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Some Interval-Valued Pythagorean Fuzzy Weighted Averaging Aggregation Operators and Their Application to Multiple Attribute Decision Making

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Abstract. The focus of our this article is to familiarize a new concept of operators including, interval-valued Pythagorean fuzzy hybrid weighted averaging (IVPFHWA) aggregation operator, interval-valued Pythagorean fuzzy ordered weighted averaging (IVPFOWA) aggregation operator and interval-valued Pythagorean fuzzy weighted averaging (IVPFWA) aggregation operator. We also discuss some of their basic properties including idempotency, boundedness, commutativity and monotonicity. We also give some examples to develop these proposed operators. The advantage of the propose operators is that these operators provide more accurate and precise results as compare to the existing method. Finally, we apply these operators to deal with multiple attribute group decision making (MAGDM) by using the Pythagorean fuzzy numbers.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: IVPFWA aggregation operator, IVPFOWA aggregation operator, IVPFHWA aggregation operator, Group decision making.

1. INTRODUCTION

Fuzzy set is introduced by Zadeh [38]. In fuzzy set Zadeh only discussed membership function. After the extensions of fuzzy set theory Atanassov generalized this concept and introduced a new concept called IFS. In [1, 2, 3, 16, 18, 20] many scholars worked in intuitionistic fuzzy set theory. In [4, 5, 6, 7] K. Atanassov presented the idea of IFS. Actually Atanassov introduced this new concept which is a generalized form of the FS. In [11] Gau and Buahrer familiarized the concept of another set called vague set. After the appearance of vague set, Hong and Choi, Chen and Tan [15, 9] respectively, developed some basic techniques to handle MADM using vague set. In [8] Burilo and Bustin developed a relation between the two famous sets called vague set, and IFS. They also mathematically proved that these sets are equivalent. In [5] K. Atanassov and Gergov presented the idea of the IV-IFS, which is a generalization of IFS. In [32] Yager familiarized the model of Pythagorean fuzzy set. The most important and central research topic is aggregation operators. There are many scholars worked in this area and introduced several operators. In [34] Yager and Kecprzyk, in [33, 35, 36, 37] Yager, in [30] Xu and Da in [10] Chen and Chen, in [12, 13] Chiclena et al., in [22] Harrera et al. in [25, 26] Xu, in [21] Tan and Chen, worked in this field. Like the other scholars, Mitchell also worked in this area. In [17] he introduced the notion of IOWA operator. In [27] Xu introduced the concept of some new averaging aggregation operators including, IFOWA operator and IFHA operator. In [14] Yager and XU also worked in this field and familiarized specific new types of geometric operators including, IFHG operator, IFOWG operator, IFWG operator, and discussed the importance of the IFHG operator to MCDM problems under the IF information. Like other scholars, in [24] Wei worked in the field of aggregation operators and introduced the notion of the two new type's aggregation operators such as, I-IFOWG operator and I-IIFOWG operator. In [39] Zhao et al. also worked in this area and introduced Specific types of new operators. Z. S. Xu. and R. R, in [31] presented the notion of DIFWA operator and UDIFWA operator. This idea used by Wei in [23] defined DIFWG operator as well as UDIFWG operator. Yager and Filav in [35] introduced the notion of the I-OWA operator, which is the extension of the OWA operator and the IIFHA operator. Z. S. Xu, in [28, 29] familiarized the notion of IIFHG operator, IIFOWG operator, and IIFHA operator, IIFOWA operator, IIFWA operator and also proved the importance of IIFHA operator to MADM problems. X. Peng and Y. Yang, in [19] developed some properties of interval-valued Pythagorean fuzzy numbers.

Thus keeping the advantages of the above mention aggregation operators in this article we introduce the notion of some new operators based on IVPFNs, such as, IVPFHA operator, IVPFOWA operator and IVPFWA operator and apply them to group decision making. We also discuss some of their basic properties including idempotency, boundedness, commutativity and monotonicity. We also give some examples to develop these proposed operators. These operators provide more accurate and precise results as compare to the existing method.

The remainder paper can be constructed as. In Section 2, we present some straightforward explanations connected to our later sections. In Section 3, we familiarize IVPFWA operator, IVPFOWA operator and IVPFHA operator. In Section 4, we developed the advantage of the propose operator. In Section 5, we have conclusion

2. PRELIMINARIES

Definition 2.1. [19] *Let K be a universal set, then an interval-valued Pythagorean fuzzy set I in K can be defined as:*

$$I = \{\langle k, \mu_I(k), \upsilon_I(k)\rangle \mid k \in K\}, \quad (1)$$

where

$$\mu_I(k) = \left[\mu_I^a(k), \mu_I^b(k)\right] \subset [0,1], \quad (2)$$

$$\upsilon_I(k) = \left[\upsilon_I^a(k), \upsilon_I^b(k)\right] \subset [0,1]. \quad (3)$$

Also

$$\mu_I^a(k) = \inf \mu_I(k), \quad (4)$$

$$\mu_I^b(k) = \sup \mu_I(k), \quad (5)$$

$$\upsilon_I^a(k) = \inf \upsilon_I(k), \quad (6)$$

$$\upsilon_I^b(k) = \sup \upsilon_I(k). \quad (7)$$

And also

$$0 \le \left(\mu_I^b(k)\right)^2 + \left(\upsilon_I^b(k)\right)^2 \le 1. \quad (8)$$

If

$$\pi_I(k) = \left[\pi_I^a(k), \pi_I^b(k)\right], \text{ for all } k \in K, \quad (9)$$

then it is said to be the interval-valued Pythagorean fuzzy index of k to I, where

$$\pi_I^a(k) = \sqrt{1 - \left(\mu_I^b(k)\right)^2 - \left(\upsilon_I^b(k)\right)^2}, \quad (10)$$

and

$$\pi_I^b(k) = \sqrt{1 - \left(\mu_I^a(k)\right)^2 - \left(\upsilon_I^a(k)\right)^2}. \quad (11)$$

Definition 2.2. [19] *Let $\lambda = \left(\left[\mu_\lambda^a, \mu_\lambda^b\right], \left[\upsilon_\lambda^a, \upsilon_\lambda^b\right]\right)$ be an interval-valued Pythagorean fuzzy number, then*

$$S(\lambda) = \frac{1}{2}\left[(\mu_\lambda^a)^2 + \left(\mu_\lambda^b\right)^2 - (\upsilon_\lambda^a)^2 - \left(\upsilon_\lambda^b\right)^2\right], \quad (12)$$

and

$$H(\lambda) = \frac{1}{2}\left[(\mu_\lambda^a)^2 + \left(\mu_\lambda^b\right)^2 + (\upsilon_\lambda^a)^2 + \left(\upsilon_\lambda^b\right)^2\right], \quad (13)$$

be the score function and accuracy degree of λ respectively.

