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A Mathematical Analysis of Magnetohydrodynamic Generalized Burger Fluid for Permeable Oscillating Plate

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Abstract. In this article, the mathematical analysis for an electrically conducting flow of generalized fractional Burgers' fluid with permeable oscillating plate is investigated. The integral transformation approach is utilized for tracing out the analytical solutions of velocity field and shear stress with and without magnetic field and porosity. The general solutions have been expressed in terms of product of Gamma functions and Fox-**H** function satisfying imposed conditions. The obtained solutions have been particularized for several limiting cases, such as (i) the solutions are retrieved in the absence of magnetic field, (ii) the solutions are retrieved in the absence of permeability, (iii) the solutions are retrieved ordinary differential operator, (iv) the solutions are particularized for fractional Burger, fractional Oldroyd-B, fractional Maxwell fluid and fractional Newtonian fluids . Finally in order to highlight the similarities and differences among various rheological parameters, the graphical illustration has been depicted for fluid flows.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Generalized Burger's model, Porosity, Magnetic Field, Analytical solutions and rheological parameters.

1. INTRODUCTION

There is no denying fact that the conventional Navier-Stokes viscous model (Newtonian) cannot validate the intrinsic characteristic of rheological (non-Newtonian) fluids. The several rheological phenomenons such as stress differences, relaxation, retardation, shear-thinning/thickening, Weissenberg effects, micro-structure, yield stress, spurt, elongation, re-coil, fading memory and many others are not characterized in the range of Newtonian fluid dynamics. The scientists and researchers have diverted their attention for knowing the similarities and differences between Newtonian and non- Newtonian fluids. Meanwhile the excellent recent studies have been performed by Irgens [5] and Chaabra and Richardson [6]. The typical non-Newtonian viscoelastic models have been investigated under different circumstances with various distinct constitutive formulations, for instance second-third-grade liquid models [33,34], Maxwell liquid model[7,2], Oldroyd-B liquid model [24], Burger liquid model [21], Microploar liquid model [8], Casson Liquid model [12],Jeffrey liquid model [38], Walter's liquid model [9], Brinkman-type liquid model [10], such models have been analyzed in various studies in medical, environmental and chemical engineering systems. The fractional derivative is the theory of differentiation to arbitrary non-integers. The concept of fractional derivative was developed by true researchers namely Abel, Fourier, Riesz, Liouville, Riemann and few others. The modeling of the problem with fractional derivatives provides the physical behavior in several applications in electrochemistry, diffusion, electromagnetism, general transport theory and engineering. In nut shell, the Caputo fractional derivative operator is most suitable time fractional derivative which useful in fluid mechanics. Caputo fractional derivative operator has become most suitable time fractional derivative, because it delivers full description of memory [24]. The applications of fractional calculus in rheological fluid include the recent studies here, Masood and Tasawar [31] analyzed fractional generalized Burger fluid with magnetic field in a porous space. They investigated closed form solutions for velocity field by employing modified Darcy's law with limiting cases for Burgers' fluids, Oldroyd-B, Maxwell, second grade and viscous fluids. Jamil and Khan [15] investigated the helical flow of fractional generalized Burger fluid in cyliderical geometries. They investigated the solutions for linear and angular velocities using Hankel and Laplace transform. They also recovered the solutions for ordinary differential operator by making fractional parameter equal to one. Jamil [16] traced out the analytical solutions of first problem of stokes for generalized Burger fluid. They explored velocity field and the adequate shear stress using Fourier sine and Laplace transform methods. Tong and Shan [37] analyzed annular pipe for unidirectional and unsteady flow of generalized Burger fluid by solving two problems, one is axial Couette flow in a annulus and second is poiseuille flow due to a constant pressure gradient. They found exact solutions by considering the geometry of two coaxial cylinders. Jamil and et al. [17] observed the effects of coaxial cylinders for rotational flow of a generalized Burgers fluid subject to the accelerated shear stress. They found out exact analytical solutions and expressed them in terms of series form and first and second kind of Bessel functions. Ilyas et al. [13] presented exact solutions of Stokes second problem for Burgers fluid subject to certain conditions for relaxation time. They analyzed the effects of Hartmann number by applying numerical integration on velocity field and tangential stress with several rheological parameters as well. In another study, Ilyas and Shafie [14] studied rotational flows of generalized Burgers' fluid for cosine and sine oscillations with magnetohydrodynamic in porous medium. They explained the strong effects of oscillating and angular frequency, Hartmann number and porosity of the medium on their obtained closed solutions Awan et al. [3] studied coaxial circular cylinders for longitudinal flow of a fractional viscoelastic fluid by invoking integral transforms. In another study, Awan et al. [4] analyzed the effects of chemical reaction for viscous incompressible unsteady flow under oscillating plate with linear concentration and temperature. Nauman [32] investigated analytical solution by invoking non-integer Caputo time fractional derivatives for unsteady rotational flow of second grade fluid within cylindrical geometry. Kobra and Alireza [28] investigated the mathematical model with porosity for steady flow of third grade fluid using numerical approach on non-linear two-point boundary value problem.

