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# Cubic B-spline Solution of Nonlinear Sixth Order Boundary Value Problems

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**Abstract.** In the present paper, we study the Cubic-B splines to find numerical solution of 6th order nonlinear boundary value problems arising in astrophysics and the narrow convecting layers bounded by stable layers. The prescribed method transforms the boundary value problem to a system of linear equations. The algorithm we are going to develop in this paper is not only simply the approximation solution of the 6th order boundary value problems using Cubic-B spline but it also describes the estimated derivatives of 1st order to 6th order of the analytic solution at the same time. This new method has lower computational cost than many other methods and is second order convergent. Some mathematical samples are presented to authenticate the speculative exploration and demonstrate the legitimacy and applicability of the technique.

**AMS (MOS) Subject Classification Codes: 34K10; 34K28; 42A10; 65D05; 65D07**
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## 1. Introduction

In the present paper, we will study the algebraic results of the given beneath nonlinear $6^{th}$ order BVP

$$w^{(6)}(z) + a_1(z)w^{(5)}(z) + a_2(z)w^{(4)}(z) + a_3(z)w^{(3)}(z) + a_4(z)w^{(2)}(z) + a_5(z) w^{(1)}(z) + a_6(z)w(z) = f(z, w(z), w^{(1)}(z), w^{(2)}(z), w^{(3)}(z), w^{(4)}(z), w^{(5)}(z)), z \in [a, b] \quad (1.\ 1)$$

and

$$w^{(m)}(a) = \alpha_m, w^{(m)}(b) = \beta_m, m = 0, 1, 2 \qquad (1.\ 2)$$

where $\alpha_0$, $\alpha_1$, $\alpha_2$, and $\beta_0$, $\beta_1$, $\beta_2$ are given real constants, $(a_i(z);\ i = 1, 2, ..., 6)$ and $f$ is continuous on the given interval [*a, b*]. Generally, it is difficult to obtain the analytic solution of equation $(1)$ for arbitrary $f(z, w(z), w^{(1)}(z), w^{(2)}(z), w^{(3)}(z), w^{(4)}(z), w^{(5)}(z))$ hence, numerical methods are desired.

In [1], the authors unraveled linear problem for $6^{th}$ order boundary value problems using spline solution. In [2], $6^{th}$ order boundary value problems were solved by using Global Phase-Integral Methods. In [3] and [37], the authors established finite difference techniques for resolving such boundary value problems. In [4], "B-spline interpolation" is equated with "finite difference", "finite element" and "finite volume" procedures. In [5], author found a class of characteristic value problems of high order (as high as twenty four) are known to arise in hydrodynamic and hydro magnetic stability. Numerical solutions of boundary value problems were presented in [6]. "Cubic-B spline" solution of nonlinear system of second-order boundary value problems were established in [7].

In [8], the authors used "Sinc-Galerkin" technique to solve $6^{th}$ order BVP. In [9], the authors summaries that "dynamo action" in some stars might be exhibited by such equations. In [10], the authors found B-spline solution of $4^{th}$ order linear boundary value problems.

As we all know that splines are generally useful for the approximation solutions of boundary value problems. "Cubic-B splines" are widely and interestingly used to solve Second Order Linear Klein-Gordon Equation in [11], singular boundary value problems in [14]. "Ritz's technique" centered on "variational theory" in [12] and variational iteration procedures were applied for $6^{th}$ order boundary value problems. Parametric quintic spline were used for solution of $6^{th}$ order two point boundary value problems in [13].

Fifth order linear boundary value problems weree solved by Cubic-B Spline method in [15]. Sixth degree B-spline functions in [16] and A class of methods based on a septic non-polynomial spline functions were used in [22] for the solution of sixth order boundary value problems. Error bounds for interpolating cubic splines under various end conditions were discussed in [17]. In [18], "Variation of parameter" technique was used for solving sixth-order boundary value problems.

Perturbation method for nonlinear engineering problems were specified in [19]. "Homotopy perturbation" technique for solving linear and nonlinear sixth order boundary value problems was described in [20]. In [21], algebraic results of $6^{th}$ order BVPs were originate by applying non-polynomial spline method. "Neural networks" mimic the learning procedure of the human brain in order to excerpt designs from ancient data as defined in [23]. These networks are rehabilitated into $6^{th}$ order boundary value problems and then resolved by diverse approaches for precise and estimated results.

In [24], the authors resolved linear problem using polynomial septic spline for $6^{th}$ order boundary value problems. Quintic spline solution of linear sixth-order boundary value problems were found in [25].