Definition 2.3. [19] *Let $\lambda = \left(\left[\mu_\lambda^a, \mu_\lambda^b\right], \left[\upsilon_\lambda^a, \upsilon_\lambda^b\right]\right), \lambda_1 = \left(\left[\mu_{\lambda_1}^a, \mu_{\lambda_1}^b\right], \left[\upsilon_{\lambda_1}^a, \upsilon_{\lambda_1}^b\right]\right), \lambda_2 = \left(\left[\mu_{\lambda_2}^a, \mu_{\lambda_2}^b\right], \left[\upsilon_{\lambda_2}^a, \upsilon_{\lambda_2}^b\right]\right)$ be the three interval-valued Pythagorean fuzzy numbers and $\delta >$*

0, *then the following operational laws hold:*

$$\delta\lambda = \left(\left[\sqrt{1-\left(1-(\mu_\lambda^a)^2\right)^\delta}, \sqrt{1-\left(1-(\mu_\lambda^b)^2\right)^\delta}\right], \left[(\upsilon_\lambda^a)^\delta, (\upsilon_\lambda^b)^\delta\right]\right), \quad (14)$$

$$(\lambda)^\delta = \left(\left[(\mu_\lambda^a)^\delta, (\mu_\lambda^b)^\delta\right], \left[\sqrt{1-\left(1-(\upsilon_\lambda^a)^2\right)^\delta}, \sqrt{1-\left(1-(\upsilon_\lambda^b)^2\right)^\delta}\right]\right), \quad (15)$$

$$\lambda_1 \otimes \lambda_2 = \left(\left[\mu_{\lambda_1}^a\mu_{\lambda_2}^a, \mu_{\lambda_1}^b\mu_{\lambda_2}^b\right], \left[\begin{array}{c}\sqrt{(\upsilon_\lambda^a)^2+(\upsilon_\lambda^a)^2-(\upsilon_\lambda^a)^2(\upsilon_\lambda^a)^2}, \\ \sqrt{(\upsilon_\lambda^b)^2+(\upsilon_\lambda^b)^2-(\upsilon_\lambda^b)^2(\upsilon_\lambda^b)^2}\end{array}\right]\right), \quad (16)$$

$$\lambda_1 \oplus \lambda_2 = \left(\begin{array}{c}\left[\begin{array}{c}\sqrt{(\mu_{\lambda_1}^a)^2+(\mu_{\lambda_2}^a)^2-(\mu_{\lambda_1}^a)^2(\mu_{\lambda_2}^a)^2}, \\ \sqrt{(\mu_{\lambda_1}^b)^2+(\mu_{\lambda_2}^b)^2-(\mu_{\lambda_1}^b)^2(\mu_{\lambda_2}^b)^2}\end{array}\right], \\ \left[\upsilon_{\lambda_1}^a\upsilon_{\lambda_2}^a, \upsilon_{\lambda_1}^b\upsilon_{\lambda_2}^b\right]\end{array}\right). \quad (17)$$

Example 2.4. *Let*

$$\begin{aligned}\lambda &= ([0.3, 0.5], [0.4, 0.8]), \\ \lambda_1 &= ([0.4, 0.6], [0.4, 0.7]), \\ \lambda_2 &= ([0.2, 0.6], [0.5, 0.7]).\end{aligned}$$

and $\delta = 2$, *then*

(1)

$$\begin{aligned}\delta\lambda &= \left(\begin{array}{c}\left[\sqrt{1-\left(1-(\mu_\lambda^a)^2\right)^\delta}, \sqrt{1-\left(1-(\mu_\lambda^b)^2\right)^\delta}\right], \\ \left[(\upsilon_\lambda^a)^\delta, (\upsilon_\lambda^b)^\delta\right]\end{array}\right) \\ &= ([0.414, 0.661], [0.16, 0.64])\end{aligned}$$

(2)

$$\begin{aligned}(\lambda)^\delta &= \left(\begin{array}{c}\left[(\mu_\lambda^a)^2, (\mu_\lambda^b)^\delta\right], \\ \left[\sqrt{1-\left(1-(\upsilon_\lambda^a)^2\right)^\delta}, \sqrt{1-\left(1-(\upsilon_\lambda^b)^2\right)^\delta}\right]\end{array}\right) \\ &= ([0.09, 0.25], [0.542, 0.932])\end{aligned}$$

(3)

$$\begin{aligned}\lambda_1 \otimes \lambda_2 &= \left(\left[\mu_{\lambda_1}^a\mu_{\lambda_2}^a, \mu_{\lambda_1}^b\mu_{\lambda_2}^b\right], \left[\begin{array}{c}\sqrt{(\upsilon_\lambda^a)^2\oplus(\upsilon_\lambda^a)^2-(\upsilon_\lambda^a)^2(\upsilon_\lambda^a)^2}, \\ \sqrt{(\upsilon_\lambda^b)^2\oplus(\upsilon_\lambda^b)^2-(\upsilon_\lambda^b)^2(\upsilon_\lambda^b)^2}\end{array}\right]\right) \\ &= ([0.08, 0.36], [0.608, 0.860])\end{aligned}$$

(4)

$$\begin{aligned}\lambda_1 \oplus \lambda_2 &= \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\left(\mu^a_{\lambda_1}\right)^2 + \left(\mu^a_{\lambda_2}\right)^2 - \left(\mu^a_{\lambda_1}\right)^2 \left(\mu^a_{\lambda_2}\right)^2}, \\ \sqrt{\left(\mu^b_{\lambda_1}\right)^2 + \left(\mu^b_{\lambda_2}\right)^2 - \left(\mu^b_{\lambda_1}\right)^2 \left(\mu^b_{\lambda_2}\right)^2} \end{array} \right], \\ \left[\upsilon^a_{\lambda_1}\upsilon^a_{\lambda_2}, \upsilon^b_{\lambda_1}\upsilon^b_{\lambda_2}\right] \end{array} \right) \\ &= \left([0.44, 0.768], [0.2, 0.49]\right)\end{aligned}$$

