 37}$$

6.3. **Ordinary Maxwell fluid.** Let $\eta\to 1$ in Eq. (6. 36) and (6. 37),

$$w_{OM}(r,t)=\frac{g_1t^d}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r-\frac{R_{2c}^2}{r}\right)-\frac{\pi g_1\Gamma(d+1)}{\mu\Lambda}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^\ell\left[G_{1,-\ell-d-1,\ell+1}\left(-\Lambda^{-1},t\right)+\Lambda G_{1,-\ell-d,\ell+1}\left(-\Lambda^{-1},t\right)\right]$$
$$+\frac{g_2R_{2c}}{{R_{2c}}^2-{R_{1c}}^2}\left(r-\frac{R_{1c}^2}{r}\right)t+\frac{1}{\Lambda}\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^\ell\Big[(\Phi_1-\Phi_2\nu r_\xi^2)G_{1,-\ell-3,\ell+1}$$
$$\left(-\Lambda^{-1},t\right)+\Lambda\Phi_1G_{1,1-\ell-3,\ell+1}\left(-\Lambda^{-1},t\right)-\Phi_2$$
$$\times G_{1,-\ell-2,\ell+1}\left(-\Lambda^{-1},t\right)-\Phi_2\Lambda G_{1,-\ell-1,\ell+1}\left(-\Lambda^{-1},t\right)\Big],\tag{6. 38}$$

$$\tau_{OM}(r,t) = g_1\Gamma(d+1)\left(\frac{R_{1c}}{r}\right)^2\left(\frac{1}{\Lambda}\right)R_{1,-d-1}\left(-\Lambda^{-1},t\right) + \frac{\pi g_1\Gamma(d+1)}{\Lambda}$$
$$\times\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^{\ell}G_{1,-\ell-d-1,\ell+1}\left(-\Lambda^{-1},t\right)$$
$$-\frac{\pi^2\mu g_2R_{2c}}{2\Lambda}\sum_{\xi=1}^{\infty}\frac{r_\xi^2J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$\times\left[R_{1,-2}\left(-\Lambda^{-1},t\right)-\sum_{\ell=0}^{\infty}\left(\frac{-\nu r_\xi^2}{\Lambda}\right)^{\ell}G_{1,-\ell-2,\ell+1}\left(-\Lambda^{-1},t\right)\right]. \quad (6.\ 39)$$

6.4. **Fractional Second grade fluid.** Applying $\Lambda^{\eta}\to 1$ and $\eta\to 1$ into Eqs. (4. 23) and (4. 31), and proceed as in section of velocity field and shear stress we get

$$w_{FSG}(r,t) = \frac{1}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r-\frac{R_{2c}^2}{r}\right)g_1t^d - \frac{\pi g_1\Gamma(d+1)}{\mu}\sum_{\xi=1}^{\infty}$$
$$\times\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^{\ell}\Big[G_{1-\gamma,-\gamma\ell-\gamma-d,\ell+1}\left(-\nu\Lambda_r^{\gamma}r_\xi^2,t\right)$$
$$+\nu\Lambda_r^{\gamma}r_\xi^2G_{1-\gamma,-\gamma\ell-1-d,\ell+1}\left(-\nu\Lambda_r^{\gamma}r_\xi^2,t\right)\Big]$$
$$+\frac{\pi^2g_2R_{2c}}{2}\sum_{\xi=1}^{\infty}\frac{r_\xi J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$\times\left[t-\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^{\ell}G_{1-\gamma,-\gamma\ell-\gamma-1,\ell+1}\left(-\nu\Lambda_r^{\gamma}r_\xi^2,t\right)\right]. \quad (6.\ 40)$$

$$\tau_{FSG}(r,t) = \left(\frac{R_{1c}}{r}\right)^2g_1t^d + \pi g_1\Gamma(d+1)\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^{\ell}$$
$$\times\left[G_{1-\gamma,-\gamma\ell-\gamma-d,\ell+1}\left(-\nu\Lambda_r^{\gamma}r_\xi^2,t\right)\right]-\frac{\pi^2\mu g_2R_{2c}}{2}\sum_{\xi=1}^{\infty}\frac{r_\xi^2J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]\left[t+\frac{\Lambda_r^{\gamma}t^{1-\gamma}}{\Gamma(-\gamma)}-\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^{\ell}\right.$$
$$\left.\times\left(G_{1-\gamma,-\gamma\ell-\gamma-1,\ell+1}\left(-\nu\Lambda_r^{\gamma}r_\xi^2,t\right)+\Lambda_r^{\gamma}G_{1-\gamma,-\gamma\ell-1,\ell+1}\left(-\nu\Lambda_r^{\gamma}r_\xi^2,t\right)\right)\right]. \quad (6.\ 41)$$