In [26] sixth-degree Splines, where the value of Spline at the middle knots of the interpolation interval were connected over stability relations to the equivalent values of the even order derivatives. A $6^{th}$ order BVP occurs in astrophysics; the thin convecting sheets bounded by steady sheets which are expected to mount "A-type" stars can be molded by

$6^{th}$ order boundary value problems as described in [27].
Numerical methods for sixth-order boundary value problems were discussed in [28]. In [29], the authors established a family of algebraic procedures for the solutions of $6^{th}$ order boundary value problems with application to "Benard layer" eigenvalue problems. Numerical solutions of fifth and sixth order nonlinear boundary value problems by daftardar jafari method were found in [30]. Wazwaz in [31] used "decomposition" and "modified domain decomposition" approaches to examine the solution of $6^{th}$ order boundary value problems.

## 2. Notations and Preliminaries

In this section fundamentals of Cubic B-spline and its application on sixth order bvp is discussed in detail. When we contrasts Cubic-B spline method with the other methods we came to know that our results are well accepted. Moreover this method has second order convergent and has comparatively lesser computational cost. Furthermore, solving with Cubic-B spline method we also can acquire the estimated derivative values of $\mathrm{w}\left(z\right), \mathrm{w}'(z), \mathrm{w}''(z), \mathrm{w}^{(3)}\left(z\right), \mathrm{w}^{(4)}\left(z\right), \mathrm{w}^{(5)}(z)$and $w^{(6)}\left(z\right)$ at the knots which is the main advantage of Cubic-B spline method, as other methods are unable to obtain these values. In the present paper, we will use cubic B-spline to resolve $6^{th}$ order BVP.
The assumed choice of independent variable is [$a$, $b$]. For this range we select equidistant points assumed by

$$\Omega = \{a = z_0, \mathrm{z}_1, \mathrm{z}_2, ..., \mathrm{z}_n = b\}$$

i.e.

$$\mathrm{z} = a + ih, (i = 0, 1, 2, ...,n)$$

where

$$h = \frac{b-a}{n}$$

Let us describe $S_3\left(\Omega\right) = \left\{p\left(t\right) \in C^2\left[a,b\right]\right\}$ such that $p\left(t\right)$ decreases to cubic polynomial on separately sub-interval$(z_i, \mathrm{z}_{i+1})$. The basis function is defined as

$$B_{-1}(z) = \tfrac{1}{6h^3}\begin{cases} (z_1 - z)^3, & if \mathrm{z} \in [z_0, z_1] \\ 0 & otherwise \end{cases}$$

$$B_0(z) = \frac{1}{6h^3}\begin{cases} h^3 + 3h^2(z_1 - z) + 3h(z_1 - z)^2 - 3(z_1 - z)^3 & if \mathrm{z} \in [z_0, z_1] \\ (z_2 - z)^3, & if \mathrm{z} \in [z_1, z_2] \\ 0 & otherwise \end{cases}$$

$$B_1(z) = \frac{1}{6h^3}\begin{cases} h^3+3h^2(z-z_0)+3h(z-z_0)^2-3(z-z_0)^3 & if \mathrm{z} \in [z_0, z_1] \\ h^3+3h^2(z_2-z)+3h(z_2-z)^2-3(z_2-z)^3 & if \mathrm{z} \in [z_1, z_2] \\ (z_3-z)^3, & if \mathrm{z} \in [z_2, \mathrm{z}_3] \\ 0 & otherwise \end{cases}$$

$$B_i(z) = \frac{1}{6h^3}\begin{cases} (z-z_{i-2})^3, & if \mathrm{z} \in [z_{i-2}, z_{i-1}] \\ h^3+3h^2(z-z_{i-1})+3h(z-z_{i-1})^2-3(z-z_{i-1})^3 & if \mathrm{z} \in [z_{i-1}, z_i] \\ h^3+3h^2(z_{i+1}-z)+3h(z_{i+1}-z)^2-3(z_{i+1}-z)^3 & if \mathrm{z} \in [z_i, z_{i+1}] \\ (z_{i+2}-z)^3, & if z \in [z_{i+1}, z_{i+2}] \\ 0 & otherwise \end{cases}$$