Definition 2.5. [29] *Let Θ be the set of all interval-valued intuitionistic fuzzy values and $\lambda_j = \left(\left[\mu^a_{\lambda_j}, \mu^b_{\lambda_j}\right], \left[\upsilon^a_{\lambda_j}, \upsilon^n_{\lambda_j}\right]\right)$ $(j = 1, 2, ..., n)$ be a collection of interval-valued intuitionistic fuzzy valves, and let $IVIFWA$: $\Theta^n \to \Theta$, if*

$$\begin{aligned}&IVIFWA_w\left(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n\right) \\ &= \left(\begin{array}{c} \left[1 - \prod_{j=1}^{n}\left(1 - \mu^a_{\lambda_j}\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - \mu^b_{\lambda_j}\right)^{w_j}\right], \\ \left[\prod_{j=1}^{n}\left(\upsilon^a_{\lambda_j}\right)^{w_j}, \prod_{j=1}^{n}\left(\upsilon^b_{\lambda_j}\right)^{w_j}\right] \end{array} \right), \quad (18)\end{aligned}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the weighted vector of λ_j $(j = 1, 2, ..., n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then $IVIFWA$ is called interval-valued intuitionistic fuzzy weighted averaging operator. Specially if $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then interval-valued intuitionistic fuzzy weighted averaging operator is reduced to an interval-valued intuitionistic fuzzy averaging operator.

Example 2.6. *Let*

$$\begin{aligned}\lambda_1 &= \left([0.3, 0.4], [0.5, 0.6]\right), \\ \lambda_2 &= \left([0.2, 0.3], [0.3, 0.6]\right), \\ \lambda_3 &= \left([0.3, 0.4], [0.3, 0.4]\right), \\ \lambda_4 &= \left([0.3, 0.5], [0.2, 0.4]\right),\end{aligned}$$

be the four interval-valued Pythagorean fuzzy values and let $w = (0.1, 0.2, 0.3, 0.4)^T$ be the weighted vector of λ_j $(j = 1, 2, 3, 4)$, then we have

$$\begin{aligned}&IVIFWA_w\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right) \\ &= \left(\begin{array}{c} \left[1 - \prod_{j=1}^{4}\left(1 - \mu^a_{\lambda_j}\right)^{w_j}, 1 - \prod_{j=1}^{4}\left(1 - \mu^b_{\lambda_j}\right)^{w_j}\right], \\ \left[\prod_{j=1}^{4}\left(\upsilon^a_{\lambda_j}\right)^{w_j}, \prod_{j=1}^{4}\left(\upsilon^b_{\lambda_j}\right)^{w_j}\right] \end{array} \right) \\ &= \left([0.281, 0.424], [0.268.0.576]\right).\end{aligned}$$

Definition 2.7. [29] *An interval-valued intuitionistic fuzzy ordered weighted averaging operator of dimension n is a mapping $IVIFOWA : \Theta^n \to \Theta$ that has an associated weighted vector $w = (w_1, w_2, ..., w_n)^T$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and is defined* *to aggregate a collection of interval-valued intuitionistic fuzzy values λ_j $(j = 1, 2, ..., n)$, according to the following expression:*

$$IVIFOWA_w(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = \left(\begin{array}{c} \left[1 - \prod_{j=1}^{n} \left(1 - \mu^a_{\lambda_{\sigma(j)}}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - \mu^b_{\lambda_{\sigma(j)}}\right)^{w_j} \right], \\ \left[\prod_{j=1}^{n} \left(\upsilon^a_{\lambda_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{n} \left(\upsilon^b_{\lambda_{\sigma(j)}}\right)^{w_j} \right] \end{array} \right). \quad (19)$$

Where $\lambda_{\sigma(j)}$ is the j^{th} largest value of λ_j. If $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then the interval-valued intuitionistic fuzzy ordered weighted averaging operator is reduced to the interval-valued intuitionistic fuzzy averaging operator.

Example 2.8. *Let*

$$\begin{aligned} \lambda_1 &= ([0.4, 0.5], [0.3, 0.4]), \\ \lambda_2 &= ([0.3, 0.6], [0.2, 0.4]), \\ \lambda_3 &= ([0.3, 0.4], [0.3, 0.5]), \\ \lambda_4 &= ([0.4, 0.5], [0.1, 0.3]), \end{aligned}$$

and let $w = (0.1, 0.2, 0.3, 0.4)^T$ be the weighted vector of λ_j $(j = 1, 2, 3, 4)$. First we calculate the score function of λ_j, we have

$$\begin{aligned} S(\lambda_1) &= 0.1, S(\lambda_2) = 0.15, \\ S(\lambda_3) &= -0.95, S(\lambda_4) = 0.25. \end{aligned}$$

Thus

$$S(\lambda_4) > S(\lambda_2) > S(\lambda_1) > S(\lambda_3)$$

Hence

$$\begin{aligned} \lambda_{\sigma(1)} &= ([0.4, 0.5], [0.1, 0.3]) \\ \lambda_{\sigma(2)} &= ([0.3, 0.6], [0.2, 0.4]) \\ \lambda_{\sigma(3)} &= ([0.4, 0.5], [0.3, 0.4]) \\ \lambda_{\sigma(4)} &= ([0.3, 0.4], [0.3, 0.5]) \end{aligned}$$

$$\begin{aligned} & IVIFOWA_w(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \\ &= \left(\begin{array}{c} \left[1 - \prod_{j=1}^{4} \left(1 - \mu^a_{\lambda_{\sigma(j)}}\right)^{w_j}, 1 - \prod_{j=1}^{4} \left(1 - \mu^b_{\lambda_{\sigma(j)}}\right)^{w_j} \right], \\ \left[\prod_{j=1}^{4} \left(\upsilon^a_{\lambda_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{4} \left(\upsilon^b_{\lambda_{\sigma(j)}}\right)^{w_j} \right] \end{array} \right) \\ &= ([0.341, 0.485], [0.247, 0.424]). \end{aligned}$$

