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Averaging Aggregation Operators with Interval Pythagorean Trapezoidal Fuzzy Numbers and Their Application to Group Decision Making

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Abstract. Pythagorean fuzzy number is a new tool for uncertainty and vagueness. It is a generalization of fuzzy numbers and intuitionistic fuzzy numbers. The paper deals with interval-valued Pythagorean trapezoidal fuzzy numbers. In this paper we introduce interval-valued Pythagorean trapezoidal fuzzy numbers and some operation on IVPTFN. We also define different types of operators for aggregating interval-valued Pythagorean trapezoidal fuzzy numbers. We present interval-valued Pythagorean trapezoidal fuzzy weighted averaging (IVPTFWA) operator, interval-valued Pythagorean trapezoidal fuzzy ordered weighted averaging (IVPTFOWA) operator and interval valued Pythagorean trapezoidal fuzzy hybrid averaging (IVPTFHA) operator. Finally we develope a general algorithm for group decision making problem.

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1. INTRODUCTION

The notion of fuzzy set theory was established by L.A. Zadeh [38] in 1965. In fuzzy set (FS) theory the degree of membership function was discussed. Fuzzy set theory has been studied in various fields such that, homoeopathic verdict, computer science, fuzzy algebra and decision making problems. In 1986 Atanassov [1] developed the idea of intuitionistic fuzzy set (IFS) and discussed the degree of membership as well as the degree of non-membership function. Intuitionistic fuzzy set is the generalization of fuzzy set theory. There are many advantages of intuitionistic fuzzy set theory such as, usage in engineering, management science and computer science [6, 7, 5, 16, 18, 8, 39, 26, 27, 28, 29, 30, 31, 23].

Atanassov also presented some relation and changed mathematically operations such as, algebraic product, sum, union, intersection and complement [2, 4]. He also introduced the thought of pseudo fixed topics of all operators defined over the intuitionistic fuzzy set [3]. In 1986, many scholars [5] have completed works in the field of intuitionistic fuzzy set and its presentations. Many scholars [15, 19, 34, 35] further extended the concept of intuitionistic fuzzy sets to introduce interval valued intuitionistic fuzzy sets $(IVIFSs)$, which enhances greatly the representation ability of uncertainty than IFs. However, the domain of intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets which are used to indicate the certain criterion does or does not belong to some fuzzy concepts.

Like the other schlor in [10, 11, 12, 13, 14, 17, 21] developed the new types of aggregation operators and applied these aggregation operators to multiple attribute group decision making $(MAGDM)$ problem. Xu and Chen [32] introduced some new types of aggregation operators including, interval-valued intuitionistic fuzzy hybrid averaging $(IVIFHA)$ operator, interval-valued intuitionistic fuzzy ordered weighted averaging $(IVIFOWA)$ operator, interval-valued intuitionistic fuzzy weighted averaging $(IVIFWA)$ operator and also proved the importance of interval-valued intuitionistic fuzzy hybrid averaging (IV-IFHA) operator to multi-criteria group decision making problems under interval-valued intuitionistic fuzzy data. Furthermore in Xu and Chen [33] introduced the idea of interval-valued intuitionistic fuzzy hybrid geometric $(IVIFHG)$ operator, interval-valued intuitionistic fuzzy ordered weighted geometric $(IVIFOWG)$ operator, interval-valued intuitionistic fuzzy weighted geometric $(IVIFWA)$ operators.

Wei and Merigó [18, 26] worked in the field of aggergation operators and introduced the notion of the two new types aggregation operators such as, the induced intuitionistic fuzzy ordered weighted geometric $(I-IFOWG)$ operator as well as the induced interval-valued intuitionistic fuzzy ordered weighted geometric $(I-IVIFOWG)$ operator Like the other scholars, Wang [24] also worked in the field of intuitionistic fuzzy set and presented the knowledge of intuitionistic trapezoidal fuzzy $(ITFNs)$ numbers and interval-valued intuitionistic trapezoidal fuzzy $(IVITFNs)$. Wang [25] not only established the idea of these numbers, but also introduced the concept of Hamming distance for trapezoidal intuitionistic fuzzy numbers $(TIFNs)$ and introduced a series of averaging aggregation operators for $ITFNs$ such as intuitionistic trapezoidal fuzzy hybrid weighted averaging $(ITFHWA)$ operator, intuitionistic trapezoidal fuzzy ordered weighted averaging $(ITFOWA)$ operator and intuitionistic trapezoidal fuzzy weighted averaging $(ITFWA)$ operator. In 2013, Yager [36] also worked in the field of Pythagorean fuzzy (PFS) set and introduced the idea of Pythagorean fuzzy set which is a generalization of intuitionistic fuzzy set in which the square of their sum less than or equal to 1. Yager [37] gave an example to state this situation, a DM gives his support for membership of an alternative as $\left(\frac{\sqrt{3}}{2}\right)$ and his support against membership is $\frac{1}{2}$ Owing to the sum of two values is bigger than 1, they are not available for IFS, but they are available for PFS since $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \leq 1$. Later on Rehman et al [20] worked on Pythagorean fuzzy ordered weighted geometric aggregation operator and their application to multiple attribute group decision making.

An advantages of the above mention aggregation operators, we develop a series of interval-valued Pythagorean trapezoidal fuzzy aggregation operators, consists of interval-vlued Pythagorean trapezoidal fuzzy weighted averaging $(IVPTFWA)$ operator, interval-valued Pythagorean trapezoidal fuzzy ordered weighted averaging $(IVPTFOWA)$ operator and the interval-valued Pythagorean trapezoidal fuzzy hybrid averaging $(IVPTFHA)$ operator. Then we construct an numerical example and find best alternative by applying score funcation.

This paper is organized as follows; In section 2, we give the concept of some basic definitions and operators which will be used in our later sections. In section 3, we develop the concept of the $IVPTFWA$ operator, $IVPTFOWA$ operator and $IVPTFHA$ operator. In section 4, we give an application of $IVPTFWA$ and $IVPTFHA$ operators to multiple attribute group decision making $(MAGDM)$ problems with interval-valued Pythagorean trapezoidal fuzzy information. We applying these operators and find out the best alternative from different alternatives. In section 5, we give numerical example. Concluding remarks are made in section 6.

2. Preliminaries

Definition 2.1. [1] Let L be a fixed set. An $IFS\ U$ in L is an object having the form:

$$U = \{\langle l, \Psi_u(l), \Upsilon_u(l)\rangle \mid l \in L\},$$

where $\Psi_u : L \to [0,1]$ and $\Upsilon_u : L \to [0,1]$ represent the degree of membership and the degree of non-membership of the element $l \in L$ to U, respectively, and for all $l \in L$:

$$0 \leq \Psi_u(l) + \Upsilon_u(l) \leq 1.$$

For each $IFs\ U$ in L,

$$\pi_U(l) = 1 - \Psi_U(l) - \Upsilon_U(l), \text{ for all } l \in L,$$

$\pi_A(l)$ is called the degree of indeterminacy of l to U.