for $(i = 0,\ 1,\ 2,\ ...,n)$. Since to each $B_i$ ( z)is also a piecewise cubic with knots at $\Omega$, collectively $B_i\left(z\right)\ \in S_3(\Omega)$. Let $\Psi = \{B_{-1}, B_0, B_1, ......, B_{n+1}\}$ and let $B_3(\Omega) = span\ \Psi$. The functions in $\Psi$ are linearly independent on [$a$, $b$], thus $B_3\left(\Omega\ \right)\ is\ (n+3)$ -dimensional and $B_3\left(\Omega\right)\ = S_3\left(\Omega\right)$.
Let $s\left(z\right)$ be the Cubic-B spline interpolating function at the nodal points and $s\left(z\right)\ \in B_3\left(\Omega\ \right)$.
Then $s\left(z\right)$ can be written as

$$s\left(z\right) = \sum_{i=-1}^{n+1} l_i B_i\left(z\right).$$

The values of $B_i(z),\ {B_i}^{(1)}\left(z\right)\ and\ {B_i}^{(2)}\left(z\right)$ at the knots are listed in Table 1.
Consequently now for a assumed function $w\left(z\right)$ there happened to be a distinctive cubic B- Spline $s\left(z\right) = \sum_{i=-1}^{n+1} l_i B_i\left(z\right)$satisfying the interpolating conditions:

$$s\left(z_i\right)\ = w\left(z_i\right), (i = 0, 1,\ \ldots, n)\ , s\left(a\right)\ = w\left(a\right),\ \ s\left(b\right)\ = w\left(b\right)$$

and $s'\left(a\right)\ = w'\left(a\right),\ s'\left(\ b\ \right)\ = w'\left(b\right)$.
Let $m_i\ = s'\left(z_i\right)$ and $M_i = s''\left(z_i\right)$ then from [19] we have

| **Table1** | $B_i\left(z\right)$ | ${B_i}^{(1)}\left(z\right)$ | ${B_i}^{(2)}\left(z\right)$ |
|---|---|---|---|
| $z_{i-2}$ | 0 | 0 | 0 |
| $z_{i-1}$ | 1 / 6 | 1 / 2h | 1 / h$^2$ |
| $z_i$ | 4 / 6 | 0 | −2 / h$^2$ |
| $z_{i+1}$ | 1 / 6 | −1 / 2h | 1 / h$^2$ |
| $z_{i-2}$ , $z_{i+2}$ and all others | | | 0 |

$$m_i = s'(z_i) = w''(z_i) - \frac{1}{180}h^4 w^{(5)}(z_i) + O(h^6) \tag{2. 3}$$

$$M_i = s''(z_i) = w''(z_i) - \frac{1}{12}h^2 w^{(4)}(z_i) + \frac{1}{360}h^4 w^{(6)}(z_i) + O(h^6) \tag{2. 4}$$

$M_i$ can be used to calculate the numerical difference formulas for $w^{(3)}(z_i)$, $\mathrm{w}^{(4)}(z_i)$ where $i = (1, 2, \ldots, n-1)$ and $w^{(5)}(z_i)$, $\mathrm{w}^{(6)}(z_i)$ where $i = (2, 3, \ldots, n-2)$ as follows, wherever the errors are acquired by the Taylor series expansion [16]

$$\frac{M_{i+1} - M_{i-1}}{2h} = \frac{s^{(3)}(z_{i-}) + s^{(3)}(z_{i+})}{2} = w^{(3)}(z_i) + \frac{1}{12}h^2 w^{(5)}(z_i) + O(h^4) \quad (2.5)$$

$$\frac{M_{i+1} - 2M_i + M_{i-1}}{h^2} = \frac{s^{(3)}(z_{i-}) - s^{(3)}(z_{i+})}{h} = w^{(4)}(z_i) - \frac{1}{720}h^4 w^{(8)}(z_i) + O(h^6) \quad (2.6)$$

$$\frac{M_{i+2} - 2M_{i+1} + 2M_{i-1} - M_{i-2}}{2h^3} = w^{(5)}(z_i) + O(h^2) \quad (2.7)$$

and following the above we have the Taylor series expansion for $w^{(6)}(z_i)$at the selected collocation points with central difference as

$$w_i^{(6)} = \frac{(w_{i+1}^{(4)} - 2w_i^{(4)} + w_{i-1}^{(4)})}{h^2} \quad (2.8)$$

$$\frac{M_{i+2} - 4M_{i+1} + 6\ M_i - 4M_{i-1} + M_{i-2}}{h^4} = w^{(6)}(z_i) + O(h^2) \quad (2.9)$$