Off course the study of generalized Burgers' fluid, differentiation to arbitrary non-integers, magnetohydrodynamic and porosity can be continued but we end here by citing very recent studies concerned with them [1,11,18,19,22,25,26,27,29,30,35,36]. Motivating from above discussions, our aim is to investigate the mathematical analysis for an electrically conducting flow of generalized fractional Burgers' fluid with permeable oscillating plate is investigated. The integral transformation approach is utilized for tracing out the analytical solutions of velocity field and shear stress with and without magnetic field and porosity. The general solutions have been expressed in terms of product of Gamma functions and Fox-H function satisfying imposed conditions. The obtained solutions have been particularized for several limiting cases, such as (i) the solutions are retrieved in the absence of magnetic field when $M \to 0$, (ii) the solutions are retrieved in the absence of permeability when $\Phi \to 0$, (iii) the solutions are retrieved for ordinary differential operator when $\psi \to 1$ (iv) the solutions are particularized for fractional Burger, fractional Oldroyd-B, fractional Maxwell fluid and fractional Newtonian fluids when $\lambda_4 \to 0$,$\lambda_4 \to \lambda_2 \to 0$, $\lambda_4 \to \lambda_3 \to \lambda_2 \to 0$, and $\lambda_4 \to \lambda_3 \to \lambda_2 \to \lambda_1 \to 0$ respectively. Finally in order to highlight the similarities and differences among various rheological parameters, the graphical illustration is has been depicted for fluid flows.

2. STATEMENT AND GOVERNING EQUATIONS OF PROBLEM

We consider the flow of generalized fractional Burgers' fluid with magnetohydrodynamic for permeable oscillating plate. The rheological equations for generalized Burgers' fluid can be characterized as [13,14,17]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \mu\lambda_4 \frac{\delta^2 \mathbf{A}_1}{\delta t^2} + \mu\lambda_3 \frac{\delta \mathbf{A}_1}{\delta t} - \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} - \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \mu \mathbf{A}_1 = \mathbf{S}, \tag{2. 1}$$

where, $p, \mathbf{I}, \mathbf{S}, \mu, A, L, \lambda_1, \lambda_3(< \lambda_1), \lambda_2, \lambda_4,$ and $\frac{\delta}{\delta t}$ the pressure, the identity tensor, the extra-stress tensor, the dynamic viscosity, the first Rivilin-Ericson tensor, velocity gradient, relaxation, retardation time, material parameter of generalized Burgers' fluid and upper convective time derivative respectively, while upper convective time derivative is stated as,

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{\delta \mathbf{S}}{\delta t} - \mathbf{LS} - \mathbf{ST}^T, \frac{\delta^2 \mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta \mathbf{S}}{\delta t}\right), \tag{2. 2}$$

Here, $\frac{d}{dt}$ and $\frac{\delta}{\delta t}$ are material time derivative and upper convected derivative. The well known governing equations for an unsteady flow of incompressible fluid are describe as below:

$$\mathbf{divV} = 0, \tag{2. 3}$$

$$\rho \frac{d\mathbf{V}}{dt} = div\mathbf{T} - \sigma B_0^2 \mathbf{V} + \mathbf{R}, \tag{2. 4}$$

Where, $\rho, \mathbf{V}, \sigma, B_0$, and **R** the density of the fluid, , the velocity, the electrical conductivity of the fluid, the applied magnetic field's magnitude and denotes the Darcy's resistance respectively. We have taken a velocity field and extra-stress tensor of the form for this problem as under

$$\mathbf{V} = \mathbf{V}(y,t) = u(y,t)i, \mathbf{S} = \mathbf{S}(y,t), \tag{2. 5}$$

Keeping initial conditions in mind, we defined as

$$\mathbf{S}(y,0) = 0, \mathbf{V}(y,0) = 0, \frac{\partial \mathbf{S}(y,0)}{\partial t} = 0, \tag{2. 6}$$

implementing equation (2.5) in equation (2.1), yields $\tau_{zz} = \tau_{xz} = \tau_{yy} = \tau_{yz} = 0$ and the suitable equation is

$$\left(\lambda_2 \frac{\partial^2 S_{xy}}{\partial t^2} + \lambda_1 \frac{\partial S_{xy}}{\partial t} + S_{xy}\right) - \frac{\partial u}{\partial y}\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\mu = 0, \tag{2. 7}$$

in which tangential stress is S_{xy}. As per previously published [34], the generalized Burger fluid has relation for **R** is

$$\left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right)\mathbf{R}$$
$$+\left(\lambda_4 \frac{\partial^2 \mathbf{V}(y,t)}{\partial t^2} + \lambda_3 \frac{\partial \mathbf{V}(y,t)}{\partial t} + \mathbf{V}(y,t)\right)\frac{\mu\phi}{k} = 0, \tag{2. 8}$$

where k, ϕ are the permeability and porosity respectively. By considering that flow direction is free from pressure gradient, balance of linear momentum, absence of body forces and implementing equation (2.5) into equation (2.4) keeping with equations (2.7, 2.8), we obtain the governing equations for the generalized Burger fluid as [13,14,17]

$$\frac{\partial u(y,t)}{\partial t}\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) + M\left(u(y,t) + \lambda_1 \frac{\partial u(y,t)}{\partial t} + \lambda_2 \frac{\partial^2 u(y,t)}{\partial t^2}\right)$$
$$+\Phi\left(u(y,t) + \lambda_3 \frac{\partial u(y,t)}{\partial t} + \lambda_4 \frac{\partial^2 u(y,t)}{\partial t^2}\right)$$
$$-\nu\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\frac{\partial^2 u(y,t)}{\partial y^2} = 0, \tag{2. 9}$$

$$\frac{\partial u(y,t)}{\partial t}\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\mu$$

$$-\left(\tau(y,t)+\lambda_1\frac{\partial \tau(y,t)}{\partial t}+\lambda_2\frac{\partial^2\tau(y,t)}{\partial t^2}\right)=0, \qquad (2.\ 10)$$