Definition 2.9. [29] *The $IVIFHA$ operator of n dimension is a mapping $IVIFHA : \Theta^n \to \Theta$, which has an associated vector $w = (w_1, w_2, ..., w_n)^T$, such that $w_j \in [0, 1]$*

and $\sum_{j=1}^{n} w_j = 1$. *Furthermore*

$$IVIFHA_{w,w}(\lambda_1, \lambda_2, \lambda_3 ..., \lambda_n) = \left(\begin{array}{c} \left[1 - \prod_{j=1}^{n} \left(1 - \mu^a_{\dot{\lambda}_{\sigma(j)}} \right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(\mu^b_{\dot{\lambda}_{\sigma(j)}} \right) \right)^{w_j} \right], \\ \left[\prod_{j=1}^{n} \left(\upsilon^a_{\dot{\lambda}_{\sigma(j)}} \right)^{w_j}, \prod_{j=1}^{n} \left(\upsilon^b_{\dot{\lambda}_{\sigma(j)}} \right)^{w_j} \right] \end{array} \right), \quad (20)$$

where $\dot{\lambda}_{\sigma(j)}$ *be the* j^{th} *largest of the weighted intuitionistic fuzzy values* $\dot{\lambda}_j \left(\dot{\lambda}_j = nw_j \lambda_j \right)$, $w = (w_1, w_2, ..., w_n)^T$ *is the weighted vector of* λ_j $(j = 1, 2, ..., n)$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *and* n *is the balancing coefficient, which plays a role of balance. If the vector* $(w_1, w_2, ..., w_n)^T$ *approaches* $\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)^T$, *then the vector* $(nw_1\lambda_1, ..., nw_n\lambda_n)^T$ *approaches* $(\lambda_1, \lambda_2, ..., \lambda_n)^T$.

Example 2.10. *Let*

$$\begin{aligned} \lambda_1 &= ([0.3, 0.5], [0.3, 0.4]), \\ \lambda_2 &= ([0.3, 0.5], [0.2, 0.4]), \\ \lambda_3 &= ([0.3, 0.4], [0.3, 0.4]), \\ \lambda_4 &= ([0.4, 0.5], [0.1, 0.2]), \end{aligned}$$

and let $w = (0.1, 0.2, 0.3, 0.4)^T$ *be the weighted vector of* λ_j $(j = 1, 2, 3, 4)$, *then*

$$\begin{aligned} \dot{\lambda}_1 &= ([0.132, 0.242], [0.617, 0.693]), \\ \dot{\lambda}_2 &= ([0.381, 0.425], [0.275, 0.480]), \\ \dot{\lambda}_3 &= ([0.348, 0.458], [0.235, 0.333]), \\ \dot{\lambda}_4 &= ([0.558, 0.670], [0.025, 0.076]). \end{aligned}$$

Now we can find the scores of $\dot{\lambda}_j$ $(j = 1, 2, 3, 4)$, *we have*

$$\begin{aligned} S\left(\dot{\lambda}_1\right) &= -0.467, S\left(\dot{\lambda}_2\right) = 0.025 \\ S\left(\dot{\lambda}_3\right) &= 0.118, S\left(\dot{\lambda}_4\right) = 0.563 \end{aligned}$$

Then

$$\begin{aligned} \dot{\lambda}_{\sigma(1)} &= ([0.558, 0.670], [0.025, 0.076]), \\ \dot{\lambda}_{\sigma(2)} &= ([0.348, 0.458], [0.235, 0.333]), \\ \dot{\lambda}_{\sigma(3)} &= ([0.381, 0.425], [0.275, 0.480]), \\ \dot{\lambda}_{\sigma(4)} &= ([0.132, 0.242], [0.617, 0.693]). \end{aligned}$$

$$
\begin{aligned}
& IVIFHA_{w,w}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\
= & \left(\begin{array}{c} \left[1-\prod_{j=1}^{4}\left(1-\mu^{a}_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}, 1-\prod_{j=1}^{4}\left(1-\left(\mu^{b}_{\dot{\lambda}_{\sigma(j)}}\right)\right)^{w_j}\right], \\ \left[\prod_{j=1}^{4}\left(\upsilon^{a}_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{4}\left(\upsilon^{b}_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}\right] \end{array} \right) \\
= & ([0.308, 0.399], [0.290, 0.429])
\end{aligned}
$$

3. Some Averaging Aggregation Operators Based on Interval-Valued Pythagorean Fuzzy Numbers

In this section, we introduce the notion of interval-valued Pythagorean fuzzy weighted averaging operator, interval-valued Pythagorean fuzzy ordered weighted averaging operator and interval-valued Pythagorean fuzzy hybrid averaging operator. We also discuss some desirable properties and give some examples.

3.1. Interval-Valued Pythagorean Fuzzy Weighted Averaging Aggregation Operator.

Interval-valued Pythagorean fuzzy weighted averaging aggregation operator and some of their properties are already defined in [19] but here we give some examples to improve the proposed operator.

Definition 3.1. *Let* $\lambda_j = \left(\left[\mu^{a}_{\lambda_j}, \mu^{b}_{\lambda_j}\right], \left[\upsilon^{a}_{\lambda_j}, \upsilon^{b}_{\lambda_j}\right]\right)(j=1,2,...,n)$ *be a collection of interval-valued Pythagorean fuzzy valves, then* $IVPFWA$ *can be defined as:*

$$
\begin{aligned}
& IVPFWA_{w}(\lambda_1,\lambda_2,\lambda_3,...,\lambda_n) \\
= & \left(\begin{array}{c} \left[\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu^{a}_{\lambda_j}\right)^2\right)^{w_j}}, \sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu^{b}_{\lambda_j}\right)^2\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{n}\left(\upsilon^{a}_{\lambda_j}\right)^{w_j}, \prod_{j=1}^{n}\left(\upsilon^{b}_{\lambda_j}\right)^{w_j}\right] \end{array} \right), \quad (21)
\end{aligned}
$$

where $w=(w_1,w_2,...,w_n)^T$ *is the weighted vector of* $\lambda_j\,(j=1,2,3,...,n)$, *with* $w_j \in [0,1]$ *and* $\sum_{j=1}^{n} w_j = 1.$