Since $s(z) = \sum_{i=-1}^{n+1} l_i B_i(z)$,by means of table 1 and beyond equations, we get estimate

values of $w(z)$ its all derivatives from 1st to 7th at the knots as

$$w(z_i) = s(z_i) = \frac{l_{i-1} + 4l_i + l_{i+1}}{6}$$

$$w'(z) = s'(z_i) = \frac{l_{i+1} - l_{i-1}}{2h}$$

$$w''(z) = s''(z_i) = \frac{l_{i+1} - 2l_i + l_{i-1}}{h^2}$$

$$w^{(3)}(z) = s^{(3)}(z_i) = \frac{l_{i+2} - 2l_{i+1} + 2l_{i-1} - l_{i-2}}{2h^3}$$

$$w^{(4)}(z) = s^{(4)}(z_i) = \frac{l_{i+2} - 4l_{i+1} + 6l_i - 4l_{i-1} + l_{i-2}}{h^4}$$

$$w^{(5)}(z) = s^{(5)}(z_i) = \frac{l_{i+3} - 4l_{i+2} + 5l_{i+1} + 5l_{i-1} + 4l_{i-2} - l_{i-3}}{2h^5}$$

and

$$w^{(6)}(z_i) = s^{(6)}(z_i) = \frac{l_{i+3} - 6l_{i+2} + 15l_{i+1} - 20l_i + 15l_{i-1} - 6l_{i-2} + l_{i-3}}{h^6} \quad (2.10)$$

## 3. CUBIC-B SPLINE SOLUTIONS OF SIXTH ORDER BVP

Let $w(z) = s(z) = \sum_{i=-1}^{n+1} l_i B_i(z)$ be the approximate solution of sixth order BVP

$$w^{(6)}(z) + a_1(z)w^{(5)}(z) + a_2(z)w^{(4)}(z) + a_3(z)w^{(3)}(z) + a_4(z)$$
$$w^{(2)}(z) + a_5(z)w^{(1)}(z) + a_6(z)w(z) = f(z, w(z), w^{(1)}(z), w^{(2)}(z),$$
$$w^{(3)}(z), w^{(4)}(z), w^{(5)}(z)), z \in [a, b]$$

where $\alpha_0$, $\alpha_1$, $\alpha_2$, and $\beta_0$, $\beta_1$, $\beta_2$ are given real constants, $(a_i(z);\ i = 1,\ 2,\ ...,\ 6)$ and $f$ is continuous on the given interval [*a, b*].

$$\begin{aligned}&w^{(6)}(z_i)+a_1(z_i)w^{(5)}(z_i)+a_2(z_i)w^{(4)}(z_i)+a_3(z_i)w^{(3)}(z_i)+a_4(z_i)\\&w^{(2)}(z_i)+a_5(z_i)w^{(1)}(z_i)+a_6(z_i)w(z_i)=f_i((z_i,w(z_i),w^{(1)}(z_i),w^{(2)}(z_i),\\&w^{(3)}(z_i),w^{(4)}(z_i),w^{(5)}(z_i)),z\in[a,b]\end{aligned}$$

Using *equation* (2. 10 ), we have

$$\begin{aligned}&\tfrac{l_{i+3}-6l_{i+2}+15l_{i+1}-20l_i+15l_{i-1}-6l_{i-2}+l_{i-3}}{h^6}+a_1(z_i)\tfrac{l_{i+3}-4l_{i+2}+5l_{i+1}+5l_{i-1}+4l_{i-2}-l_{i-3}}{2h^5}\\&+a_2(z_i)\tfrac{l_{i+2}-4l_{i+1}+6l_i-4l_{i-1}+l_{i-2}}{h^4}+a_3(z_i)\tfrac{l_{i+2}-2l_{i+1}+2l_{i-1}-l_{i-2}}{2h^3}+a_4(z_i)\\&\tfrac{l_{i+1}-2l_i+l_{i-1}}{h^2}+a_5(z_i)\tfrac{l_{i+1}-l_{i-1}}{2h}+a_6(z_i)\tfrac{l_{i-1}+4l_i+l_{i+1}}{6}=f_i(z_i,\ \tfrac{l_{i-1}+4l_i+l_{i+1}}{6},\\&\tfrac{l_{i+1}-l_{i-1}}{2h},\ \tfrac{l_{i+1}-2l_i+l_{i-1}}{h^2},\ \tfrac{l_{i+2}-2l_{i+1}+2l_{i-1}-l_{i-2}}{2h^3},\ \tfrac{l_{i+2}-4l_{i+1}+6l_i-4l_{i-1}+l_{i-2}}{h^4},\\&\tfrac{l_{i+3}-4l_{i+2}+5l_{i+1}+5l_{i-1}+4l_{i-2}-l_{i-3}}{2h^5}\ ),\quad z\in[a,\ b]\end{aligned}\tag{3. 11}$$