By making $\nu\Lambda_r^{\gamma}=\alpha$ into Eqs. (6. 40) and (6. 41) we get the solution for second grade fluid with fractional derivatives.The reduced form of velocity field and shear stress are

$$
\begin{aligned}
w_{FSG}(r,t) = & \frac{1}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r - \frac{R_{2c}^2}{r}\right)g_1 t^d - \frac{\pi g_1 \Gamma(d+1)}{\mu} \\
& \times \sum_{\xi=1}^{\infty} \frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \sum_{\ell=0}^{\infty} \left(-\nu r_\xi^2\right)^\ell \\
& \times \left[G_{1-\gamma,-\gamma\ell-\gamma-d,\ell+1}\left(-\alpha r_\xi^2, t\right) + \alpha r_\xi^2 G_{1-\gamma,-\gamma\ell-1-d,\ell+1}\left(-\alpha r_\xi^2, t\right)\right] \\
& + \frac{\pi^2 g_2 R_{2c}}{2} \sum_{\xi=1}^{\infty} \frac{r_\xi J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \\
& \times \left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi) - J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right] \\
& \times \left[t - \sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell G_{1-\gamma,-\gamma\ell-\gamma-1,\ell+1}\left(-\alpha r_\xi^2, t\right)\right]. \qquad (6.\ 42)
\end{aligned}
$$

$$
\begin{aligned}
\tau_{FSG}(r,t) = & \left(\frac{R_{1c}}{r}\right)^2 g_1 t^d + \pi g_1 \Gamma(d+1) \sum_{\xi=1}^{\infty} \frac{J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell \\
& \times \left[G_{1-\gamma,-\gamma\ell-\gamma-d,\ell+1}\left(-\alpha r_\xi^2, t\right)\right] - \frac{\pi^2 \mu g_2 R_{2c}}{2} \sum_{\xi=1}^{\infty} \frac{r_\xi^2 J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \\
& \times \left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi) - J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]\left[t + \frac{\Lambda_r^\gamma t^{1-\gamma}}{\Gamma(1-\gamma)} - \sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell\right. \\
& \left.\times\left(G_{1-\gamma,-\gamma\ell-\gamma-1,\ell+1}\left(-\alpha r_\xi^2, t\right) + \Lambda_r^\gamma G_{1-\gamma,-\gamma\ell-1,\ell+1}\left(-\alpha r_\xi^2, t\right)\right)\right]. \qquad (6.\ 43)
\end{aligned}
$$

6.5. **Ordinary second grade fluid.** Let $\gamma \to 1$ in Eq. (6. 42), (6. 43)

$$
\begin{aligned}
w_{OSG}(r,t) = & \frac{1}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r - \frac{R_{2c}^2}{r}\right)g_1 t^d - \frac{\pi g_1 \Gamma(d+1)}{\mu} \\
& \times \sum_{\xi=1}^{\infty} \frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell \\
& \times \left[G_{0,-\ell-1-d,\ell+1}\left(-\alpha r_\xi^2, t\right) + \alpha r_\xi^2 G_{0,-\ell-1-d,\ell+1}\left(-\alpha r_\xi^2, t\right)\right] \\
& + \frac{\pi^2 g_2 R_{2c}}{2} \sum_{\xi=1}^{\infty} \frac{r_\xi J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi) - J_1^2(R_{2c}r_\xi)]} \\
& \times \left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi) - J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right] \\
& \times \left[t - \sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell G_{0,-\ell-2,\ell+1}\left(-\alpha r_\xi^2, t\right)\right]. \qquad (6.\ 44)
\end{aligned}
$$