Where, $M=\frac{\sigma B_0}{\rho}$ is applied magnetic field, $\Phi=\frac{\mu\phi}{k}$, is porous medium and, $\nu=\frac{\mu}{\rho}$, is kinematic viscosity of the fluid. Meanwhile, using the concept of non-integer order derivative, so called Caputo fractional operator, we investigated governing equations for the generalized Burger fluid in the form of non-integer order derivative or Caputo fractional derivative as

$$\frac{\partial u(y,t)}{\partial t}\left(1+\lambda_1\frac{\partial^{\psi}}{\partial t^{\psi}}+\lambda_2\frac{\partial^{2\psi}}{\partial t^{2\psi}}\right)+$$

$$M\left(u(y,t)+\lambda_1\frac{\partial^{\psi}u(y,t)}{\partial t^{\psi}}+\lambda_2\frac{\partial^{2\psi}u(y,t)}{\partial t^{2\psi}}\right)$$

$$+\Phi\left(u(y,t)+\lambda_3\frac{\partial^{\psi}u(y,t)}{\partial t^{\psi}}+\lambda_4\frac{\partial^{2\psi}u(y,t)}{\partial t^{2\psi}}\right)$$

$$-\nu\left(\lambda_4\frac{\partial^{2\psi}}{\partial t^{2\psi}}+\lambda_3\frac{\partial^{\psi}}{\partial t^{\psi}}+1\right)\frac{\partial^2 u(y,t)}{\partial y^2}=0, \qquad (2.\ 11)$$

$$\frac{\partial u(y,t)}{\partial t}\left(\lambda_4\frac{\partial^{2\psi}}{\partial t^{2\psi}}+\lambda_3\frac{\partial^{\psi}}{\partial t^{\psi}}+1\right)$$

$$\mu-\left(\tau(y,t)+\lambda_1\frac{\partial^{\psi}\tau(y,t)}{\partial t^{\psi}}+\lambda_2\frac{\partial^{2\psi}\tau(y,t)}{\partial t^{2\psi}}\right)=0, \qquad (2.\ 12)$$

Where, $\frac{\partial^{\psi}u(y,t)}{\partial t^{\psi}}$ is so called Caputo fractional operator described as [20]

$$\frac{\partial^{\psi}u(y,t)}{\partial t^{\psi}}=\int_0^t\frac{u^{'}(y,t)d\chi}{\Gamma(1-\psi)(t-\chi)^{\psi}},0<\chi<1. \qquad (2.\ 13)$$

3. Statement of the problem

Here an electrically conducting incompressible generalized fractional Burgers fluid is considered with porous medium occupying the space above an oscillating plate which is situated perpendicular to the y-axis. The plate is saturated under an influence of magnetic field B_0 and permeability. At the moment $t=0^+$ the plate moves to oscillations in its own plane with velocity $u(0,t)=u_0H(t)cos(\Omega t)$ or $u(0,t)=u_0H(t)sin(\Omega t)$. Owing to shear, the fluid above the plate is gradually moved, while the fractional governing equations are given by the equations (2.11) and (2.12) and corresponding to the imposed conditions are:

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=\frac{\partial^2 u(y,0)}{\partial t^2}=0,\ \ y>0, \qquad (3.\ 14)$$

$$u(0,t) = u_0 H(t) cos(\Omega t) \ \ or \ \ u(0,t) = u_0 H(t) sin(\Omega t), \ \ t \geq 0, \tag{3. 15}$$

$$u(y,t), \frac{\partial u(y,0)}{\partial t} \to 0 \ \ as \ \ y \to \infty \ \ and \ \ t > 0. \tag{3. 16}$$

In order to have analytical solutions of fractional differential equations (2.11-2.12) and above imposed conditions (3.14-3.16), we shall apply integral transform methods

4. Analysis of the problem

4.1. **Mathematical analysis of velocity field.**

Case-I: Cosine Oscillations $u(0,t) = u_0 H(t) cos(\Omega t)$

Apply the Laplace transform to equation (2.11) and having in mind the imposed conditions (3.14-3.16), we arrive at

$$\frac{\partial^2 \overline{u}(y,s)}{\partial t^2} = \frac{(s + M + \Phi)(1 + s^{\psi}\lambda_1 + s^{2\psi}\lambda_2)}{(1 + s^{\psi}\lambda_3 + s^{2\psi}\lambda_4)\nu} \overline{u}(y,s), \tag{4. 17}$$

subject to the imposed conditions $\overline{u}(0,s) = \frac{su_0}{s^2+\Omega^2}$ and $\overline{u}(y,s) \to \frac{\partial \overline{u}(y,s)}{\partial y} \to 0 \ \ as \ \ y \to \infty$ and $\overline{u}(y,s) = L\bigg(u(y,t)\bigg)$. the solution of the equation(4.17) is obtained as

$$\overline{u}(y,s) = \frac{su_0}{s^2+\Omega^2} Exp\Bigg(- y\sqrt{\frac{(s + M + \Phi)(1 + s^{\psi}\lambda_1 + s^{2\psi}\lambda_2)}{(1 + s^{\psi}\lambda_3 + s^{2\psi}\lambda_4)\nu}}\Bigg), \tag{4. 18}$$