Let

$$\begin{aligned}&f_i(z_i,\tfrac{l_{i-1}+4l_i+l_{i+1}}{6},\tfrac{l_{i+1}-l_{i-1}}{2h},\tfrac{l_{i+1}-2l_i+l_{i-1}}{h^2},\tfrac{l_{i+2}-2l_{i+1}+2l_{i-1}-l_{i-2}}{2h^3},\\&\tfrac{l_{i+2}-4l_{i+1}+6l_i-4l_{i-1}+l_{i-2}}{h^4},\tfrac{l_{i+3}-4l_{i+2}+5l_{i+1}-5l_{i-1}+4l_{i-2}-l_{i-3}}{2h^5})=\mathrm{L}_i\end{aligned}$$

Simplifying it becomes

$$\begin{aligned}&6\ (l_{i+3}-6\ l_{i+2}+15\ l_{i+1}-20\ l_i+15\ l_{i-1}-6\ l_{i-2}+l_{i-3})+3\ h\ a_1(z_i)(l_{i+3}-\\&4\ l_{i+2}+5\ l_{i+1}-5\ l_{i-1}+4\ l_{i-2}-l_{i-3})+6\ h^2\ a_2(z_i)(l_{i+2}-4\ l_{i+1}+6\ l_i-4\ l_{i-1}\\&+l_{i-2})+3\ h^3a_3(z_i)(l_{i+2}-2\ l_{i+1}+2\ l_{i-1}-l_{i-2})+6\ h^4\ a_4(z_i)(l_{i+1}-2\ l_i+l_{i-1})\\&+3\ h^5\ a_5(z_i)(l_{i+1}-l_{i-1})+h^6\ a_6(z_i)(l_{i-1}+4\ l_i+l_{i+1})=6\ h^6\ L_i,\ z\in[a,\ b]\end{aligned}\tag{3. 12}$$

By solving *equation* (3. 12 ) we will have a linear system of (*n - 3)* equations $(i = 2,\ 3,\ ...,\ n - 2)$ with (*n + 3)* unknowns $l_i$ where$(i = -1,0,1,\ ...,n + 1)$, so six more equations are desirable. By the boundary conditions at *z = a*, we get:

$$\begin{aligned}&w\,(a)=\alpha_0 &&\Rightarrow l_{-1}+4l_0+l_1=6\alpha_0\\&\mathrm{w}'\,(a)=\alpha_1 &&\Rightarrow -l_{-1}+l_1=2\alpha_1 h\\&w\,''\,(a)=\alpha_2 &&\Rightarrow l_{-1}-2l_0+l_1=\alpha_2 h^2\end{aligned}\tag{3. 13}$$

Similarly for *z = b*

$$\begin{aligned}&w\,(b)=\beta_0 &&\Rightarrow l_{n-1}+4l_n+l_{n+1}=6\beta_0\\&w'\,(b)=\beta_1 &&\Rightarrow -l_{n-1}+l_{n+1}=2\beta_1 h\\&w\,''\,(b)=\beta_2 &&\Rightarrow l_{n-1}-2l_n+l_{n+1}=\beta_2 h^2\end{aligned}\tag{3. 14}$$

The approximate solution $w(z) = s(z) = \sum_{i=-1}^{n+1} l_i B_i(z)$ is attained by resolving the overhead system of *(n+3)* equations in *(n+3)* unknowns using the above set of equations (3. 12 )-(3. 14 ) as it was prescribed that this method transforms the boundary value problem to a system of linear equations. The algorithm we have developed in this paper is not only simply the approximation solution of the 6th order boundary value problems using Cubic-B spline but it also describes the estimated derivatives of 1st order to 6th order of the analytic solution at the same time.

## 4. Results

In the numerical section, we have solved three examples to show the efficiency of Cubic-B spline method. Obviously, the results of our method are very encouraging because this method signifies the fastest convergence as well as an incredible low error. These three problems are presented to authenticate the speculative exploration and demonstrate the legitimacy and applicability of the technique.