Definition 2.2. [25] Let p be intuitionistic trapezoidal fuzzy number, its membership function be

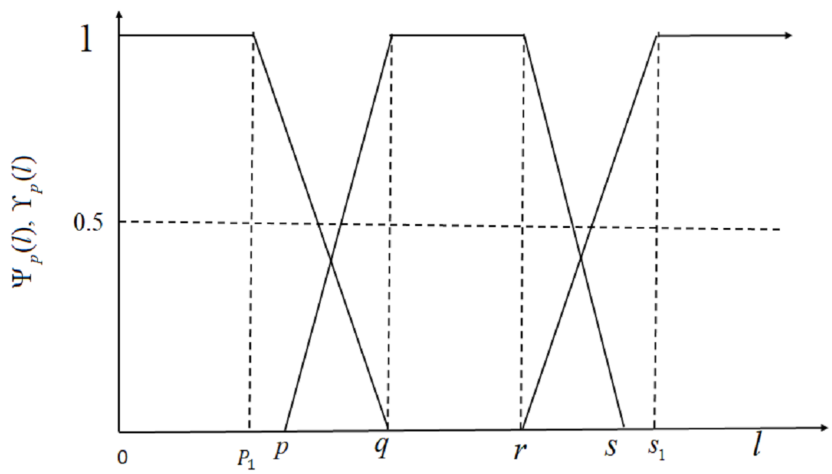
$$\Psi_p(l) = \begin{cases} \frac{l-p}{q-p}\Psi_p, & p \leq l \leq q; \\ \Psi_p, & q \leq l \leq r; \\ \frac{s-l}{s-r}\Psi_p, & r \leq l \leq s; \\ 1, & \text{otherwise}, \end{cases} \tag{1}$$

and its non-membership function be

$$\Upsilon_p(l) = \begin{cases} \frac{s-l+\Upsilon_p(l-p_1)}{q-p} & p_1 \leq l \leq q; \\ \Upsilon_p, & q \leq l \leq r; \\ \frac{l-r+\Upsilon_p(s_1-l)}{s_1-r}, & r \leq l \leq s_1; \\ 0, & \text{otherwise}, \end{cases} \qquad (2)$$

where $0 \leq \Psi_{\tilde{\alpha}} \leq 1$; $0 \leq \Upsilon_{\tilde{\alpha}} \leq 1$; $0 \leq (\Psi_{\tilde{\alpha}}) + (\Upsilon_{\tilde{\alpha}}) \leq 1$; $p, q, r, s \in R$. Then $\tilde{p} = \langle ([p, q, r, s]; \Psi_{\tilde{\alpha}}), ([p_1, q, r, s_1]; \Upsilon_{\tilde{\alpha}}) \rangle$ is called intuitionistic trapezoidal fuzzy number. For convenience we write, $\tilde{p} = ([p, q, r, s]; \Psi_{\tilde{\alpha}}, \Upsilon_{\tilde{\alpha}})$.

Figure 1. Intuitionistic trapezoidal fuzzy number



Definition 2.3. [25] Let $\tilde{p}_1 = ([p_1, q_1, r_1, s_1]; \Psi_{\tilde{\alpha}_1}, \Upsilon_{\tilde{\alpha}_1})$, and $\tilde{p}_2 = ([p_2, q_2, r_2, s_2]; \Psi_{\tilde{\alpha}_2}, \Upsilon_{\tilde{\alpha}_2})$, be two trapezoidal fuzzy numbers, and $\delta \geq 0$. Then,

(1) $\tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [p_1 + p_2, q_1 + q_2, r_1 + r_2, s_1 + s_2]; \\ (\Psi_{\tilde{\alpha}_1}) + (\Psi_{\tilde{\alpha}_2}) - (\Psi_{\tilde{\alpha}_1}\Psi_{\tilde{\alpha}_2}), \Upsilon_{\tilde{\alpha}_1}\Upsilon_{\tilde{\alpha}_2} \end{pmatrix}$,

(2) $\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [p_1p_2, q_1q_2, r_1r_2, s_1s_2]; \Psi_{\tilde{\alpha}_1}\Psi_{\tilde{\alpha}_2}, \\ (\Upsilon_{\tilde{\alpha}_1}) + (\Upsilon_{\tilde{\alpha}_2}) - (\Upsilon_{\tilde{\alpha}_1}\Upsilon_{\tilde{\alpha}_2}) \end{pmatrix}$,

(3) $\delta\tilde{p} = ([\delta p, \delta q, \delta r, \delta s]; 1 - (1 - \Psi_{\tilde{\alpha}})^{\delta}; (\Upsilon_{\tilde{\alpha}})^{\delta})$,

(4) $\tilde{p}^{\delta} = ([p^{\delta}, q^{\delta}, r^{\delta}, s^{\delta}]; \Psi_{\tilde{\alpha}}^{\delta}, 1 - (1 - \Upsilon_{\tilde{\alpha}})^{\delta})$.

Example 2.4. Let $\tilde{p} = ([0.5, 0.4, 0.6, 0.9]; 0.3, 0.5), \tilde{p}_1 = ([0.3, 0.4, 0.5, 0.3]; 0.4, 0.6), \tilde{p}_2 = ([0.4, 0.5, 0.4, 0.3]; 0.5, 0.4)$ be trapezoidal fuzzy numbers, and $\delta = 0.5$. Then, we verify the above results such that,

(1)

$$\tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [0.3 + 0.4, 0.4 + 0.5, 0.5 + 0.4, 0.3 + 0.3]; \\ (0.4) + (0.5) - (0.4)(0.5), (0.6)(0.4) \end{pmatrix},$$
$$= ([0.7, 0.9, 0.9, 0.6]; 0.7, 0.24).$$

(2)

$$\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [(0.3)\,(0.4)\,,(0.4)\,(0.5)\,,(0.5)\,(0.4)\,,(0.3)\,(0.3)]\,; \\ (0.4)(0.5),(0.6+0.4)-(0.6)\,(0.4) \end{pmatrix}, \\ = ([0.12, 0.2, 0.2, 0.9]\,; 0.2, 0.76)\,.$$

(3)

$$\delta\tilde{p} = \begin{pmatrix} [(0.5)\,(0.5)\,,(0.5)\,(0.4)\,,(0.5)\,(0.6)\,,(0.5)\,(0.9)]\,; \\ (1-(1-0.3)^{0.5}\,(0.5)^{0.5} \end{pmatrix}, \\ = ([0.25, 0.2, 0.3, 0.45]\,; 0.16, 0.70)\,.$$

(4)

$$\tilde{p}^{\delta} = \begin{pmatrix} \left[(0.5)^{0.5}\,,(0.4)^{0.5}\,,(0.6)^{0.5}\,,(0.9)^{0.5}\right]; \\ (0.3)^{0.5}\,,1-(1-0.5)^{0.5} \end{pmatrix}, \\ = ([0.70, 0.63, 0.77, 0.94]\,; 0.59, 0.29)\,.$$

Definition 2.5. [36] Let L be a fixed set. The PFs U in L is an object having the form:

$$U = \{\langle l, \Psi_U(l), \Upsilon_U(l)\rangle \mid l \in L\},$$

where $\Psi_A : L \to [0,1]$ and $\Upsilon_A : L \to [0,1]$ represented the degree of membership and the degree of non-membership of the element $l \in L$ to A, respectively, and for every $l \in L$,

$$0 \leq \Psi_U \leq 1, 0 \leq \Upsilon_U \leq 1, 0 \leq \Psi_U^2(l) + \Upsilon_U^2(l) \leq 1.$$

For each PFS U in L,

$$\pi_U(l) = \sqrt{1 - \Psi_U^2(l) - \Upsilon_U^2(l)}, \text{ for all } l \in L,$$

$\pi_U(l)$ is called the degree of indeterminacy of l to U.