In order to justify the initial and boundary condition, we present equation (4.18) equivalently as

$$\overline{u}(y,s) = \frac{su_0}{s^2+\Omega^2} + \frac{su_0}{s^2+\Omega^2} \sum_{\xi_1=1}^{\infty} \frac{(-y\sqrt{\lambda_1})^{\xi_1}}{\xi_1!\sqrt{\nu}} \sum_{\xi_2=0}^{\infty} \frac{(-1)^{\xi_2}}{\xi_2!} \sum_{\xi_3=0}^{\infty} \frac{1}{\xi_3!}\bigg(-\frac{\lambda_2}{\lambda_1}\bigg)^{\xi_3}$$

$$\times \sum_{\xi_4=0}^{\infty} \frac{(-M-\Phi)^{\xi_4}}{\xi_4!} \sum_{\xi_5=0}^{\infty} \frac{(-1)^{\xi_5}}{\xi_5!} \sum_{\xi_6=0}^{\infty} \frac{1}{\lambda_6!}\bigg(\frac{-\lambda_4}{\lambda_3}\bigg)^{\lambda_6}$$

$$\times \frac{\Gamma(1+\xi_1)\Gamma(1+\frac{\xi_1}{2})\Gamma(1+\frac{\xi_1}{2})\Gamma(1+\xi_4)\Gamma(\xi_5+\frac{\xi_1}{2})}{\Gamma(\xi_1-\xi_2+1)\Gamma(1+\xi_3+\frac{\xi_1}{2})\Gamma(1+\xi_4+\frac{\xi_1}{2})\Gamma(\frac{\xi_1}{2})\Gamma(\xi_4-\xi_6+1)}$$

$$\times \frac{1}{s^{\frac{\xi_1}{2}-\frac{\psi\xi_1}{2}-\psi\xi_3-\psi\xi_4+\xi_4-\psi\xi_6}}, \tag{4. 19}$$

Inverting equation (4.19) by means of Laplace transform, and expressing final expression of velocity field in terms of Fox-H function [20,23] and convolution theorem, we get

$$u(y,t) = u_0 H(t) cos(\Omega t) + u_0 H(t) \sum_{\xi_1=1}^{\infty} \frac{(-y\sqrt{\lambda_1})^{\xi_1}}{\xi_1!\sqrt{\nu}} \sum_{\xi_2=0}^{\infty} \frac{(-1)^{\xi_2}}{\xi_2!}$$

$$\times \sum_{\xi_3=0}^{\infty} \frac{1}{\xi_3!}\bigg(-\frac{\lambda_2}{\lambda_1}\bigg)^{\xi_3} \sum_{\xi_4=0}^{\infty} \frac{(-M-\Phi)^{\xi_4}}{\xi_4!} \sum_{\xi_5=0}^{\infty} \frac{(-1)^{\xi_5}}{\xi_5!} \int_0^t cos\Omega(t-\delta)$$

$$\times\mathbf{H}^{1,5}_{5,7}\left[\frac{\lambda_4}{t^{\psi}\lambda_3}\Bigg|\begin{matrix}(-\xi_1,0)\left(-\frac{\xi_1}{2},0\right)\left(-\frac{\xi_1}{2},0\right)\left(1-\frac{\xi_1}{2}-\xi_5,0\right)(-\xi_4,0)\\ (0,1)\,(\xi_2-\xi_1,0)\left(\xi_3-\frac{\xi_1}{2},0\right)\left(\xi_4-\frac{\xi_1}{2},0\right)(-\xi_4,0)\\ \left(1-\frac{\xi_1}{2},0\right)\left(\frac{\psi\xi_1}{2}-\frac{\xi_1}{2}+\psi\xi_3+\psi\xi_4-\xi_4,\psi\right)\end{matrix}\right]$$

$$\times t^{\frac{\psi\xi_1}{2}-\frac{\xi_1}{2}+\psi\xi_3+\psi\xi_4-\xi_4-\psi\xi_6-1}d\delta. \tag{4. 20}$$

4.2. **Mathematical analysis of the shear stress.**

Apply the Laplace transform to equation (2.12) and having in mind the imposed conditions (3.14-3.16), we arrive at

$$\overline{\tau}(y,s)=-\mu\frac{\partial\overline{u}(y,s)}{\partial y}\left(\frac{1+s^{\psi}\lambda_3+s^{2\psi}\lambda_4}{1+s^{\psi}\lambda_1+s^{2\psi}\lambda_2}\right), \tag{4. 21}$$

differentiating equations (4.18) with respect "y" partially, we obtain simplified expression as

$$\overline{\tau}(y,s)=-\frac{\mu s u_0}{s^2+\Omega^2}\sqrt{\frac{(s+M+\Phi)(1+s^{\psi}\lambda_1+s^{2\psi}\lambda_2)}{(1+s^{\psi}\lambda_3+s^{2\psi}\lambda_4)\nu}}$$

$$\times\left(\frac{1+s^{\psi}\lambda_3+s^{2\psi}\lambda_4}{1+s^{\psi}\lambda_1+s^{2\psi}\lambda_2}\right), \tag{4. 22}$$