**Problem 1.**

$$w^{(6)}(z) = e^{-z}(w(z))^2, 0 \leq z \leq 1$$

subject to

$$\begin{aligned} &w(0) = 1, && w(1) = e, \\ &w^{(1)}(0) = 1, && w^{(1)}(1) = e, \\ &w^{(2)}(0) = 1, && w^{(2)}(1) = e. \end{aligned}$$

The exact solution for the above problem is given by $w(z) = e^z$. The proposed technique is verified on this problem where the domain [0, 1] is distributed into 10 equivalent subintervals. We will have 13 equations in 13 unknowns using the above set of equations (3. 12 )-(3. 14 ), 6 equations from (3. 12 ), three equations fom (3. 13 ) and three equations from (3. 14 ). Arithmetical outcomes for this problem are revealed in Table 2 and Table 3 for $h = \frac{1}{10}$. The values of unknowns $l_i$ where$(i = -1, 0, , \ldots, n + 1)$ are

$l_{-1}$ = 0.903333333333333, $l_0$ = 0.998333333333333, $l_1$ = 1.103333333333333,
$l_2$ = 1.219382535625033, $l_3$ = 1.347638945427880, $l_4$ = 1.489381893988865,
$l_5$ = 1.646025783113693, $l_6$ = 1.819135179612557, $l_7$ = 2.010441401651985,
$l_8$ = 2.221860753925479, $l_9$ = 2.455514585041338, $l_{10}$ = 2.713751358744947,
$l_{11}$ = 2.999170950733147.

**Table 2 of Problem 1.**

| z | Exact Solution | Cubic B-Spline Solution | Cubic B-Spline Error | In [24] and [41] | In [39] |
|---|---|---|---|---|---|
| 0.1 | 1.10517091807564 | 1.10517486704861 | -3.95E-06 | 1.233E-04 | -1.2E-04 |
| 0.2 | 1.22140275816017 | 1.22141707021022 | -1.43E-05 | 2.354E-04 | -2.3E-04 |
| 0.3 | 1.34985880757600 | 1.34988670188757 | -2.79E-05 | 3.257E-04 | -3.2E-04 |
| 0.4 | 1.49182469764127 | 1.49186538408283 | -4.07E-05 | 3.855E-04 | -3.8E-04 |
| 0.5 | 1.64872127070012 | 1.64877003434269 | -4.88E-05 | 4.086E-04 | - 4.0E-04 |
| 0.6 | 1.82211880039050 | 1.82216798386931 | -4.92E-05 | 3.919E-04 | -3.9E-04 |
| 0.7 | 2.01375270747047 | 2.01379359002433 | -4.09E-05 | 3.361E-04 | -3.3E-04 |
| 0.8 | 2.22554092849246 | 2.22556650039920 | -2.56E-05 | 2.459E-04 | -2.4E-04 |
| 0.9 | 2.45960311115695 | 2.45961174213929 | -8.63E-06 | 1.299E-04 | -1.2E-04 |

Error* = Exact solution - Approximate solution.

**Table 3 of Problem 1.**

| z | $w^{(1)}(z)$ | $w^{(2)}(z)$ | $w^{(3)}(z)$ | $w^{(4)}(z)$ | $w^{(5)}(z)$ |
|---|---|---|---|---|---|
| 0.1 | -7.51E-05 | 2.51E-04 | 1.71E-02 | 1.57E-03 | |
| 0.2 | -1.25E-04 | 6.82E-04 | 8.14E-03 | 2.73E-03 | -2.08E-01 |
| 0.3 | -1.38E-04 | 1.20E-03 | -8.47E-04 | 2.99E-03 | -2.21E-01 |
| 0.4 | -1.09E-04 | 1.73E-03 | -9.83E-03 | 2.34E-03 | -2.34E-01 |
| 0.5 | -4.52E-05 | 2.17E-03 | -1.88E-02 | 7.79E-04 | -2.49E-01 |
| 0.6 | 4.07E-05 | 2.44E-03 | -2.77E-02 | -1.69E-03 | -2.66E-01 |
| 0.7 | 1.25E-04 | 2.44E-03 | -3.67E-02 | -5.07E-03 | -2.84E-01 |
| 0.8 | 1.75E-04 | 2.09E-03 | -4.56E-02 | -9.37E-03 | -3.05E-01 |
| 0.9 | 1.50E-04 | 1.31E-03 | -5.45E-02 | -1.46E-02 | |

**Problem 2.**

$$w^{(6)}(z) = 20e^{-36w(z)} - 40(1+z)^{-6}, 0 \leq z \leq 1$$

subject to

$$w\left(0\right) = 0, \qquad w\left(1\right) = \frac{1}{6}\log 2,$$
$$w^{(1)}\left(0\right) = \frac{1}{6}, \qquad w^{(1)}\left(1\right) = \frac{1}{12},$$
$$w^{(2)}\left(0\right) = -\frac{1}{6}, \qquad w^{(2)}\left(1\right) = -\frac{1}{24}.$$