Definition 2.6. [40] Let $\tilde{p} = (\Psi_\alpha, \Upsilon_\alpha)$, $\tilde{p}_1 = (\Psi_{\alpha_1}, \Upsilon_{\alpha_1})$ and $\tilde{p}_2 = (\Psi_{\alpha_2}, \Upsilon_{\alpha_2})$ be three $PFNs$ and $\delta > 0$. Then

(1) $\tilde{p}^c = (\Upsilon_\alpha, \Psi_\alpha)$,

(2) $\tilde{p}_1 \oplus \tilde{p}_2 = \left(\sqrt{(\Psi_{\tilde{\alpha}_1})^2 + (\Psi_{\tilde{\alpha}_2})^2 - (\Psi_{\tilde{\alpha}_1}^2 \Psi_{\tilde{\alpha}_2}^2)}, \Upsilon_{\tilde{\alpha}_1}\Upsilon_{\tilde{\alpha}_2}\right)$,

(3) $\tilde{p}_1 \otimes \tilde{p}_2 = \left(\Psi_{\tilde{\alpha}_1}\Psi_{\tilde{\alpha}_2}, \sqrt{(\Upsilon_{\tilde{\alpha}_1})^2 + (\Upsilon_{\tilde{\alpha}_2})^2 - (\Upsilon_{\tilde{\alpha}_1}^2 \Upsilon_{\tilde{\alpha}_2}^2)}\right)$,

(4) $\delta\tilde{p} = \sqrt{1 - (1 - \Psi_{\tilde{\alpha}}^2)^{\delta}}; (\Upsilon_{\tilde{\alpha}})^{\delta})$,

(5) $\tilde{p}^{\delta} = (\Psi_{\tilde{\alpha}}^{\delta}, \sqrt{1 - (1 - \Upsilon_{\tilde{\alpha}}^2)^{\delta}})$.

Example 2.7. Let $\tilde{p} = (0.5, 0.6)$, $\tilde{p}_1 = (0.6, 0.4)$ $\tilde{p}_2 = (0.7, 0.4)$ be trapezoidal fuzzy numbers, and $\delta = 0.6$. Then we verify the above results such that,

(2)

$$\tilde{p}_1 \oplus \tilde{p}_2 = \left(\sqrt{(0.6)^2 + (0.7)^2 - (0.6^2)(0.7^2)}, (0.4)\,(0.4)\right), \\ = (0.85, 0.17).$$

(3)

$$\tilde{p}_1 \otimes \tilde{p}_2 = \left((0.6)\,(0.7)\,,\sqrt{(0.4)^2+(0.7)^2-(0.4)^2(0.7^2)}\right), \\ = (0.42, 0.75).$$

(4)

$$\delta\tilde{p} = \sqrt{1-(1-0.5^2)^{0.6}}; (0.6)^{0.6}), \\ = (0.39, 0.73).$$

(5)

$$\tilde{p}^{\delta} = ((0.5)^{0.6}, \sqrt{1-(1-0.6^2)^{0.6}}), \\ = (0.65, 0.0.48).$$

Definition 2.8. Let $\tilde{p} = ([p,q,r,s]\,;\Psi,\Upsilon) = ([p,q,r,s]\,;[\underline{\Psi},\bar{\Psi}],[\underline{\Upsilon},\bar{\Upsilon}])$ be an interval-valued Pythagorean trapezoidal fuzzy number, where $\Psi = [\underline{\Psi},\bar{\Psi}]$ and $\Upsilon = [\underline{\Upsilon},\bar{\Upsilon}]$ represent an interval-valued, hence $\Psi \subset [0,1]$ and $\Upsilon \subset [0,1]$, such that $0 \leq \Psi^2+\Upsilon^2 \leq 1$.

Definition 2.9. Let $\tilde{p}_1 = ([p_1,q_1,r_1,s_1]\,;[\underline{\Psi}_1,\bar{\Psi}_1],[\underline{\Upsilon}_1,\bar{\Upsilon}_1])$, and $\tilde{p}_2 = ([p_2,q_2,r_2,s_2]\,;[\underline{\Psi}_1,\bar{\Psi}_2],[\underline{\Upsilon}_2,\bar{\Upsilon}_2])$, be any two $IVPTF$ numbers, and $\delta \geq 0$. Then

$$(1)\ \tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [p_1+p_2, q_1+q_2, r_1+r_2, s_1+s_2]; \\ \left[\sqrt{(\underline{\Psi}_1)^2+(\underline{\Psi}_2)^2-(\underline{\Psi}_1\underline{\Psi}_2)^2}, \underline{\Upsilon}_1\underline{\Upsilon}_2\right] \\ ,\left[\sqrt{(\bar{\Psi}_1)^2+(\bar{\Psi}_2)^2-(\bar{\Psi}_1\bar{\Psi}_2)^2}, \bar{\Upsilon}_1\bar{\Upsilon}_2\right] \end{pmatrix},$$

$$(2)\ \tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [p_1p_2, q_1q_2, r_1r_2, s_1s_2]\,; \\ \left[\underline{\Psi}_1\underline{\Psi}_2, \sqrt{\underline{\Upsilon}_1^2+\underline{\Upsilon}_2^2-(\underline{\Upsilon}_1\underline{\Upsilon}_2)^2}\right], \\ \left[\bar{\Psi}_1\bar{\Psi}_2, \sqrt{\bar{\Upsilon}_1^2+\bar{\Upsilon}_2^2-(\bar{\Upsilon}_1\bar{\Upsilon}_2)^2}\right] \end{pmatrix},$$

$$(3)\ \delta\tilde{p} = \begin{pmatrix} [\delta p, \delta q, \delta r, \delta s]\,; \left[\sqrt{1-\left(1-\underline{\Psi}_{\tilde{\alpha}}^2\right)^{\delta}}, (\underline{\Upsilon}_{\tilde{\alpha}})^{\delta}\right], \\ \left[\sqrt{1-\left(1-\bar{\Psi}_{\tilde{\alpha}}^2\right)^{\delta}}, (\bar{\Upsilon}_{\tilde{\alpha}})^{\delta}\right] \end{pmatrix},$$

$$(4)\ \tilde{p}^{\delta} = \begin{pmatrix} [p^{\delta}, q^{\delta}, r^{\delta}, s^{\delta}]\,; \left[\underline{\Psi}_{\tilde{\alpha}}^{\delta}, \sqrt{1-\left(1-\underline{\Upsilon}_{\tilde{\alpha}}^2\right)^{\delta}}\right], \\ \left[\bar{\Psi}_{\tilde{\alpha}}^{\delta}, \sqrt{1-\left(1-\bar{\Upsilon}_{\tilde{\alpha}}^2\right)^{\delta}}\right] \end{pmatrix}.$$