In order to avoid lengthy and cumbersome calculation, we present equation (4.22) equivalently as

$$\overline{\tau}(y,s)=-\frac{\mu s u_0}{s^2+\Omega^2\sqrt{\nu}}\sum_{\xi_1=1}^{\infty}\left(\frac{(-y)^{\xi_1}}{\xi_1!}\right)\sum_{\xi_2=0}^{\infty}\frac{(-\lambda_1)^{\xi_2}}{\xi_2!}\sum_{\xi_3=0}^{\infty}\frac{(-M-\Phi)}{\xi_{3!}}$$

$$\times\sum_{\xi_4=0}^{\infty}\frac{(-1)^{\xi_4}}{\xi_4!}\sum_{\xi_5=0}^{\infty}\frac{-1}{\xi_5!}\left(-\frac{\lambda_2}{\lambda_1}\right)^{\xi_5}\sum_{\xi_6=0}^{\infty}\frac{1}{\xi_6!}\left(\frac{-\lambda_4}{\lambda_3}\right)^{\xi_6}$$

$$\times\frac{\Gamma(1+\frac{\xi_1-1}{2})\Gamma(1+\frac{\xi_1}{2})\Gamma(\xi_4+\frac{\xi_1-1}{2})\Gamma(\xi_2-\xi_5)\Gamma(1+\xi_4)}{\Gamma(\frac{\xi_1-1}{2}-\xi_2+1)\Gamma(1+\xi_3+\frac{\xi_1}{2})\Gamma(\frac{\xi_1-1}{2})\Gamma(\xi_2-\xi_3+1)\Gamma(\xi_4-\xi_6+1)}$$

$$\times\frac{1}{s^{-\frac{\xi_1}{2}-\psi\xi_2-\psi\xi_4-\psi\xi_5+\xi_3-\psi\xi_6}}, \tag{4. 23}$$

Inverting equation (4.23) by means of Laplace transform, and expressing final expression of shear stress in terms of Fox-H function and convolution theorem, we get

$$\tau(y,t)=-\frac{\mu u_0 H(t)}{\sqrt{\nu}}\sum_{\xi_1=1}^{\infty}\left(\frac{(-y)^{\xi_1}}{\xi_1!}\right)\sum_{\xi_2=0}^{\infty}\frac{(-\lambda_1)^{\xi_2}}{\xi_2!}\sum_{\xi_3=0}^{\infty}\frac{(-M-\Phi)}{\xi_{3!}}$$

$$\sum_{\xi_4=0}^{\infty}\frac{(-1)^{\xi_4}}{\xi_4!}\sum_{\xi_5=0}^{\infty}\frac{-1}{\xi_5!}\left(-\frac{\lambda_2}{\lambda_1}\right)^{\xi_5}\int_0^t cos\Omega(t-\delta)$$

$$\times \mathbf{H}^{1,5}_{5,7}\left[\frac{\lambda_4}{t^{\psi}\lambda_3}\left|\begin{matrix}\left(\frac{\xi_1-1}{2},0\right)\left(-\frac{\xi_1}{2},0\right)\left(1-\xi_4-\frac{\xi_1-1}{2},0\right)(1-\xi_2-\xi_5,0)(-\xi_4,0)\\(0,1)\,,\left(\xi_2-\frac{\xi_1-1}{2},0\right),\left(-\xi_3-\frac{\xi_1}{2},0\right)\left(1-\frac{\xi_1-1}{2},0\right)(-\xi_4,0)\\(\xi_5-\xi_2,0)\left(\frac{-\xi_1}{2}+\psi\xi_2+\xi_3+\psi\xi_4+\psi\xi_5+2,-\psi\right)\end{matrix}\right.\right]$$

$$\times t^{\frac{-\xi_1}{2}-\psi\xi_2+\xi_3-\psi\xi_4-\psi\xi_5-\psi\xi_6-1}d\delta. \tag{4. 24}$$

Case-II: Sine Oscillations $u(0,t)=u_0H(t)sin(\Omega t)$

$$u(y,t)=u_0H(t)sin(\Omega t)+u_0H(t)\sum_{\xi_1=1}^{\infty}\frac{(-y\sqrt{\lambda_1})^{\xi_1}}{\xi_1!\sqrt{\nu}}\sum_{\xi_2=0}^{\infty}\frac{(-1)^{\xi_2}}{\xi_2!}$$

$$\times\sum_{\xi_3=0}^{\infty}\frac{-1}{\xi_3!}\left(-\frac{\lambda_2}{\lambda_1}\right)^{\xi_3}\sum_{\xi_4=0}^{\infty}\frac{(-M-\Phi)^{\xi_4}}{\xi_{4!}}\sum_{\xi_5=0}^{\infty}\frac{(-1)^{\xi_5}}{\xi_5!}\int_0^t sin\Omega(t-\delta)$$

$$\times \mathbf{H}^{1,5}_{5,7}\left[\frac{\lambda_4}{t^{\psi}\lambda_3}\left|\begin{matrix}(-\xi_1,0)\left(-\frac{\xi_1}{2},0\right)\left(-\frac{\xi_1}{2},0\right)\left(1-\frac{\xi_1}{2}-\xi_5,0\right)(-\xi_4,0)\\(0,1)\,,(\xi_2-\xi_1,0)\,,\left(\xi_3-\frac{\xi_1}{2},0\right)\left(\xi_4-\frac{\xi_1}{2},0\right)\left(1-\frac{\xi_1}{2},0\right)\\(-\xi_4,0)\left(\frac{\psi\xi_1}{2}-\frac{\xi_1}{2}+\psi\xi_3+\psi\xi_4-\xi_4,\psi\right)\end{matrix}\right.\right]$$