The precise solution for the overhead problem is specified by $w\left(z\right) = \frac{1}{6}\log(1+z)$. The suggested technique is confirmed on this problem where the domain [0, 1] is divided into 10 equivalent subintervals. We will have 13 equations in 13 unknowns using the above set of equations (3. 12 )-(3. 14 ), 6 equations from (3. 12 ), three equations fom (3. 13 ) and three equations from (3. 14 ). Algebraic outcomes for this problem are presented in Table 4 and Table 5 for$h = \frac{1}{10}$. The maximum absolute errors of problem 2 is discussed in [14] and [35] as follows in Table 6. The values of unknowns $l_i$ where$(i = -1, 0, 1, \ldots, n + 1)$ are

$l_{-1}$ = -0.017222222222222, $l_0$ = 0.000277777777778, $l_1$ = 0.016111111111111, $l_2$ = 0.030570248061547, $l_3$ = 0.043877250319776, $l_4$ = 0.056204036753635, $l_5$ = 0.067685953317041, $l_6$ = 0.078431201448422, $l_7$ = 0.088527611743725, $l_8$ = 0.098047662706358, $l_9$ = 0.107052307871102, $l_{10}$ = 0.115593974537769, $l_{11}$ = 0.123718974537769.

**Table 4 of Problem 2.**

| z | Exact Solution | Cubic B-Spline Solution | Cubic B-Spline Error |
|---|---|---|---|
| 0.1 | 0.015885029967387 | 0.015882078380628 | 2.95E-06 |
| 0.2 | 0.030386926132326 | 0.030378225612846 | 8.70E-06 |
| 0.3 | 0.043727377411249 | 0.043713881015714 | 1.35E-05 |
| 0.4 | 0.056078706103535 | 0.056063225108559 | 1.55E-05 |
| 0.5 | 0.067577518018027 | 0.067563175245037 | 1.43E-05 |
| 0.6 | 0.078333938207623 | 0.078323061809076 | 1.09E-05 |
| 0.7 | 0.088438041843695 | 0.088431551854947 | 6.49E-06 |
| 0.8 | 0.097964444150353 | 0.097961761740043 | 2.68E-06 |
| 0.9 | 0.106975647695399 | 0.106975144788089 | 5.03E-07 |

Error* = Exact solution - Approximate solution.

**Table 5 of Problem 2.**

| z | $w^{(1)}(z)$ | $w^{(2)}(z)$ | $w^{(3)}(z)$ | $w^{(4)}(z)$ | $w^{(5)}(z)$ |
|---|---|---|---|---|---|
| 0.1 | 5.28E-05 | -3.21E-04 | -6.83E-03 | 2.11E-02 | |
| 0.2 | 5.82E-05 | -5.27E-04 | -4.09E-03 | 1.92E-02 | -8.43E-02 |
| 0.3 | 3.62E-05 | -5.98E-04 | -1.91E-03 | 1.56E-02 | -7.26E-02 |
| 0.4 | 4.10E-06 | -5.47E-04 | -2.97E-04 | 1.11E-02 | -6.64E-02 |
| 0.5 | -2.47E-05 | -4.07E-04 | 7.49E-04 | 6.18E-03 | -6.29E-02 |
| 0.6 | -4.16E-05 | -2.20E-04 | 1.23E-03 | 9.33E-04 | -6.08E-02 |
| 0.7 | -4.31E-05 | -3.42E-05 | 1.13E-03 | -4.48E-03 | -5.96E-02 |
| 0.8 | -3.09E-05 | 1.00E-04 | 4.66E-04 | -1.00E-02 | -5.88E-02 |
| 0.9 | -1.23E-05 | 1.30E-04 | -7.72E-04 | -1.56E-02 | |

**Table 6 of Problem 2.**

| | h=1/8 | h=1/16 | h=1/32 |
|---|---|---|---|
| Second Order Method [14] | $1.627 \times 10^{-4}$ | $7.31 \times 10^{-6}$ | $1.87 \times 10^{-6}$ |
| Second Order Method [34] | $2.3 \times 10^{-3}$ | $3.4 \times 10^{-4}$ | $1.54 \times 10^{-4}$ |

**Problem 3.**

$$w^{(6)}(z) = e^{z}(w(z))^{3}, 0 \leq z \leq 1$$

subject to

$$w\left(0\right) = 1, \qquad w\left(1\right) = e^{-1/2},$$

$$w^{(1)}\left(0\right) = -\frac{1}{2}, \qquad w^{(1)}\left(1\right) = -\frac{1}{2}e^{-1/2},$$

$$w^{(2)}\left(0\right) = \frac{1}{4}, \qquad w^{(2)}\left(1\right) = \frac{1}{4}e^{-1/2}.$$