Example 3.2. *Let*

$$
\begin{aligned}
\lambda_1 &= ([0.3,0.4],[0.5,0.7]), \\
\lambda_2 &= ([0.2,0.6],[0.3,0.6]), \\
\lambda_3 &= ([0.3,0.6],[0.3,0.5]), \\
\lambda_4 &= ([0.4,0.7],[0.2,0.6]),
\end{aligned}
$$

be the four interval-valued Pythagorean fuzzy values and let $w = (0.1, 0.2, 0.3, 0.4)^T$ *be the weighted vector of* $\lambda_j\,(j = 1, 2, 3, 4)$ *, then we have*

$$\begin{aligned} & IVPFWA_w(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \\ = & \left(\begin{array}{c} \left[\sqrt{1 - \prod_{j=1}^{4}\left(1 - \left(\mu_{\lambda_j}^a\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{4}\left(1 - \left(\mu_{\lambda_j}^b\right)^2\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{4}\left(\upsilon_{\lambda_j}^a\right)^{w_j}, \prod_{j=1}^{4}\left(\upsilon_{\lambda_j}^b\right)^{w_j} \right] \end{array} \right) \\ = & ([0.330, 0.632], [0.268, 0.576]). \end{aligned}$$

Theorem 3.3. *(Commutativity):*

$$\textit{IVPFWA}_w(\lambda_1, \lambda_2, ..., \lambda_n) = \textit{IVPFWA}_w\left(\acute{\lambda}_1, \acute{\lambda}_2, ..., \acute{\lambda}_n\right), \quad (22)$$

where $\left(\acute{\lambda}_1, \acute{\lambda}_2, ..., \acute{\lambda}_n\right)$ *is any permutation of* $(\lambda_1, \lambda_2, ..., \lambda_n)$ *and* $w = (w_1, w_2, ..., w_n)^T$ *the weighted vector of* λ_j, $\acute{\lambda}_j$ *where* $j = 1, 2, ..., n$.

Proof. Straightforward. □

Theorem 3.4. *(Idempotency): If* $\lambda_j = \lambda$ *for all* $j\,(j = 1, 2, ..., n)$ *, then*

$$\textit{IVPFWA}_w(\lambda_1, \lambda_2, ..., \lambda_n) = \lambda. \quad (23)$$

Proof. Straightforward. □

Example 3.5. *Let*

$$\begin{aligned} \lambda_1 &= ([0.3, 0.4], [0.6, 0.7]), \\ \lambda_2 &= ([0.3, 0.4], [0.6, 0.7]) \\ \lambda_3 &= ([0.3, 0.4], [0.6, 0.7]), \end{aligned}$$

and $w = (0.2, 0.3, 0.5)$ *be the weighted vector of* λ_j*, then*

$$\begin{aligned} & IVPFWA_w(\lambda_1, \lambda_2, \lambda_3) \\ = & \left(\begin{array}{c} \left[\sqrt{1 - \prod_{j=1}^{3}\left(1 - \left(\mu_{\lambda_j}^a\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{3}\left(1 - \left(\mu_{\lambda_j}^b\right)^2\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{3}\left(\upsilon_{\lambda_j}^a\right)^{w_j}, \prod_{j=1}^{3}\left(\upsilon_{\lambda_j}^b\right)^{w_j} \right] \end{array} \right) \\ = & ([0.3, 0.4], [0.6, 0.7]). \end{aligned}$$

Theorem 3.6. *(Boundedness): Let* $\lambda_j = \left(\left[\mu_{\lambda_j}^a, \mu_{\lambda_j}^b\right], \left[\upsilon_{\lambda_j}^a, \upsilon_{\lambda_j}^b\right]\right)(j = 1, 2, ..., n)$ *be a collection of interval-valued Pythagorean fuzzy valves and let* $w = (w_1, w_2, ...w_n)^T$ *be the weighted vector of* $\lambda_j\,(j = 1, 2, ..., n)$ *,such that* $\sum_{j=1}^{n} w_j = 1$*,then*

$$\lambda_{\min} \leq \textit{IVPFWA}_w(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) \leq \lambda_{\max}.$$

Proof. Straightforward. □

Theorem 3.7. *(Monotonicity): If $\lambda_j \leq \lambda_j^{'}$ for all $j\,(j = 1, 2, ..., n)$, then*

$$\mathit{IVPFWA}_w\left(\lambda_1, \lambda_2, ..., \lambda_n\right) \leq \mathit{IVPFWA}_w\left(\lambda_1^{'}, \lambda_2^{'}, ..., \lambda_n^{'}\right). \quad (24)$$

Proof. Straightforward. □

Example 3.8. *Let*

$$\begin{aligned} \lambda_1 &= ([0.3, 0.5], [0.5, 0.7]), \\ \lambda_2 &= ([0.4, 0.5], [0.6, 0.7]), \\ \lambda_3 &= ([0.2, 0.4], [0.6, 0.8]), \end{aligned}$$

and

$$\begin{aligned} \lambda_1^{'} &= ([0.3, 0.6], [0.3, 0.5]), \\ \lambda_2^{'} &= ([0.5, 0.7], [0.2, 0.6]), \\ \lambda_3^{'} &= ([0.6, 0.8], [0.4, 0.5]), \end{aligned}$$

be the three interval-valued Pythagorean fuzzy values and let $w = (0.2, 0.3, 0.5)^T$ be the weighted vector of then we have

$$\begin{aligned} & IVPFWA_w\left(\lambda_1, \lambda_2, \lambda_3\right) \\ = & \left(\begin{array}{c} \left[\sqrt{1 - \prod_{j=1}^{3}\left(1 - \left(\mu_{\lambda_j}^{a}\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{3}\left(1 - \left(\mu_{\lambda_j}^{b}\right)^2\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{3}\left(\upsilon_{\lambda_j}^{a}\right)^{w_j}, \prod_{j=1}^{3}\left(\upsilon_{\lambda_j}^{b}\right)^{w_j}\right] \end{array}\right) \\ = & ([0.295, 0.454], [0.578, 0.748]). \end{aligned}$$

Again

$$\begin{aligned} & IVPFWA_w\left(\lambda_1^{'}, \lambda_2^{'}, \lambda_3^{'}\right) \\ = & \left(\begin{array}{c} \left[\sqrt{1 - \prod_{j=1}^{3}\left(1 - \left(\mu_{\lambda_j}^{a}\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{3}\left(1 - \left(\mu_{\lambda_j}^{b}\right)^2\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{3}\left(\upsilon_{\lambda_j}^{a}\right)^{w_j}, \prod_{j=1}^{3}\left(\upsilon_{\lambda_j}^{b}\right)^{w_j}\right] \end{array}\right) \\ = & ([0.528, 0.742], [0.306, 0.528]). \end{aligned}$$