$$\times t^{\frac{\psi\xi_1}{2}-\frac{\xi_1}{2}+\psi\xi_3+\psi\xi_4-\xi_4-\psi\xi_6-1}d\delta, \tag{4. 25}$$

$$\tau(y,t)=-\frac{\mu u_0H(t)}{\sqrt{\nu}}\sum_{\xi_1=1}^{\infty}\left(\frac{(-y)^{\xi_1}}{\xi_1!}\right)\sum_{\xi_2=0}^{\infty}\frac{(-\lambda_1)^{\xi_2}}{\xi_2!}\sum_{\xi_3=0}^{\infty}\frac{(-M-\Phi)}{\xi_{3!}}$$

$$\sum_{\xi_4=0}^{\infty}\frac{(-1)^{\xi_4}}{\xi_4!}\sum_{\xi_5=0}^{\infty}\frac{1}{\xi_5!}\left(-\frac{\lambda_2}{\lambda_1}\right)^{\xi_5}\int_0^t sin\Omega(t-\delta)$$

$$\times \mathbf{H}^{1,5}_{5,7}\left[\frac{\lambda_4}{t^{\psi}\lambda_3}\left|\begin{matrix}\left(\frac{\xi_1-1}{2},0\right)\left(-\frac{\xi_1}{2},0\right)\left(1-\xi_4-\frac{\xi_1-1}{2},0\right)(1-\xi_2-\xi_5,0)(-\xi_4,0)\\(0,1)\,,\left(\xi_2-\frac{\xi_1-1}{2},0\right),\left(\xi_3-\frac{\xi_1}{2},0\right)\left(1-\frac{\xi_1-1}{2},0\right)(\xi_5-\xi_2,0)\\(-\xi_4,1)\left(-\frac{\xi_1}{2}+\psi\xi_2+\xi_3+\psi\xi_4+\psi\xi_5+2,-\psi\right)\end{matrix}\right.\right]$$

$$\times t^{\frac{-\xi_1}{2}-\psi\xi_2+\xi_3-\psi\xi_4-\psi\xi_5-\psi\xi_6-1}d\delta. \tag{4. 26}$$

5. RESULTS AND DISCUSSIONS

In this section, we highlight the main features and concluding remarks for the present mathematical analysis of magnetohydrodynamic and permeability. The analytical solutions of velocity field and shear stress with and without magnetic field and porosity have been established by employing discrete Laplace transform with its inversion. The general solutions have been expressed in terms of product of Gamma functions and Fox-**H** function satisfying imposed conditions. The equations (4.23-4.26) are the analytical solutions of generalized fractional Burger flow with magnetic field and permeability. In brevity, the generalized fractional Burger model has several special cases which depends upon concerned rheological parameters, for instance, letting $M \to 0$ in equations (4.23-4.26) then the solutions are termed in the absence of fluid magnetic field, making $\Phi \to 0$ in equations (4.23-4.26) then the solutions are termed in the absence of fluid porosity, putting $\psi \to 1$ in equations (4.23-4.26) then the solutions are termed in the ordinary differential operator, setting $\lambda_4 \to 0$in equations (4.23-4.26) then the solutions are termed for fractional Burger fluid, substituting $\lambda_4 \to \lambda_2 \to 0$ in equations (4.23-4.26) then the solutions are termed for fractional Oldroyd-B fluid, employing $\lambda_4 \to \lambda_3 \to \lambda_2 \to 0$ in equations (4.23-4.26) then the solutions are termed for fractional Maxwell fluid, applying,$\lambda_4 \to \lambda_3 \to \lambda_2 \to \lambda_1 \to 0$ in equations (4.23-4.26) then the solutions are termed for fractional Newtonian fluid and finally taking $\Omega \to 0$in equations (4.23-4.26) then the solutions are termed for first problem of stokes investigated by Ilyas and Sharidan [14, see equation 20]. In short, the main features concerned with results are enumerated below

• Figure 1 is prepared for time parameter, it is observed that as time increases then velocity field and shear stress are increasing function of time. Meanwhile it is noted that both velocity field and shear stress have qualitatively identical behavior of fluid flows over the boundary.

• Figure 2 is depicted to show the impact of viscosity on fluid flow, an interesting fact is achieved that velocity field and shear stress have increasing behavior of fluid flow as smaller value of viscosity is increased. In general, this phenomenon meets with true facts of shear thickening and shear thinning.

• The influence of relaxation time, retardation time and material parameters are shown in figures 3 and 4. Here, it is noted that velocity field has reciprocal trend of fluid flow. Meanwhile, velocity field is increasing and decreasing with respect to rheological parameter vice versa.

• Figure. 5 elaborates that effects of magnetic field on fluid flow, the variation in magnetic field decreases the velocity field and increases the shear stress. This may be due to the fact that resistive force is generated by applied magnetic field which is alike a drag force. Consequently, velocity field and shear stress have opposite trend due to applied magnetic field which slows down the motion of fluid flow.

• Figure. 6 is plotted for porosity; here velocity field and shear stress show scattering behavior reciprocally.

• Comparison of four ordinary and fractional models namely (i) Burger fluid model, (ii) Oldroyd-B fluid model, (iii) Maxwell fluid model and (iv) Newtonian fluid model is shown in Fig. 7. It is noted that either ordinary or fractional Newtonian fluid moves faster in comparison with remaining other models. It is also noted that all ordinary models have sequestrating behavior and fractional models have scattering one.