$$\tau_{OSG}(r,t)=\left(\frac{R_{1c}}{r}\right)^2 g_1 t^d+\pi g_1\Gamma(d+1)\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell$$
$$\times\left[G_{0,-\ell-1-d,\ell+1}\left(-\alpha r_\xi^2,t\right)\right]-\frac{\pi^2\mu g_2 R_{2c}}{2}\sum_{\xi=1}^{\infty}\frac{r_\xi^2 J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]\left[1+\Lambda_r t^{-1}-\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell\right.$$
$$\left.\times\left(G_{0,-\ell-1,\ell+1}\left(-\alpha r_\xi^2,t\right)+G_{0,-\ell,\ell+1}\left(-\alpha r_\xi^2,t\right)\right)\right]. \tag{6. 45}$$

6.6. **Newtonian fluid.** Let $\Lambda^\eta,\Lambda_r^\gamma\to 0$ into Eqs. (4. 28) and (4. 33), we find

$$w_N(r,t)=\frac{1}{2\mu}\left(\frac{R_{1c}}{R_{2c}}\right)^2\left(r-\frac{R_{2c}^2}{r}\right)g_1 t^d-\frac{\pi g_1\Gamma(d+1)}{\mu}$$
$$\times\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{r_\xi[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell\frac{t^{\ell+d}}{\Gamma(\ell+d+1)}+\frac{\pi^2 g_2 R_{2c}}{2}\times$$
$$\sum_{\xi=1}^{\infty}\frac{r_\xi J_1^2(R_{2c}r_\xi)B_{1w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]$$
$$\times\left[t-\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell\frac{t^{\ell+1}}{\Gamma(\ell+2)}\right]. \tag{6. 46}$$

$$\tau_N(r,t)=\left(\frac{R_{1c}}{r}\right)^2 g_1 t^d+\pi g_1\Gamma(d+1)\sum_{\xi=1}^{\infty}\frac{J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\sum_{\ell=0}^{\infty}\left(-\nu r_\xi^2\right)^\ell\frac{t^{\ell+d}}{\Gamma(\ell+d+1)}-\frac{\pi^2\mu g_2 R_{2c}}{2}\sum_{\xi=1}^{\infty}\frac{r_\xi^2 J_1^2(R_{2c}r_\xi)B_{2w}(r,r_\xi)}{[J_2^2(R_{1c}r_\xi)-J_1^2(R_{2c}r_\xi)]}$$
$$\times\left[Y_2(R_{1c}r_\xi)J_2(R_{2c}r_\xi)-J_2(R_{1c}r_\xi)Y_2(R_{2c}r_\xi)\right]\left[t-\sum_{\ell=0}^{\infty}\frac{\left(-\nu r_\xi^2\right)^\ell t^{\ell+1}}{\Gamma(\ell+2)}\right]. \tag{6. 47}$$

7. Numerical Results and discussion

In this analysis motion of an IFOBF in an annulus of concentric cylinders is studied. The movement in fluid is due to the motion of both cylinders. The analysis of velocity field and adequate shear stress are made by using most efficient and powerful integral transform techniques. For the simplicity and better perception, generalized R and G functions are used to represent the obtained results. These solutions are satisfied all the initial and boundary conditions. By imposing favourable limits, similar solutions for ordinary Oldroyd-B, Maxwell, second grade and Newtonian fluids for fractional and ordinary derivatives are also obtained.