Example 2.10. Let

$$\tilde{p} = ([0.3, 0.4, 0.5, 0.6]\,; [0.7, 0.4]\,, [0.5, 0.6])\,, \\ \tilde{p}_1 = ([0.3, 0.4, 0.5, 0.6]\,; [0.8, 0.5]\,, [0.6, 0.4])\,, \\ \tilde{p}_2 = ([0.5, 0.3, 0.4, 0.4]\,; [0.8, 0.4]\,, [0.8, 0.3])\,,$$

be any three interval-valued Pythagorean trapezoidal fuzzy numbers, and let $\delta = 0.4$. Then, we verify the results as follows;

(1)

$$\tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [0.3+0.5, 0.4+0.3, 0.5+0.4, 0.6+0.4]; \\ \left[\sqrt{(0.8)^2+(0.8)^2-(0.8)^2(0.8)^2}, (0.6)(0.8)\right] \\ , \left[\sqrt{(0.5)^2+(0.4)^2-(0.5)^2(0.4)^2}, (0.4)(0.3)\right] \end{pmatrix},$$
$$= ([0.8, 0.7, 0.9, 1.0]; [0.80, 0.48], [0.45, 0.12]).$$

(2)

$$\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [(0.8)(0.5), (0.4)(0.3), (0.5)(0.4), (0.6)(0.4)]; \\ \left[(0.8)(0.8), \sqrt{0.6^2+0.8^2-((0.6)^2(0.8)^2}\right], \\ \left[(0.5)(0.4), \sqrt{0.4^2+0.3^2-(0.4)^2(0.3)^2}\right] \end{pmatrix},$$
$$= ([0.40, 0.12, 0.20, 0.24]; [0.64, 0.72], [0.20, 0.36]).$$

(3)

$$\delta\tilde{p} = \begin{pmatrix} [(0.4)(0.3), (0.4)(0.4), (0.4)(0.5), (0.4)(0.6)]; \\ \left[\sqrt{1-(1-0.7^2)^{0.4}}, (0.5)^{0.4}\right], \\ \left[\sqrt{1-(1-0.4^2)^{0.4}}, (0.6)^{0.4}\right] \end{pmatrix},$$
$$= ([0.12, 0.16, 0.20, 0.24]; [0.48, 0.75], [0.25, 0.81]).$$

(4)

$$\tilde{p}^{\delta} = \begin{pmatrix} [0.3^{0.4}, 0.4^{0.4}, 0.5^{0.4}, 0.6^{0.4}]; \left[0.7^{0.4}, \sqrt{1-(1-0.5^2)^{0.4}}\right], \\ \left[0.4^{0.4}, \sqrt{1-(1-0.6^2)^{0.4}}\right] \end{pmatrix},$$
$$= ([0.61, 0.69, 0.75, 0.85]; [0.86, 0.32], [0.69, 0.40]).$$

Definition 2.11. [22] Let $\tilde{p} = ([p, q, r, s]; [\underline{\Psi}, \bar{\Psi}], [\underline{\Upsilon}, \bar{\Upsilon}])$ be an interval-valued Pythagorean trapezoidal fuzzy number. Then a score function S can be defined as follows:

$$s(\tilde{p}) = \left(\frac{p+q+r+s}{4} \cdot \frac{\underline{\Psi}^2 - \underline{\Upsilon}^2 + \bar{\Psi}^2 - \bar{\Upsilon}^2}{2}\right) \quad s(\tilde{p}) \in [0, 1] \qquad (3)$$

Example 2.12. Let $\tilde{p} = ([0.8, 0.6, 0.5, 0.7]; [0.7, 0.5], [0.8, 0.6])$ be an interval-valued Pythagorean trapezoidal fuzzy number. Then we verify the above result as follows;

$$s(\tilde{p}) = \left(\frac{0.8+0.6+0.5+0.7}{4} \cdot \frac{0.7^2 - 0.8^2 + 0.5^2 - 0.1^2}{2}\right),$$
$$= -0.0845.$$

Definition 2.13. [22] Let $\tilde{p} = ([p, q, r, s]\,; [\underline{\Psi}, \bar{\Psi}], [\underline{\Upsilon}, \bar{\Upsilon}])$ be an interval-valued Pythagorean trapezoidal fuzzy number. Then an accuracy function H can be defined as follows:

$$H(\tilde{p}) = \left(\frac{p+q+r+s}{4} \cdot \frac{\underline{\Psi}^2 + \underline{\Upsilon}^2 + \bar{\Psi}^2 + \bar{\Upsilon}^2}{2}\right) \quad H(\tilde{p}) \in [0,1], \qquad (4)$$

determine the degree of an accuracy of the interval-valued Pythagorean trapezoidal fuzzy number $\tilde{p}$, where $H(\tilde{p}) \in [0,1]$.

Example 2.14. Let $\tilde{p} = ([0.8, 0.6, 0.5, 0.7]\,; [0.7, 0.5]\,, [0.8, 0.6])$ be interval-valued Pythagorean trapezoidal fuzzy number. Then we verify the above result as follows;

$$\begin{aligned} s(\tilde{p}) &= \left(\frac{0.8+0.6+0.5+0.7}{4} \cdot \frac{0.7^2+0.8^2+0.5^2+0.1^2}{2}\right), \\ &= 0.56. \end{aligned}$$

Theorem 2.15. Let $\tilde{p}_1 = ([p_1, q_1, r_1, s_1]\,; [\underline{\Psi}_1, \bar{\Psi}_1], [\underline{\Upsilon}_1, \bar{\Upsilon}_1])$, and $\tilde{p}_2 = ([p_2, q_2, r_2, s_2]\,;$ $[\underline{\Psi}_1, \bar{\Psi}_2], [\underline{\Upsilon}_2, \bar{\Upsilon}_2])$, be any two $IVPTF$, numbers and $\delta, \delta_1, \delta_2$ are any scalar numbers. Then,

(1) $\tilde{p}_1 \otimes \tilde{p}_2 = \tilde{p}_2 \otimes \tilde{p}_1$,

(2) $(\tilde{p}_1 \otimes \tilde{p}_2)^{\delta} = \tilde{p}_2^{\delta} \otimes \tilde{p}_1^{\delta}$,

(3) $\tilde{p}^{\delta_1} \otimes \tilde{p}^{\delta_2} = \tilde{p}^{(\delta_1+\delta_2)}$.

Proof. (1) Proof is obvious.