The exact solution for the above problem is given by $w\left(z\right) = e^{-z/2}$. The proposed technique is verified on this problem where the domain [0, 1] is distributed into 10 equivalent subintervals. We will have 13 equations in 13 unknowns using the above set of equations (3. 12 )-(3. 14 ), 6 equations from (3. 12 ), three equations fom (3. 13 ) and three equations from (3. 14 ). Arithmetical outcomes for this problem are revealed in Table 7 and Table 8 for$h = \frac{1}{10}$. The values of unknowns $l_i$ where$(i = \ -1, 0, 1, \ ..., n \ + \ 1)$ are

$l_{-1}$ = 1.0508333333333, $l_0$ = 0.9995833333333, $l_1$ = 0.9508333333333, $l_2$ = 0.9044410535516, $l_3$ = 0.8602973847039, $l_4$ = 0.8183086432231, $l_5$ = 0.7783842551151, $l_6$ = 0.7404296031360, $l_7$ = 0.7043437849497, $l_8$ = 0.6700220425020, $l_9$ = 0.6373626349147, $l_{10}$ = 0.6062779386044, $l_{11}$ = 0.5767095689434.

**Table 7 of Problem 3.**

| z | Exact Solution | Cubic B-Spline Solution | Cubic B-Spline Error | DTM and ADM[6] |
|---|---|---|---|---|
| 0.1 | 0.9512294245007 | 0.9512262867030 | 3.14E-06 | 1.89E-03 |
| 0.2 | 0.9048374180360 | 0.9048158220406 | 2.16E-05 | 0.90E-02 |
| 0.3 | 0.8607079764251 | 0.8606565392650 | 5.14E-05 | 3.52E-02 |
| 0.4 | 0.8187307530780 | 0.8186527021186 | 7.81E-05 | 6.49E-02 |
| 0.5 | 0.7788007830714 | 0.7787125444699 | 8.82E-05 | 9.44E-02 |
| 0.6 | 0.7408182206817 | 0.7407410754348 | 7.71E-05 | 1.15E-01 |
| 0.7 | 0.7046880897187 | 0.7046377975728 | 5.03E-05 | -3.28E-04 |
| 0.8 | 0.6703200460356 | 0.6702990983120 | 2.09E-05 | -2.41E-04 |
| 0.9 | 0.6376281516218 | 0.6376250867942 | 3.06E-06 | -1.28E-04 |

Error* = Exact solution - Approximate solution.

**Table 8 of Problem 3.**

| z | $w^{(1)}(z)$ | $w^{(2)}(z)$ | $w^{(3)}(z)$ | $w^{(4)}(z)$ | $w^{(5)}(z)$ |
|---|---|---|---|---|---|
| 0.1 | 9.67E-05 | 2.04E-03 | 6.79E-03 | -2.72E-01 | |
| 0.2 | 2.61E-04 | 1.35E-03 | -1.17E-02 | -9.77E-02 | 1.47E+00 |
| 0.3 | 3.08E-04 | -3.16E-04 | -1.55E-02 | 2.27E-02 | 9.47E-01 |
| 0.4 | 2.00E-04 | -1.75E-03 | -9.75E-03 | 9.16E-02 | 4.44E-01 |
| 0.5 | -5.19E-06 | -2.27E-03 | 4.10E-04 | 1.12E-01 | -3.37E-02 |
| 0.6 | -2.07E-04 | -1.68E-03 | 1.02E-02 | 8.49E-02 | -4.88E-01 |
| 0.7 | -3.06E-04 | -2.36E-04 | 1.52E-02 | 1.39E-02 | -9.21E-01 |
| 0.8 | -2.54E-04 | 1.35E-03 | 1.09E-02 | -9.93E-02 | -1.33E+00 |
| 0.9 | -9.36E-05 | 1.94E-03 | -6.70E-03 | -2.53E-01 | |

## 5. Conclusion

The preceding segments demonstrate that the cubic-B spline technique is a sensible tactic to the numerical solution of nonlinear sixth order BVP. The computations associated with the examples discussed above were performed by using Matlab R2015a. The algebraic outcomes validate the effectiveness and accurateness of the anticipated scheme. The proposed algorithm produced a rapidly convergent series. We recommend that Cubic-B spline technique can also be accommodating when we investigate further higher order boundary value problems. It works well for higher order problems and represents the fastest convergence as well as a remarkable low error.

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