3.2. **Interval-Valued Pythagorean Fuzzy Ordered Weighted Averaging Aggregation Operator.**

Definition 3.9. *Let* $\lambda_j = \left(\left[\mu^a_{\lambda_j}, \mu^b_{\lambda_j}\right], \left[\upsilon^a_{\lambda_j}, \upsilon^b_{\lambda_j}\right]\right) (j = 1, 2, ..., n)$ *be a collection of interval valued Pythagorean fuzzy valves, then an IVPFOWA operator can be define as:*

$$IVPFOWA_w (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = \left(\begin{array}{c} \left[\sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\mu^a_{\lambda_{\sigma(j)}}\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\mu^b_{\lambda_{\sigma(j)}}\right)^2\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{n}\left(\upsilon^a_{\lambda_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{n}\left(\upsilon^b_{\lambda_{\sigma(j)}}\right)^{w_j}\right] \end{array} \right), \quad (25)$$

where $w = (w_1, w_2, ..., w_n)^T$ *be the weighted vector with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$ *and* $\lambda_{\sigma(j)}$ *is the* j^{th} *largest value of* λ_j.

Theorem 3.10. *Let* $\lambda_j = \left(\left[\mu^a_{\lambda_j}, \mu^b_{\lambda_j}\right], \left[\upsilon^a_{\lambda_j}, \upsilon^b_{\lambda_j}\right]\right) (j = 1, 2, ..., n)$ *be a collection of interval valued Pythagorean fuzzy valves, by applying the IVPFOWA operator, then their aggregated value is also IVPFV.*

Proof. Proof is easy so it is omitted here. □

Theorem 3.11. *(Commutativity):*

$$\textit{IVPFOWA}_w (, \lambda_1, \lambda_2, ..., \lambda_n) = \textit{IVPFOWA}_w \left(\acute{\lambda_1}, \acute{\lambda_2}, ..., \acute{\lambda_n}\right), \quad (26)$$

where $\left(\acute{\lambda_1}, \acute{\lambda_2}, ..., \acute{\lambda_n}\right)$ *is any permutation of* $(\lambda_1, \lambda_2, ..., \lambda_n)$ *, and* $w = (w_1, w_2, ..., w_n)^T$ *the weighted vector of* λ_j*,* $\acute{\lambda_j}$ *where* $j = 1, 2, ..., n$.

Proof. Straightforward. □

Theorem 3.12. *(Idempotency): If* $\lambda_j = \lambda$ *for all* j $(j = 1, 2, ..., n)$ *, then*

$$\textit{IVPFOWA}_w (, \lambda_1, \lambda_2, ..., \lambda_n) = \lambda. \quad (27)$$

Proof. Straightforward. □

Theorem 3.13. *(Boundedness): Let* $\lambda_j = \left(\left[\mu^a_{\lambda_j}, \mu^b_{\lambda_j}\right], \left[\upsilon^a_{\lambda_j}, \upsilon^b_{\lambda_j}\right]\right) (j = 1, 2, ..., n)$ *be a collection of interval-valued Pythagorean fuzzy valves and let* $w = (w_1, w_2, ...w_n)^T$ *be the weighted vector of* $\lambda_{\sigma(j)}$ $(j = 1, 2, ..., n)$ *, such that* $\sum_{j=1}^{n} w_j = 1$*, then*

$$\lambda_{\min} \leq \textit{IVPFOWA}_w (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) \leq \lambda_{\max}. \quad (28)$$

Proof. Straightforward. □

Theorem 3.14. *(Monotonicity): If* $\lambda_j \leq \acute{\lambda_j}$ *for all* j $(j = 1, 2, ..., n)$ *, then*

$$\textit{IVPFOWA}_w (, \lambda_1, \lambda_2, ..., \lambda_n) \leq \textit{IVPFOWA}_w \left(\acute{\lambda_1}, \acute{\lambda_2}, ..., \acute{\lambda_n}\right). \quad (29)$$

Proof. Straightforward. □

Example 3.15. *Let*

$$\begin{aligned}\lambda_1 &= ([0.4, 0.6], [0.3, 0.7]),\\ \lambda_2 &= ([0.3, 0.6], [0.2, 0.7]),\\ \lambda_3 &= ([0.3, 0.8], [0.3, 0.5]),\\ \lambda_4 &= ([0.4, 0.9], [0.1, 0.3]),\end{aligned}$$

be the four interval-valued Pythagorean fuzzy values and let $w = (0.1, 0.2, 0.3, 0.4)^T$ *be the weighted vector of* λ_j $(j = 1, 2, 3, 4)$. *First we calculate the scores of* λ_j $(j = 1, 2, 3, 4)$, *thus we have*

$$\begin{aligned}S(\lambda_1) &= -0.03, S(\lambda_2) = -0.04\\ S(\lambda_3) &= 0.19, S(\lambda_4) = 0.43\end{aligned}$$

Thus

$$S(\lambda_4) > S(\lambda_3) > S(\lambda_1) > S(\lambda_2)$$

Hence

$$\begin{aligned}\lambda_{\sigma(1)} &= ([0.4, 0.9], [0.1, 0.3]),\\ \lambda_{\sigma(2)} &= ([0.3, 0.8], [0.3, 0.5]),\\ \lambda_{\sigma(3)} &= ([0.4, 0.6], [0.3, 0.7]),\\ \lambda_{\sigma(4)} &= ([0.3, 0.6], [0.2, 0.7]).\end{aligned}$$

Thus

$$\begin{aligned}& IVPFOWA_w(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\\ =& \left(\begin{array}{c}\left[\sqrt{1-\prod_{j=1}^{4}\left(1-\left(\mu^a_{\lambda_{\sigma(j)}}\right)^2\right)^{w_j}}, \sqrt{1-\prod_{j=1}^{4}\left(1-\left(\mu^b_{\lambda_{\sigma(j)}}\right)^2\right)^{w_j}}\right],\\ \left[\prod_{j=1}^{4}\left(\upsilon^a_{\lambda_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{4}\left(\upsilon^b_{\lambda_{\sigma(j)}}\right)^{w_j}\right]\end{array}\right)\\ =& ([0.344, 0.703], [0.228, 0.601]).\end{aligned}$$

3.3. **Interval-Valued Pythagorean Fuzzy Hybrid Weighted Averaging Aggregation Operator.**