(2) Using Definition 2.8 and operational law 2, we have

$$\tilde{p}_1 \otimes \tilde{p}_2 = \left(\begin{array}{c} [p_1p_2, q_1q_2, r_1r_2, s_1s_2]\,; \\ \left[\underline{\Psi}_1\underline{\Psi}_2, \sqrt{\underline{\Upsilon}_1^2 + \underline{\Upsilon}_2^2 - (\underline{\Upsilon}_1\underline{\Upsilon}_2)^2}\right] \\ , \left[\bar{\Psi}_1\bar{\Psi}_2, \sqrt{\bar{\Upsilon}_1^2 + \bar{\Upsilon}_2^2 - (\bar{\Upsilon}_1\bar{\Upsilon}_2)^2}\right] \end{array}\right).$$

Then, from Definition 2.8 and operational law 4, it follows that

$$(\tilde{p}_1 \otimes \tilde{p}_2)^{\delta} = \left(\begin{array}{c} \left[(p_1p_2)^{\delta}, (q_1q_2)^{\delta}, (r_1r_2)^{\delta}, (s_1s_2)^{\delta}\right]; \\ \left[(\underline{\Psi}_1\underline{\Psi}_2)^{\delta}, \sqrt{1-(1-(\underline{\Upsilon}_1^2 + \underline{\Upsilon}_2^2 - (\underline{\Upsilon}_1\underline{\Upsilon}_2)^2)^{\delta}}\right] \\ , \left[(\bar{\Psi}_1\bar{\Psi}_2)^{\delta}, \sqrt{1-(1-(\bar{\Upsilon}_1^2 + \bar{\Upsilon}_2^2 - (\bar{\Upsilon}_1\bar{\Upsilon}_2)^2)^{\delta}}\right] \end{array}\right).$$

Since

$$(\tilde{p}_1)^\delta = \left(\begin{array}{c} \left[(p_1)^\delta, (q_1)^\delta, (r_1)^\delta, (s_1)^\delta\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_1}\right)^\delta, \sqrt{(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}_1})^\delta}\right] \\ , \left[\left(\bar{\Psi}_{\tilde{\alpha}_1}\right)^\delta, \sqrt{(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}_1})^\delta}\right] \end{array} \right),$$

$$(\tilde{p}_2)^\delta = \left(\begin{array}{c} \left[(p_2)^\delta, (q_2)^\delta, (r_2)^\delta, (s_2)^\delta\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_2}\right)^\delta, \sqrt{(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}_2})^\delta}\right] \\ , \left[\left(\bar{\Psi}_{\tilde{\alpha}_2}\right)^\delta, \sqrt{(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}_2})^\delta}\right] \end{array} \right).$$

Therefore we can write $(\tilde{p}_1)^\delta$ and $(\tilde{p}_2)^\delta$ as follows;

$$(\tilde{p}_2)^\delta \otimes (\tilde{p}_1)^\delta = \left(\begin{array}{c} \left[(p_1p_2)^\delta, (q_1q_2)^\delta, (r_1r_2)^\delta, (s_1s_2)^\delta\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_1}\underline{\Psi}_{\tilde{\alpha}_2}\right)^\delta, \sqrt{\begin{array}{c}(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}_1})^\delta + (1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}_2})^\delta - \\ (1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}_1})^\delta(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}_2})^\delta\end{array}}\right] \\ , \left[\left(\bar{\Psi}_{\tilde{\alpha}_1}\bar{\Psi}_{\tilde{\alpha}_2}\right)^\delta, \sqrt{\begin{array}{c}(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}_1})^\delta + (1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}_2})^\delta - \\ (1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}_1})^\delta(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}_2})^\delta\end{array}}\right] \end{array} \right),$$

$$= \left(\begin{array}{c} \left[(p_1p_2)^\delta, (q_1q_2)^\delta, (r_1r_2)^\delta, (s_1s_2)^\delta\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_1}\underline{\Psi}_{\tilde{\alpha}_2}\right)^\delta, \sqrt{(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}_1} + \underline{\Upsilon}^2_{\tilde{\alpha}_2} - \underline{\Upsilon}^2_{\tilde{\alpha}_1}\underline{\Upsilon}^2_{\tilde{\alpha}_2})^\delta}\right] \\ , \left[\left(\bar{\Psi}_{\tilde{\alpha}_1}\bar{\Psi}_{\tilde{\alpha}_2}\right)^\delta, \sqrt{(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}_1} + \bar{\Upsilon}^2_{\tilde{\alpha}_2} - \bar{\Upsilon}^2_{\tilde{\alpha}_1}\bar{\Upsilon}^2_{\tilde{\alpha}_2})^\delta}\right] \end{array} \right).$$

.

Hence, $(\tilde{p}_1 \otimes \tilde{p}_2)^\delta = \tilde{p}_2^\delta \otimes \tilde{p}_1^\delta$.

(3) Using Definition 2.8 and operational law 4,we have

$$(\tilde{p})^{\delta_1} = \left(\begin{array}{c} \left[(p)^{\delta_1}, (q)^{\delta_1}, (r)^{\delta_1}, (s)^{\delta_1}\right]; \\ \left[(\underline{\Psi}_{\tilde{\alpha}})^{\delta_1}, \sqrt{(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1}}\right], \\ \left[(\bar{\Psi}_{\tilde{\alpha}})^{\delta_1}, \sqrt{(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1}}\right] \end{array} \right),$$

$$(\tilde{p})^{\delta_2} = \left(\begin{array}{c} \left[(p)^{\delta_2}, (q)^{\delta_2}, (r)^{\delta_2}, (s)^{\delta_2}\right]; \\ \left[(\underline{\Psi}_{\tilde{\alpha}})^{\delta_2}, \sqrt{(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2}}\right], \\ \left[(\bar{\Psi}_{\tilde{\alpha}})^{\delta_2}, \sqrt{(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2}}\right] \end{array} \right).$$

Then,

$$(\tilde{p})^{\delta_1} \otimes (\tilde{p})^{\delta_2} = \begin{pmatrix} \left[p^{\delta_1} p^{\delta_2}, q^{\delta_1} q^{\delta_2}, r^{\delta_1} r^{\delta_2}, s^{\delta_1} s^{\delta_2}\right]; \\ \left[(\underline{\Psi}_{\tilde{\alpha}})^{\delta_1} (\underline{\Psi}_{\tilde{\alpha}})^{\delta_2}, \sqrt{\begin{array}{c} (1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1}) + (1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2}) \\ -(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1})(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2}) \end{array}} \right], \\ \left[(\bar{\Psi}_{\tilde{\alpha}})^{\delta_1} (\bar{\Psi}_{\tilde{\alpha}})^{\delta_2}, \sqrt{\begin{array}{c} (1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1}) + (1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2}) \\ -(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1})(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2}) \end{array}} \right] \end{pmatrix},$$

$$= \begin{pmatrix} \left[p^{\delta_1} p^{\delta_2}, q^{\delta_1} q^{\delta_2}, r^{\delta_1} r^{\delta_2}, s^{\delta_1} s^{\delta_2}\right]; \\ \left[(\underline{\Psi}_{\tilde{\alpha}})^{\delta_1} (\underline{\Psi}_{\tilde{\alpha}})^{\delta_2}, \sqrt{(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1})(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2})} \right], \\ \left[(\bar{\Psi}_{\tilde{\alpha}})^{\delta_1} (\bar{\Psi}_{\tilde{\alpha}})^{\delta_2}, \sqrt{(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1})(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_2})} \right] \end{pmatrix},$$

$$= \begin{pmatrix} \left[p^{\delta_1+\delta_2}, q^{\delta_1+\delta_2}, r^{\delta_1+\delta_2}, s^{\delta_1+\delta_2}\right]; \\ \left[\underline{\Psi}^{\delta_1+\delta_2}_{\tilde{\alpha}}, \sqrt{(1-(1-\underline{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1+\delta_2}} \right], \\ \left[\bar{\Psi}^{\delta_1+\delta_2}_{\tilde{\alpha}}, \sqrt{(1-(1-\bar{\Upsilon}^2_{\tilde{\alpha}})^{\delta_1+\delta_2}} \right] \end{pmatrix},$$

$$= (\tilde{p})^{\delta_1+\delta_2}.$$

□

3. Averaging Aggregation Operators With Interval-Valued Pythagorean Trapezoidal Fuzzy Numbers

In this section, we introduce the notion of interval-valued Pythagorean trapezoidal fuzzy weighted averaging ($IVPTFWA$) operator, interval-valued Pythagorean trapezoidal fuzzy ordered weighted averaging ($IVPTFOWA$) operator, and interval-valued Pythagorean trapezoidal fuzzy hybrid averaging ($IVPTFHA$) operator. We also discuss various properties of these operators including idempotency, bounded and monotonicity as follows.