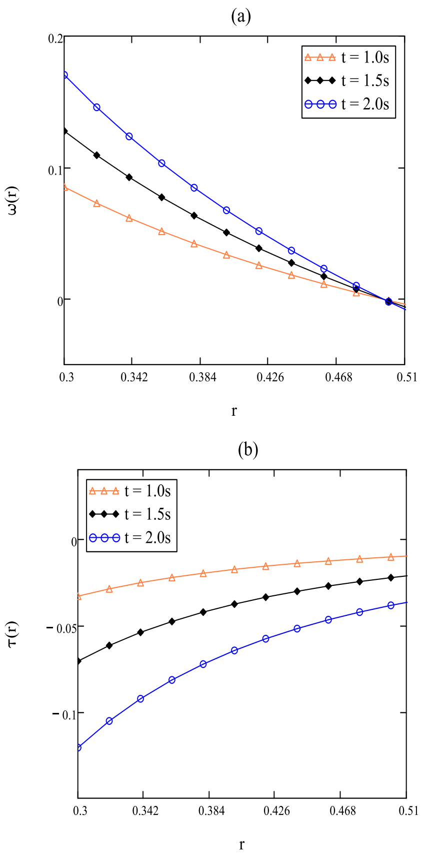

FIGURE 2. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (4. 28) and (4. 33), for different values of t and $[R_{1c} = 0.3, R_{2c} = 0.51, g_1 = -1, g_2 = -0.01, \Lambda = 15, \Lambda_r = 4, \eta = 0.9, \gamma = 0.1, \mu = 1, \nu = 0.01, d = 1]$

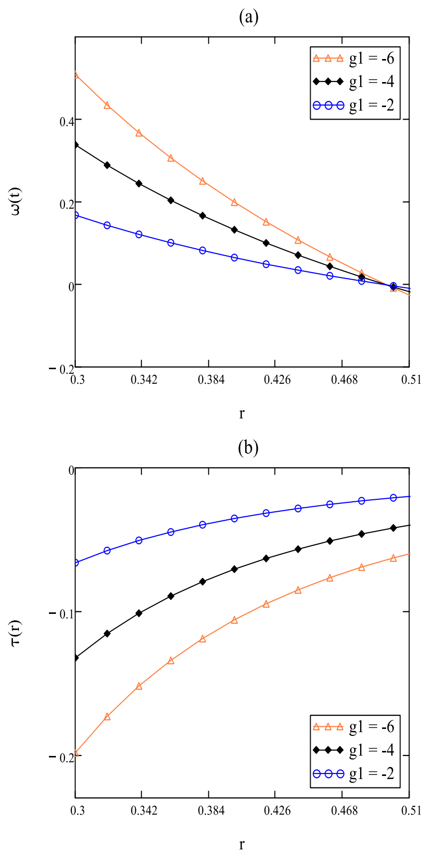

FIGURE 3. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (4. 28) and (4. 33), for different values of g_1 and $[R_{1c} = 0.3, R_{2c} = 0.51, t = 1s, g_2 = -0.01, \Lambda = 15, \Lambda_r = 4, \eta = 0.9, \gamma = 0.1, \mu = 1, \nu = 0.01, d = 1]$

Furthermore, we recover the solution obtained by Kamran et al. [17] represented by Eqs. (4. 23) and (4. 32), by taking $d = 0 = g_2$ into Eqs. (4. 28) and (4. 33) and If we substitute $d = 0$ and $g_2 = 0$ into Eqs. (6. 34) and (6. 35), we recover the solutions obtained by Fetecau et al. [10, Eqs. (17) and (19)].

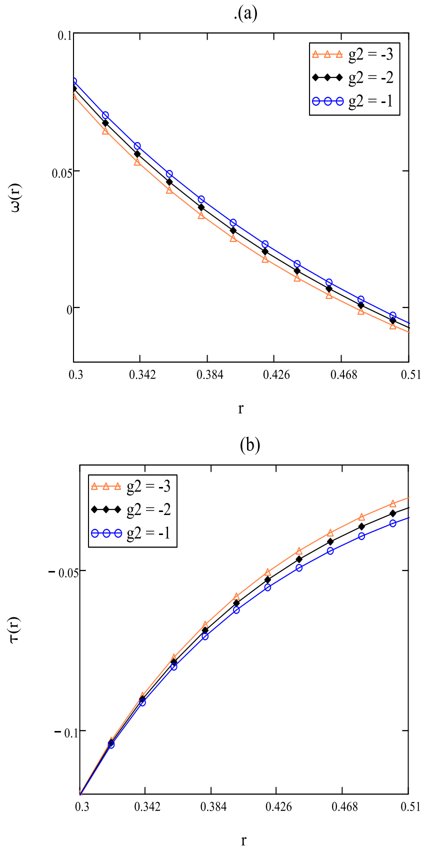

FIGURE 4. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (4. 28) and (4. 33), for different values of g_2 and[$R_{1c} = 0.3, R_{2c} = 0.51, t = 1s, g1 = -1, \Lambda = 15, \Lambda_r = 4, \eta = 0.9, \gamma = 0.1, \mu = 1, \nu = 0.01, d = 1$]