Definition 3.16. *An interval-valued Pythagorean fuzzy hybrid averaging operator of dimension* n *is a mapping* $IVPFHA : \Theta^n \to \Theta$, *which has an associated vector* $w = (w_1, w_2, ..., w_n)^T$, *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *Furthermore*

$$\begin{aligned}& IVPFHA_{w,w}(\lambda_1, \lambda_2, ..., \lambda_n)\\ =& \left(\begin{array}{c}\left[\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu^a_{\dot{\lambda}_{\sigma(j)}}\right)^2\right)^{w_j}}, \sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu^b_{\dot{\lambda}_{\sigma(j)}}\right)^2\right)^{w_j}}\right],\\ \left[\prod_{j=1}^{n}\left(\upsilon^a_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{n}\left(\upsilon^b_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}\right]\end{array}\right), \quad (30)\end{aligned}$$

where $\dot{\lambda}_{\sigma(j)}$ *is the* j^{th} *largest of the weighted Pythagorean fuzzy values* $\dot{\lambda}_j$ $\left(\dot{\lambda}_j = nw_j\lambda_j\right)$, $w = (w_1, w_2, ..., w_n)^T$ *is the weighted vector of* λ_j $(j = 1, 2, ..., n)$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *and* n *is the balancing coefficient, which plays a role of balance. If the vector* $(w_1, w_2, ..., w_n)^T$ *approaches* $\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, *then the vector* $(nw_1\lambda_1, nw_2\lambda_2, ..., nw_n\lambda_n)^T$ *approaches* $(\lambda_1, \lambda_2, ..., \lambda_n)^T$.

Theorem 3.17. *Let* $\lambda_j = \left(\left[\mu^a_{\lambda_j}, \mu^b_{\lambda_j}\right], \left[\upsilon^a_{\lambda_j}, \upsilon^b_{\lambda_j}\right]\right)$ $(j = 1, 2, ..., n)$ *be a collection of IVPFVs, by the applying of IVPFHA operator, then their aggregated value is also IVPFV.*

Proof. Proof is easy so it is omitted here. □

Example 3.18. *Let*

$$\begin{aligned}
\lambda_1 &= ([0.4, 0.7], [0.3, 0.4]), \\
\lambda_2 &= ([0.3, 0.6], [0.2, 0.4]), \\
\lambda_3 &= ([0.3, 0.7], [0.3, 0.5]), \\
\lambda_4 &= ([0.4, 0.8], [0.1, 0.3]),
\end{aligned}$$

be the four interval-valued Pythagorean fuzzy values and let $w = (0.1, 0.2, 0.3, 0.4)^T$ *be the weighted vector of* λ_j $(j = 1, 2, 3, 4)$, *thus*

$$\begin{aligned}
\dot{\lambda}_1 &= ([0.259, 0.485], [0.617, 0.693]), \\
\dot{\lambda}_2 &= ([0.269, 0.547], [0.275, 0.480]), \\
\dot{\lambda}_3 &= ([0.327, 0.744], [0.235, 0.435]), \\
\dot{\lambda}_4 &= ([0.493, 0.897], [0.025, 0.145]).
\end{aligned}$$

Now we can find the scores of $\dot{\lambda}_j$ $(j = 1, 2, 3, 4)$.

$$\begin{aligned}
S\left(\dot{\lambda}_1\right) &= -0.279, S\left(\dot{\lambda}_2\right) = 0.032 \\
S\left(\dot{\lambda}_3\right) &= 0.208, S\left(\dot{\lambda}_4\right) = 0.513
\end{aligned}$$

Thus

$$S\left(\dot{\lambda}_4\right) > S\left(\dot{\lambda}_3\right) > S\left(\dot{\lambda}_2\right) > S\left(\dot{\lambda}_1\right)$$

Hence

$$\begin{aligned}
\dot{\lambda}_{\sigma(1)} &= ([0.493, 0.897], [0.025, 0.145]), \\
\dot{\lambda}_{\sigma(2)} &= ([0.327, 0.744], [0.235, 0.435]), \\
\dot{\lambda}_{\sigma(3)} &= ([0.269, 0.547], [0.275, 0.480]), \\
\dot{\lambda}_{\sigma(4)} &= ([0.259, 0.485], [0.617, 0.693]).
\end{aligned}$$

Thus

$$\begin{aligned} & IVPFHA_{w,w}\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right) \\ = & \left(\begin{array}{c} \left[\sqrt{1-\prod_{j=1}^{4}\left(1-\left(\mu^{a}_{\dot{\lambda}_{\sigma(j)}}\right)^2\right)^{w_j}}, \sqrt{1-\prod_{j=1}^{4}\left(1-\left(\mu^{b}_{\dot{\lambda}_{\sigma(j)}}\right)^2\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{4}\left(\upsilon^{a}_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{4}\left(\upsilon^{b}_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}\right] \end{array}\right) \\ = & \left(\left[0.705, 0.793\right], \left[0.109, 0.300\right]\right). \end{aligned}$$

Theorem 3.19. *An IVPFWA operator is a specials case of IVPFHA operator.*

Proof. Let $w=\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n},\right)^T$, then we have

$$\begin{aligned} \text{IVPFHA}_{w,w}\left(\lambda_1,\lambda_2,...,\lambda_n\right) &= w_1\dot{\lambda}_{\sigma(1)} \oplus w_2\dot{\lambda}_{\sigma(2)} \oplus ... \oplus w_n\dot{\lambda}_{\sigma(n)} \\ &= \frac{1}{n}\left(\dot{\lambda}_{\sigma(1)} \oplus \dot{\lambda}_{\sigma(2)} \oplus ... \oplus \dot{\lambda}_{\sigma(n)}\right) \\ &= \frac{1}{n}\left(nw_1\lambda_1 \oplus nw_2\lambda_2 \oplus ... \oplus nw_n\lambda_n\right) \\ &= w_1\lambda_1 \oplus w_2\lambda_2 \oplus ... \oplus w_n\lambda_n \\ &= \text{IVPFWA}_w\left(\lambda_1,\lambda_2,...,\lambda_n\right). \end{aligned}$$