Definition 3.1. Let $\tilde{p}_{\pounds}$ $(j = \pounds = 1, 2, ..., \Phi)$ be a group of $IVPTF$ numbers, let Ω be set of $IVPTF$ numbers, such that $IVPTFWA$, $\Omega^{\Phi} \to \Omega$, if

$$IVPTFWA\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_{\Phi}\right) = \left(\hbar_1 \tilde{p}_1 \oplus \hbar_2 \tilde{p}_2 ... \oplus \hbar_{\Phi} \tilde{p}_{\Phi}\right). \quad (5)$$

Then $IVPTFWA$ called interval-valued Pythagorean trapezoidal fuzzy weighted averaging operator of dimension Φ. Especially, if $\hbar = (\hbar_1, \hbar_2, ..., \hbar_{\Phi})^T$ is the weighting vector such that $\tilde{p}_{\pounds}$ $(j = \pounds = 1, 2, ..., \Phi)$, with $\hbar_{\pounds} \in [0, 1]$ and $\sum_{\pounds=1}^{\Phi} \hbar_{\pounds} = 1$, if $\hbar = \left(\frac{1}{\Phi}, \frac{1}{\Phi}, ..., \frac{1}{\Phi}\right)^T$. Then, interval-valued Pythagorean trapezoidal fuzzy weighted averaging ($IVPTFWA$) operator is reduced to interval-valued Pythagorean trapezoidal fuzzy averaging ($IVPTFA$) operator of measurement Φ, which is defined as follows:

$$IVPTFA_w\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_{\Phi}\right) = \left(\tilde{p}_1 \oplus \tilde{p}_2 ... \oplus \tilde{p}_{\Phi}\right)^{\frac{1}{\Phi}}. \quad (6)$$

By Definition 3.10 and Theorem 2.15, we can obtain the following result. In order to proof, we use mathematical induction.

Theorem 3.2. let $\tilde{p}_{\pounds}\ (j=\pounds=1,2,...,\Phi)$ be a group of $IVPTF$ numbers. Then, their aggregated values by using $IVPTFWA$ operator is an also $IVPTF$ number such that,

$$IVPTFWA\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right) = \left(\begin{array}{c} \left[\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}p_{\pounds},\ \sum_{\pounds=1}^{\Phi}\hbar_{\pounds}q_{\pounds},\ \sum_{\pounds=1}^{\Phi}\hbar_{\pounds}r_{\pounds},\ \sum_{\pounds=1}^{\Phi}\hbar_{\pounds}s_{\pounds}\right]; \\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_{\pounds}}\right)^{\hbar_{\pounds}}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_{\pounds}}\right)^{\hbar_{\pounds}}}\right], \\ \left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}_{\pounds}^{\hbar_{\pounds}},\ \prod_{\pounds=1}^{\Phi}\bar{\Upsilon}_{\pounds}^{\hbar_{\pounds}}\right] \end{array}\right), \tag{7}$$

here $\hbar=\left(\frac{1}{\Phi},\frac{1}{\Phi},...,\frac{1}{\Phi}\right)^T$ is the weighting vector of $\tilde{p}_{\pounds}\ (\pounds=1,2,...,\Phi)$ with $\hbar_{\pounds}\in[0,1]$ and $\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}=1$.

Proof. The result first follows from Definition 3.10 and Theorem 2.15, by mathematical induction we prove the second result when $\Phi=2$.

$$\hbar_1\tilde{p}_1 = \left(\begin{array}{c} \left[\hbar_1p_1,\hbar_1q_1,\hbar_1r_1,\hbar_1s_1\right]; \\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}}\right], \\ \left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}_{\tilde{\alpha}_1}^{\hbar_1},\ \prod_{\pounds=1}^{\Phi}\bar{\Upsilon}_{\tilde{\alpha}_1}^{\hbar_1}\right] \end{array}\right),$$

$$\hbar_2\tilde{p}_2 = \left(\begin{array}{c} \left[\hbar_2p_2,\hbar_2q_2,\hbar_2r_2,\hbar_2s_2\right]; \\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}}\right], \\ \left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}_{\tilde{\alpha}_2}^{\hbar_2},\ \prod_{\pounds=1}^{\Phi}\bar{\Upsilon}_{\tilde{\alpha}_2}^{\hbar_2}\right] \end{array}\right).$$

Then,

$$\begin{aligned} &PTFWA(\tilde{p}_1,\tilde{p}_2)=\hbar_1\tilde{p}_1\oplus\hbar_2\tilde{p}_2 \\ &= \left(\begin{array}{c} \left[\hbar_1p_1+\hbar_2p_2,\hbar_1q_1+\hbar_2q_2,\hbar_1r_1+\hbar_2r_2,\hbar_1s_1+\hbar_2s_2\right]; \\ \left[\sqrt{\begin{array}{c}1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}+1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2} \\ -(1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1})(1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2})\end{array}},\left(\underline{\Upsilon}_{\tilde{\alpha}_1}^{\hbar_1}\right)\left(\underline{\Upsilon}_{\tilde{\alpha}_2}^{\hbar_2}\right)\right] \\ ,\left[\sqrt{\begin{array}{c}1-\left(1-\bar{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}+1-\left(1-\bar{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2} \\ -(1-\left(1-\bar{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1})(1-\left(1-\bar{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2})\end{array}},\left(\bar{\Upsilon}_{\tilde{\alpha}_1}^{\hbar_1}\right)\left(\bar{\Upsilon}_{\tilde{\alpha}_2}^{\hbar_2}\right)\right] \end{array}\right), \\ &= \left(\begin{array}{c} \left[\hbar_1p_1+\hbar_2p_2,\hbar_1q_1+\hbar_2q_2,\hbar_1r_1+\hbar_2r_2,\hbar_1s_1+\hbar_2s_2\right]; \\ \left[\sqrt{1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2})},\left(\underline{\Upsilon}_{\tilde{\alpha}_1}^{\hbar_1}\right)\left(\underline{\Upsilon}_{\tilde{\alpha}_2}^{\hbar_2}\right)\right] \\ ,\left[\sqrt{1-\left(1-\bar{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2})},\left(\bar{\Upsilon}_{\tilde{\alpha}_1}^{\hbar_1}\right)\left(\bar{\Upsilon}_{\tilde{\alpha}_2}^{\hbar_2}\right)\right] \end{array}\right). \end{aligned}$$