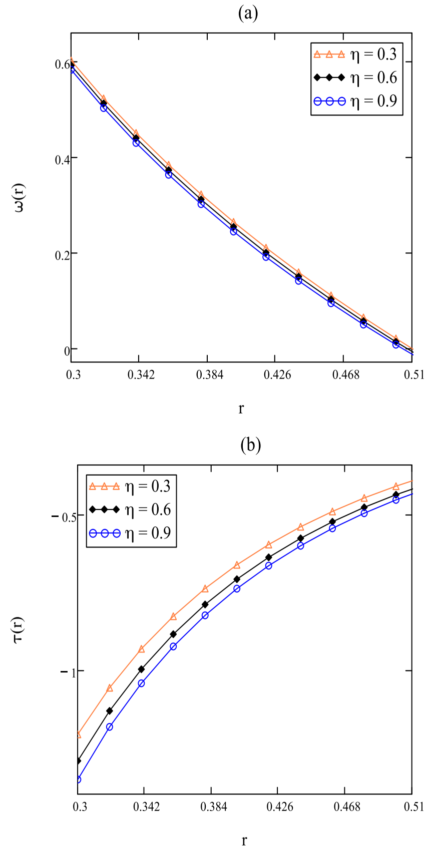

FIGURE 5. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (4. 28) and (4. 33)), for different values of η and [$R_{1c} = 0.3, R_{2c} = 0.51, t = 2.2s, g1 = -3, g2 = -0.01\Lambda = 4.5, \Lambda_r = 10, \gamma = 0.1, \mu = 1, \nu = 0.01, d = 1$]

Now, here are results obtained from above analysis. The velocity profiles and shear stress are illustrate in Figures 2 - 11. Fig. 2 for effect of t, Figs. 3 and 4 for the effect of g_1 and g_2 on the fluid motion. Figs. 5 and 6 for the effect of fractional parameters η and γ on the fluid motion. Figs. 7 and 8 shows the effect of the relaxation time and retardation time on the fluid motion. Figs. 9 and 10 shows the effect of the kinematics and dynamic

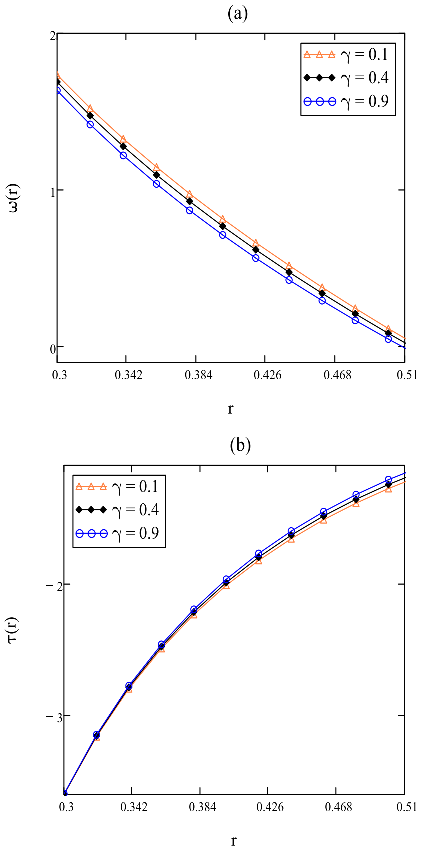

FIGURE 6. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (4. 28) and (4. 33)), for different values of γ and $[R_{1c} = 0.3, R_{2c} = 0.51, t = 9s, g1 = -2, g2 = -0.001\Lambda = 15, \Lambda_r = 4, \eta = 0.9, \mu = 1, \nu = 0.01, d = 1]$