• Once again comparison of only ordinary and fractional Burger model with and without magnetic field and porosity is prepared in Fig. 8. It pointed out that either ordinary or fractional Burger fluid model without magnetic field and porosity moves faster in comparison with remaining other models with and without magnetic field and porosity. On the other hand, models with magnetic field have sequestrating behavior and models with porosity have scattering behavior over the whole domain of plate.

6. Conclusion

The mathematical analysis of fractional Burger fluid under the influence of magnetohydrodynamics and permeability is carried out successfully. The analytical solutions of velocity field and shear stress with and without magnetic field and porosity have been established by employing discrete Laplace transform with its inversion. The following main points are extracted from the mathematical study of fractional Burger fluid:

• The general solutions have been investigated for velocity field and shear stress and expressed in terms of product of Gamma functions and Fox-H function.

• The velocity field and shear stress are increasing function with respect to time and viscosity.

• The magnetic field, porosity, relaxation time, retardation time and material parameters have reciprocal behavior on the velocity field and shear stress.

• The comparative analysis of fractional Burger fluid for four ordinary and fractional models has been carried out for velocity field and shear stress having several similarities and differences on the fractional Burger fluid flow.

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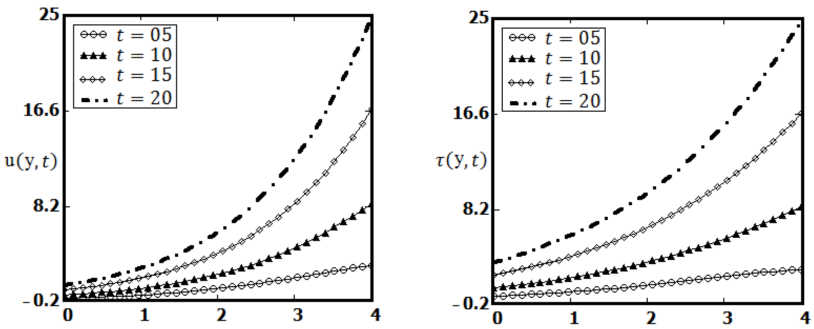

FIGURE 1. Profiles of the velocity field and shear stress for $u_0 = 1,\ \ \Omega = 0.5, \nu = 0.679, \mu = 18, \lambda_1 = 0.7,\ \lambda_2 = 1.5,\ \lambda_3 = 4,\ \lambda_4 = 3.7, \Phi = 0.5, M = 0.2, \psi = 0.3$ with different values of t.

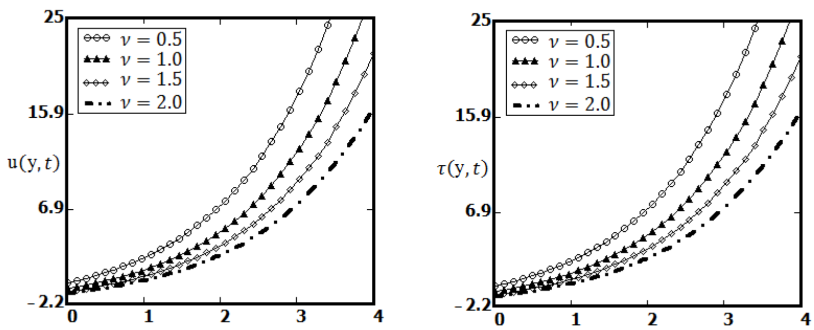

FIGURE 2. Profiles of the velocity field and shear stress for $u_0 = 1,\ \ \Omega = 0.5, t = 5s, \mu = 12\lambda_1 = 0.1,\ \lambda_2 = 1.1,\ \lambda_3 = 12,\ \lambda_4 = 6, \Phi = 0.5, M = 0.2, \psi = 0.3$ with different values of ν.

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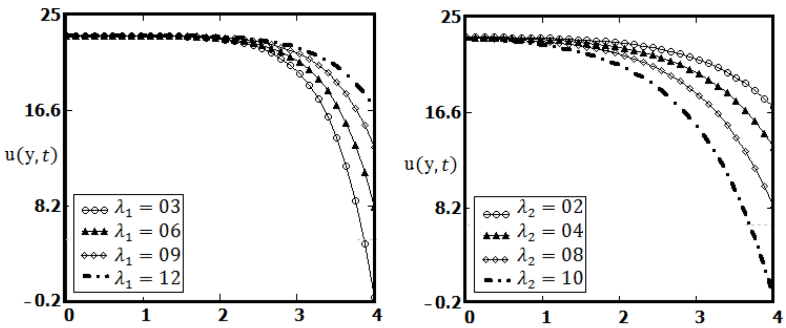

FIGURE 3. Profiles of the velocity field and shear stress for $u_0 = 2, \ \Omega = 0.1, \nu = 8, \mu = 20, \ \lambda_3 = 18, \lambda_4 = 5.5, \Phi = 0.5, M = 0.2, \psi = 0.3$ with different values of λ_1 and λ_2.