If E.q. (7) holds for $\Phi = k$, that is

$$PTFWA_w\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_k\right) = \left(\begin{array}{c} \left[\sum_{\pounds=1}^{k} \hbar_\pounds p_\pounds, \sum_{\pounds=1}^{k} \hbar_\pounds q_\pounds, \sum_{\pounds=1}^{k} \hbar_\pounds r_\pounds, \sum_{\pounds=1}^{k} \hbar_\pounds s_\pounds\right]; \\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_\pounds}\right)^{\hbar_\pounds}}, \sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_\pounds}\right)^{\hbar_\pounds}}\right] \\ , \left[\prod_{\pounds=1}^{\Phi} \underline{\Upsilon}_\pounds^{\hbar_\pounds}, \prod_{\pounds=1}^{\Phi} \bar{\Upsilon}_\pounds^{\hbar_\pounds}\right] \end{array} \right).$$

Therefore $\Phi = k+1$ and operational laws in Definition 2.8, we have

$$PTFWA_w\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_{k+1}\right)$$

$$= \left(\begin{array}{c} \left[\sum_{\pounds=1}^{k+1} \hbar_\pounds p_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds q_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds r_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds s_\pounds\right]; \\ \left[\sqrt{\begin{array}{c} 1-\prod_{\pounds=1}^{k}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_\pounds}+1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_{k+1}}\right)^{\hbar_{k+1}}- \\ \left(1-\prod_{\pounds=1}^{k}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_\pounds}\right)^{\hbar_\pounds}(1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_{k+1}}\right)^{\hbar_{k+1}}\right) \end{array}}, \prod_{\pounds=1}^{k+1}\underline{\Upsilon}^{\hbar_\pounds}_{\tilde{\alpha}_\pounds}\right], \\ \left[\sqrt{\begin{array}{c} 1-\prod_{\pounds=1}^{k}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_\pounds}+1-\left(1-\bar{\Psi}^2_{\tilde{\alpha}_{k+1}}\right)^{\hbar_{k+1}}- \\ \left(1-\prod_{\pounds=1}^{k}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_\pounds}\right)^{\hbar_\pounds}(1-\left(1-\bar{\Psi}^2_{\tilde{\alpha}_{k+1}}\right)^{\hbar_{k+1}}\right) \end{array}}, \prod_{\pounds=1}^{k+1}\bar{\Upsilon}^{\hbar_\pounds}_{\tilde{\alpha}_\pounds}\right] \end{array} \right),$$

$$= \left(\begin{array}{c} \left[\sum_{\pounds=1}^{k+1} \hbar_\pounds p_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds q_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds r_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds s_\pounds\right]; \\ , \left[\sqrt{1-\prod_{\pounds=1}^{k+1}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_\pounds}\right)^{\hbar_\pounds}}, \prod_{\pounds=1}^{k+1}\underline{\Upsilon}^{\hbar_\pounds}_{\tilde{\alpha}_\pounds}\right], \left[\sqrt{1-\prod_{\pounds=1}^{k+1}\left(1-\bar{\Psi}^2_{\tilde{\alpha}_\pounds}\right)^{\hbar_\pounds}}, \prod_{\pounds=1}^{k+1}\bar{\Upsilon}^{\hbar_\pounds}_{\tilde{\alpha}_\pounds}\right] \end{array} \right).$$

Therefore E.q. (7) holds for $\Phi = k+1$. Hence $E.q.$ (7) holds $\forall\ \Phi$. □

Theorem 3.3. Let $\tilde{p}_\pounds\ (j = \pounds = 1, 2, ..., \Phi)$ be a collection of $IVPTF$ numbers and $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$ be the weighting vector of $\tilde{p}_\pounds\ (\pounds = 1, 2, ..., \Phi)$, with $\hbar_\pounds \in [0, 1]$ and $\sum_{\pounds=1}^{\Phi} \hbar_\pounds = 1$. Then we have following properties;

(1) (Idempotent): If all $\tilde{p}_\pounds\ (\pounds = 1, 2, ..., \Phi)$ are equal such that $\tilde{p}_\pounds = \tilde{p}\ \forall \pounds$, then

$$IVPTFWA_w\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right) = \tilde{p}. \qquad (8)$$

(2) (Bounded):

$$\tilde{p}^- \leq IVPTFWA\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right) \leq \tilde{p}^+,$$

here

$$\tilde{p}^- = \min_\pounds\left(\tilde{p}_\pounds\right) \text{ and } \tilde{p}^+ = \max_\pounds\left(\tilde{p}_\pounds\right).$$

(3) (Monotonicity): Let $\tilde{p}^*_\pounds\ (\pounds = 1, 2, ..., \Phi)$ be a collection of $IVPTF$ numbers. If $\tilde{p}_\pounds \leq \tilde{p}^*_\pounds\ \forall\ \pounds$. Then,

$$IVPTFWA_w\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right) \leq IVPTFWA_w\left(\tilde{p}^*_1, \tilde{p}^*_2, ..., \tilde{p}^*_\Phi\right) \forall \pounds. \qquad (9)$$

Example 3.4. Let

$$\begin{aligned}
\tilde{p}_1 &= ([0.3, 0.4, 0.5, 0.6] ; [0.7, 0.4] , [0.8, 0.3]) , \\
\tilde{p}_2 &= ([0.4, 0.5, 0.6, 0.4] ; [0.9, 0.2] , [0.8, 0.6]) , \\
\tilde{p}_3 &= ([0.5, 0.4, 0.6, 0.9] ; [0.7, 0.6] , [0.6, 0.4]) , \\
\tilde{p}_4 &= ([0.4, 0.3, 0.2, 0.1] ; [0.5, 0.6] , [0.7, 0.5]) , \\
\tilde{p}_5 &= ([0.5, 0.6, 0.4, 0.3] ; [0.8, 0.3] , [0.6, 0.5])
\end{aligned}$$

be five interval-valued Pythagorean trapezoidal fuzzy numbers and let $\hbar =$ $(0.10, 0.20, 0.30, 0.15, 0.25)$ be the weighting vector of $\tilde{p}_{£}$ $(£ = 1, 2, 3, 4, 5)$. We apply E.q. (7) , such that

$$\left(\begin{array}{c}
IVPTFWA (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5) = \\
\left[\begin{array}{c}
0.10 (0.3) + 0.20 (0.4) + 0.30 (0.5) + 0.15 (0.4) + 0.25 (0.5) , \\
0.10 (0.4) + 0.20 (0.5) + 0.30 (0.4) + 0.15 (0.3) + 0.25 (0.6) , \\
0.10 (0.5) + 0.20 (0.6) + 0.30 (0.6) + 0.15 (0.2) + 0.25 (0.4) , \\
0.10 (0.6) + 0.20 (0.4) + 0.30 (0.9) + 0.15 (0.1) + 0.25 (0.3)
\end{array} \right] ; \\
\left[\begin{array}{c}
\sqrt{1 - (1 - 0.7^2)^{.10} (1 - 0.9^2)^{.20} (1 - 0.7^2)^{.30} (1 - 0.5^2)^{.15} (1 - 0.8^2)^{.25}} \\
, \sqrt{1 - (1 - 0.4^2)^{.10} (1 - 0.2^2)^{.20} (1 - 0.6^2)^{.30} (1 - 0.6^2)^{.15} (1 - 0.3^2)^{.25}}
\end{array} \right] \\
, \left[\begin{array}{c}
(0.8)^{.10} (0.8)^{.20} (0.6)^{.30} (0.7)^{.15} (0.6)^{.25}, \\
(0.3)^{.10} (0.6)^{.20} (0.4)^{.30} (0.5)^{.15} (0.5)^{.25}
\end{array} \right]
\end{array} \right)$$

$$IVPTFWA (\tilde{p}_1, ..., \tilde{p}_5) = ([0.44, 0.45, 0.48, 0.5] ; [0.77, 0.48] , [0.66, 0.46]) .$$