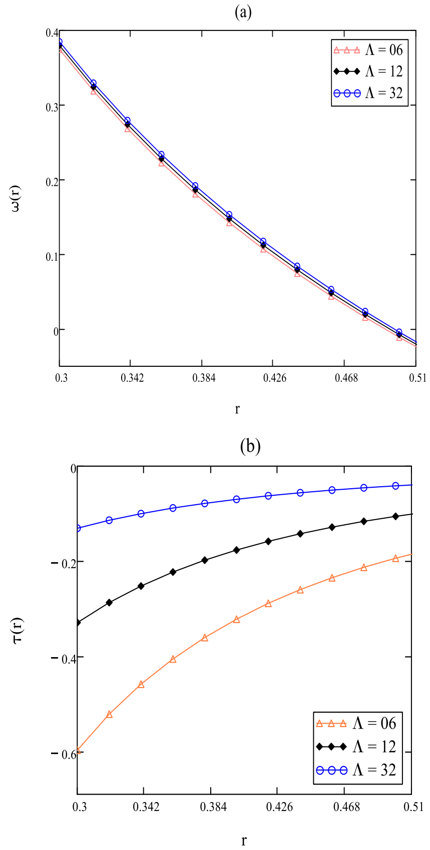

FIGURE 7. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (4. 28) and (4. 33), for different values of Λ and $[R_{1c} = 0.3, R_{2c} = 0.51, t = 3s, g1 = -0.5, g2 = -0.003\gamma = 0.1, \Lambda_r = 3, \eta = 0.2, \mu = 1, \nu = 0.01, d = 1]$

viscosity on the fluid and Fig. 11 shows the effect of power parameter. SI units system is used within all figures, and the roots are approximated by $r_n = \frac{(2n-1)\pi}{2(R_2-R_1)}$.

- It is found that velocity field and absolute shear stress increases with the passage of time.

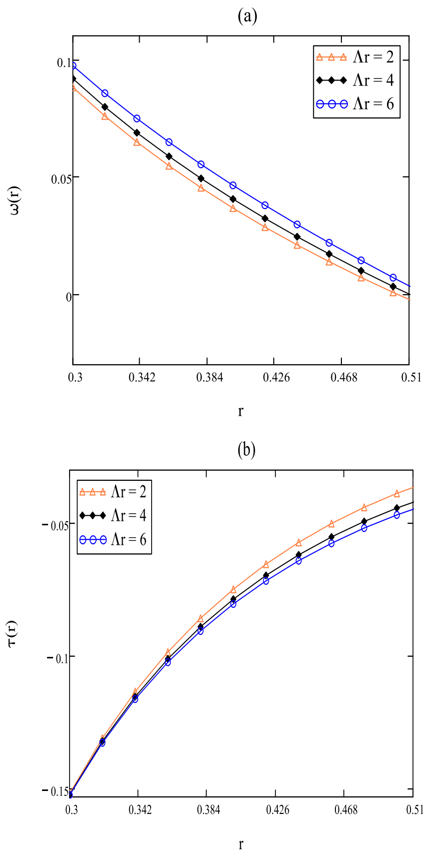

FIGURE 8. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (4. 28) and (4. 33), for different values of Λ_r and $[R_{1c} = 0.3, R_{2c} = 0.51, t = 2.2s, g1 = -0.5, g2 = -0.001, \gamma = 0.1, \Lambda = 1, \eta = 0.9, \mu = 1.1, \nu = 0.01, d = 1]$

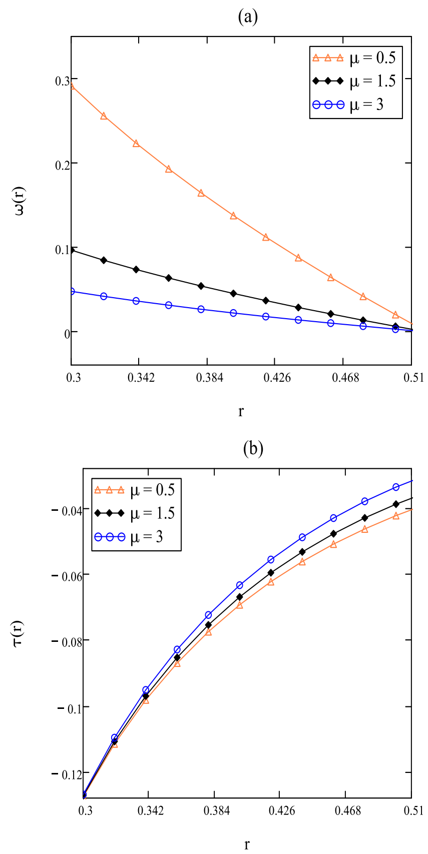

FIGURE 9. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs.(4. 28) and (4. 33), for different values of μ and $[R_{1c} = 0.3, R_{2c} = 0.51, t = 3s, g1 = -0.5, g2 = -0.003, \gamma = 0.1, \Lambda = 15, \Lambda_r = 8, \eta = 0.9, \mu = 1, \nu = 0.01, d = 1]$