□

Theorem 3.20. *The IVPFOWA operator is a specials case of the IVPFHA operator.*

Proof. Let $w=\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n},\right)^T$, and $\dot{\lambda}_j = nw_j\lambda_j = n\left(\frac{1}{n}\lambda_j\right) = \lambda_j$, then

$$\begin{aligned} \text{IVPFHA}_{w,w}\left(\lambda_1,\lambda_2,...,\lambda_n\right) &= w_1\dot{\lambda}_{\sigma(1)} \oplus w_2\dot{\lambda}_{\sigma(2)} \oplus ... \oplus w_n\dot{\lambda}_{\sigma(n)} \\ &= w_1\lambda_{\sigma(1)} \oplus w_2\lambda_{\sigma(2)} \oplus ... \oplus w_n\lambda_{\sigma(n)} \\ &= \text{IVPFOWA}_w\left(\lambda_1,\lambda_2,...,\lambda_n\right). \end{aligned}$$

□

4. An Application of the Proposed Aggregation Operators to Multiple Attribute Decision Making problem

Algorithm: Let $S=\{S_1,S_2,...,S_n\}$ be a set of n alternatives, and $F=\{F_1,F_2,...,F_m\}$ be the set of m attributes and $w=(w_1,w_2,...,w_m)^T$ be the weighted vector of the attributes $F_i\ (i=1,2,...,m)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{m} w_i = 1$

Step 1: In this step the decisions makers provide the decision information in the following form:

$$D_{m\times n} = [\lambda_{ij}]_{m\times n}\left(\begin{array}{c} i=1,2,3,...,m \\ j=1,2,3,...,n \end{array}\right).$$

Step 2: Compute $\lambda_j\ (j=1,2,...,n)$ by using the $IVPFWA$ aggregation operator.

Step 3: Compute the scores of $\lambda_j\ (j=1,2,...,n)$. If there is no difference between two or more than two scores, then have we must to calculate the accuracy degrees.

Step 4: Arrange the scores function of the all alternatives in the form of descending order and select that alternative, which has the highest score function value.

Example 4.1. *Suppose a customer wants to buy a laptop from different laptops, let S_1, S_2, S_3, represent the three laptops of different companies. Let F_1, F_2, F_3, F_4, be the criteria of these laptops. In the process of choosing one of the best laptops, four factors are consider. F_1 : price of each laptop. F_2 : model of each laptop. F_3 : design of each laptop. F_4 : betree of the laptop. Suppose the weighted vector of $F_i\ (i = 1, 2, 3, 4)$ is $w = (0.1, 0.2, 0.3, 0.4)^T$, and the interval-valued Pythagorean fuzzy values of the alternative $A_j\ (j = 1, 2, 3, 4)$ are represented by the following decision matrix*

For IPFWA Operator

Step 1: The decision maker give his decision in table 1.

Table1 Pythagorean Fuzzy Decision Matrix

	S_1	S_2	S_3
F_1	$([0.3, 0.5], [0.4, 0.8])$	$([0.3, 0.6], [0.2, 0.7])$	$([0.3, 0.5], [0.5, 0.8])$
F_2	$([0.2, 0.6], [0.3, 0.7])$	$([0.4, 0.5], [0.3, 0.6])$	$([0.2, 0.5], [0.2, 0.6])$
F_3	$([0.3, 0.7], [0.2, 0.5])$	$([0.2, 0.6], [0.2, 0.7])$	$([0.3, 0.7], [0.4, 0.7])$
F_4	$([0.4, 0.5], [0.4, 0.6])$	$([0.4, 0.6], [0.4, 0.5])$	$([0.4, 0.4], [0.2, 0.8])$

Step 2: Compute $\lambda_j, (j = 1, 2, 3)$

$$\begin{aligned}
\lambda_1 &= ([0.330, 0.593], [0.306, 0.602]) \\
\lambda_2 &= ([0.344, 0.582], [0.303, 0.593]) \\
\lambda_3 &= ([0.330, 0.548], [0.269, 0.725])
\end{aligned}$$

Step 3: in this step we can find the scores of $\lambda_j\ (j = 1, 2, 3)$

$$\begin{aligned}
S(\lambda_1) &= 0.002, S(\lambda_2) = 0.007, \\
S(\lambda_3) &= -0.094,
\end{aligned}$$

Step 4: Arrange the scores of the all alternatives in the form of descending order and select that alternative, which has the highest score function. Since $\lambda_2 > \lambda_1 > \lambda_3$. Hence $S_2 > S_1 > S_3$. Thus S_2 is the best option for the customer.

For IPFOWA Operator

Step 1: In this step we construct the Pythagorean fuzzy ordered decision matrix.

Table1 Pythagorean Fuzzy Ordered Decision Matrix

	S_1	S_2	S_3
F_1	$([0.3, 0.7], [0.2, 0.5])$	$([0.4, 0.6], [0.4, 0.5])$	$([0.3, 0.7], [0.4, 0.7])$
F_2	$([0.4, 0.5], [0.4, 0.6])$	$([0.4, 0.5], [0.3, 0.6])$	$([0.2, 0.5], [0.2, 0.6])$
F_3	$([0.2, 0.6], [0.3, 0.7])$	$([0.3, 0.6], [0.2, 0.7])$	$([0.4, 0.4], [0.2, 0.8])$
F_4	$([0.3, 0.5], [0.4, 0.8])$	$([0.2, 0.6], [0.2, 0.7])$	$([0.3, 0.5], [0.5, 0.8])$

Step 2: Compute $\lambda_j \ (j = 1, 2, 3)$

$$\begin{aligned}\lambda_1 &= ([0.299, 0.558], [0.342, 0.692]) \\ \lambda_2 &= ([0.303, 0.582], [0.232, 0.656]) \\ \lambda_3 &= ([0.319, 0.503], [0.309, 0.745])\end{aligned}$$

Step 3: in this step we can find the scores of $\lambda_j \ (j = 1, 2, 3, 4)$

$$\begin{aligned}S(\lambda_1) &= -0.097, S(\lambda_2) = -0.026 \\ S(\lambda_3) &= -0.147,\end{aligned}$$

Step 4: Arrange the scores of the all alternatives in the form of descending order and select that alternative, which has the highest score function. Since $\lambda_2 > \lambda_1 > \lambda_3$. Hence $S_2 > S_1 > S_3$.. Thus S_2 is the best option for the customer.

5. Conclusion

n this article, we have introduced the notion of IVPFWA operator, IVPFOWA operator, and IVPFHA operator. We have also discussed some of their basic properties and give some examples to develop the proposed operators. At the last we presented an application of these proposed operators.

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