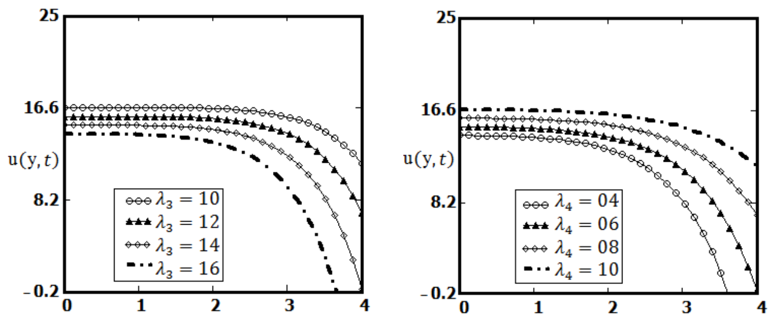

FIGURE 4. Profiles of the velocity field and shear stress for $u_0 = 5, \ \Omega = 0.5, \nu = 25, \mu = 4.3, \ \lambda_1 = 0.7, \lambda_2 = 2.4, \Phi = 0.5, M = 0.2, \psi = 0.3$ with different values of λ_3 and λ_4.

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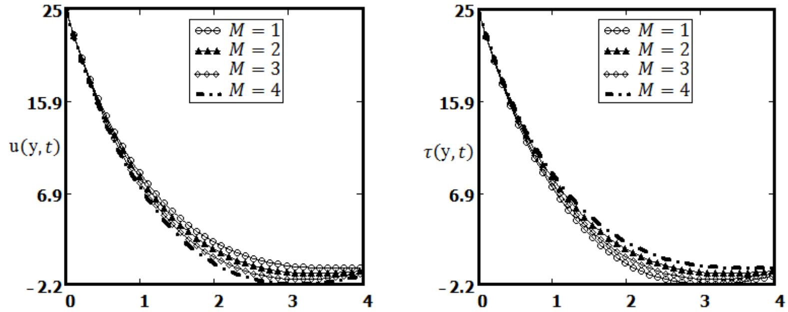

FIGURE 5. Profiles of the velocity field and shear stress for $u_0 = 1, \ \Omega = 0.5, \nu = 0.679, \mu = 11, \ \lambda_1 = 0.5, \lambda_2 = 3.2, \lambda_3 = 9, \lambda_4 = 11.7\Phi = 0.5, \psi = 0.3$ with different values of M.

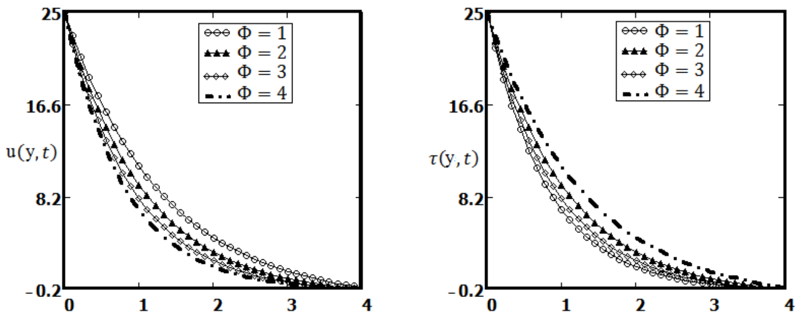

FIGURE 6. Profiles of the velocity field and shear stress for $u_0 = 1, \ \Omega = 0.5, \nu = 0.679, \mu = 3.2, \ \lambda_1 = 2.9, \lambda_2 = 4, \lambda_3 = 25, \lambda_4 = 29.25, M = 1, \psi = 0.3$ with different values of Φ.

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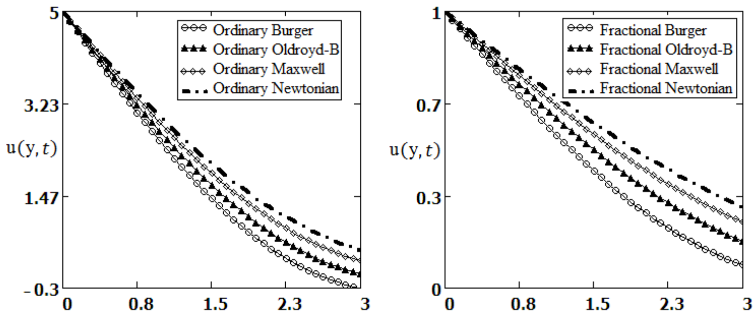

FIGURE 7. Comparison of velocity fields for four models for $u_0 = 6, \ \Omega = 0.5, \nu = 1.679, \mu = 11, \ \lambda_1 = 4, \lambda_2 = 9, \lambda_3 = 11, \lambda_4 = 20, M = 2, \Phi = 3.5$.

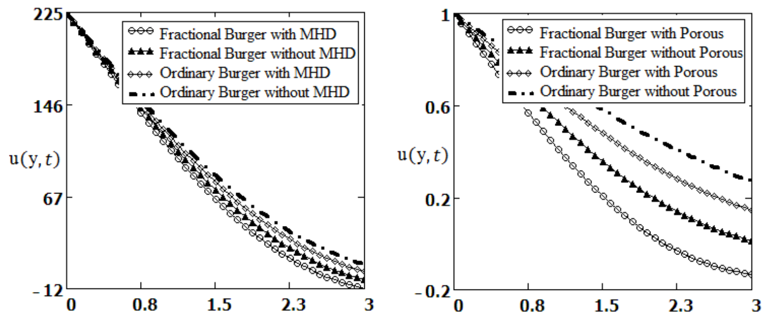

FIGURE 8. Comparison of velocity fields for four models with and without magnetic field and porous medium for $u_0 = 6, \ \Omega = 0.5, \nu = 1.679, \mu = 11, \ \lambda_1 = 4, \lambda_2 = 9, \lambda_3 = 11, \lambda_4 = 20$.

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