- Retardation parameter Λ_r, kinematic viscosity ν and constant g_2 shows the similar behaviour like time for velocity field and absolute shear stress.
- The velocity and absolute stress of the fluid decreases as fluid becomes more thick.
- Fluids velocity and absolute value of stress are decreasing functions of fractional parameter γ and constant g_1.

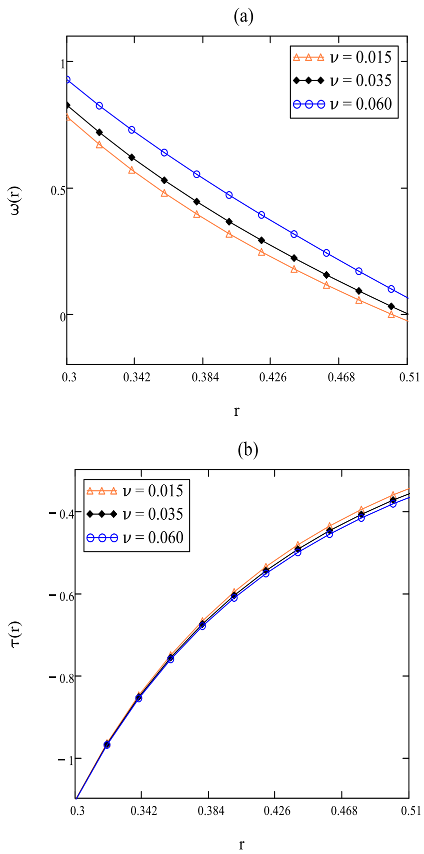

FIGURE 10. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (27) and (32), for different values of ν and $[R_{1c} = 0.3, R_{2c} = 0.51, t = 3s, g1 = -3, g2 = -0.001, \gamma = 0.2, \Lambda = 10, \eta = 0.9, \mu = 1, \Lambda_r = 4, d = 1]$

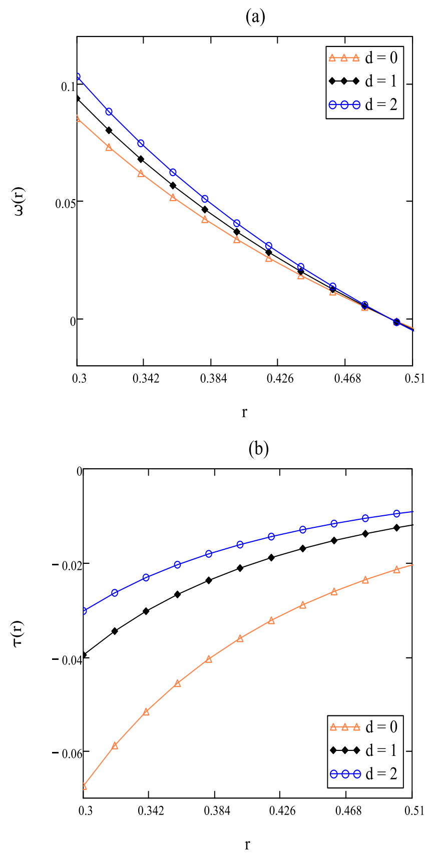

FIGURE 11. Variation in velocity field $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (27) and (32), for different values of d and $[R_{1c} = 0.3, R_{2c} = 0.51, t = 1.1s, g1 = -1, g2 = -0.01, \gamma = 0.1, \Lambda = 15, \eta = 0.9, \mu = 1, \nu = 0.01, \Lambda_r = 4]$

- The Velocity field is increasing function of Relaxation parameter Λ and exponent d but shear stress in absolute value have opposite effect.
- The behaviour of fractional parameter η is opposite to Λ.

8. Declarations

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8.2. **Competing interests.** The authors declare that they have no competing interests.

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