Definition 3.5. Let $\tilde{p}_{£}$ $(j = £ = 1, 2, ..., \Phi)$ be a the group of interval-valued Pythagorean trapezoidal fuzzy $(IVPTF)$ numbers, interval-valued Pythagorean trapezoidal fuzzy ordered weighted averaging $(IVPTFOWA)$ operator of measurement Φ is a mapping and let $IVPTFOWA$: $\Omega^{\Phi} -> \Omega$, is the weighting vector $\hbar = (\hbar_1, \hbar_2, ..., \hbar_{\Phi})^T$ such that $\hbar_{£} \in [0, 1]$ and $\sum_{£=1}^{\Phi} \hbar_{£} = 1$.

$$IVPTFOWA (\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_{\Phi}) = (\hbar_1 \tilde{p}_{\sigma_{(1)}} \oplus \hbar_2 \tilde{p}_{\sigma_{(2)}} ... \oplus \hbar_{\Phi} \tilde{p}_{\sigma_{(\Phi)}}), \qquad (10)$$

here $(\sigma (1) , \sigma (2) , ..., \sigma (\Phi))$ is a permutation of $(1, 2, ..., \Phi)$ such that $\tilde{p}_{\sigma_{(£-1)}} \geq \tilde{p}_{\sigma_{(£)}}$ for all $£$. If $\hbar = (\hbar_1, \hbar_2, ..., \hbar_{\Phi})^T$, then interval-valued Pythagorean trapezoidal fuzzy ordered weighted averaging $(IVPTFOWA)$ operator is reduced to be interval-valued Pythagorean trapezoidal fuzzy averaging $(IVPTFA)$ operator of dimension Φ.

Theorem 3.6. Let $\tilde{p}_{£}$ $(£ = 1, 2, ..., \Phi)$ be a collection of interval-valued Pythagorean trapezoidal fuzzy $(IVPTF)$ numbers, then their aggregated value by using the $IVPTFOWG$

operator is also $IVPTF$ number such that,

$$IVPTFOWA\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right) = \left(\begin{array}{c}\left[\sum_{\pounds=1}^{\Phi}\hbar_\pounds p_{\sigma(\pounds)},\sum_{\pounds=1}^{\Phi}\hbar_\pounds q_{\sigma(\pounds)},\sum_{\pounds=1}^{\Phi}\hbar_\pounds r_{\sigma(\pounds)},\sum_{\pounds=1}^{\Phi}\hbar_\pounds s_{\sigma(\pounds)}\right];\\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}_{\sigma(\pounds)}^2\right)^{\hbar_\pounds}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\bar{\Psi}_{\sigma(\pounds)}^2\right)^{\hbar_\pounds}}\right],\\ \left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}_{\sigma(\pounds)}^{\hbar_\pounds},\prod_{\pounds=1}^{\Phi}\bar{\Upsilon}_{\sigma(\pounds)}^{\hbar_\pounds}\right]\end{array}\right), \tag{11}$$

here $\hbar=\left(\frac{1}{\Phi},\frac{1}{\Phi},...,\frac{1}{\Phi}\right)^T$ be the weighting vector of $\tilde{p}_\pounds\ (j=\pounds=1,2...,\Phi)$ with $\hbar_\pounds\in[0,1]$ and $\sum_{\pounds=1}^{\Phi}\hbar_\pounds=1$.

Example 3.7. Let

$$\begin{aligned}
\tilde{p}_1 &= ([0.3,0.4,0.5,0.6];[0.8,0.4],[0.6,0.3]),\\
\tilde{p}_2 &= ([0.4,0.5,0.6,0.4];[0.9,0.4],[0.8,0.3]),\\
\tilde{p}_3 &= ([0.6,0.5,0.4,0.8];[0.7,0.6],[0.6,0.4]),\\
\tilde{p}_4 &= ([0.4,0.3,0.2,0.1];[0.6,0.6],[0.6,0.5]),\\
\tilde{p}_5 &= ([0.6,0.4,0.2,0.1];[0.8,0.5],[0.6,0.5]).
\end{aligned}$$

be five interval-valued Pythagorean trapezoidal fuzzy numbers, suppose $\hbar=(0.15,0.25,0.10,0.20,0.30)$ be the weighting vector of $\tilde{p}_\pounds\ (\pounds=1,2,3,4,5)$. By using score function we determine $S(\tilde{p}_1)=0.078$, $S(\tilde{p}_2)=0.057, S(\tilde{p}_3)=0.094, S(\tilde{p}_4)=0.013, S(\tilde{p}_5)=0.20$ and we can write in ordered form such that, $S(\tilde{p}_5)\geq S(\tilde{p}_3)\geq S(\tilde{p}_1)\geq S(\tilde{p}_2)\geq S(\tilde{p}_4)$, also we can write $\tilde{p}_{\sigma(1)}=\tilde{p}_5$, $\tilde{p}_{\sigma(2)}=\tilde{p}_3$, $\tilde{p}_{\sigma(3)}=\tilde{p}_1$, $\tilde{p}_{\sigma(4)}=\tilde{p}_2, \tilde{p}_{\sigma(5)}=\tilde{p}_4$.

Therefore we can write,

$$\begin{aligned}
\tilde{p}_1 &= ([0.6,0.4,0.2,0.1];[0.8,0.5],[0.6,0.5]),\\
\tilde{p}_2 &= ([0.6,0.5,0.4,0.8];[0.7,0.6],[0.6,0.4]),\\
\tilde{p}_3 &= ([0.3,0.4,0.5,0.6];[0.8,0.4],[0.6,0.3]),\\
\tilde{p}_1 &= ([0.4,0.5,0.6,0.4];[0.9,0.4],[0.8,0.3]),\\
\tilde{p}_1 &= ([0.4,0.3,0.2,0.1];[0.6,0.6],[0.6,0.5])
\end{aligned}$$

By using e.q.(11) we have,

$$\left(\begin{array}{c} IVPTFOWA\left(\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4,\tilde{p}_5\right) = \\ \left[\begin{array}{c} 0.15\,(0.6)+0.25\,(0.6)+0.10\,(0.3)+0.20\,(0.4)+0.30\,(0.4)\,, \\ 0.15\,(0.4)+0.25\,(0.5)+0.10\,(0.4)+0.20\,(0.5)+0.30\,(0.3)\,, \\ 0.15\,(0.2)+0.25\,(0.4)+0.10\,(0.5)+0.20\,(0.6)+0.30\,(0.2)\,, \\ 0.15\,(0.1)+0.25\,(0.8)+0.10\,(0.6)+0.20\,(0.4)+0.30\,(0.1)\,, \end{array}\right]; \\ \left[\begin{array}{c} \sqrt{1-\left(1-0.8^2\right)^{.15}\left(1-0.7^2\right)^{.25}\left(1-0.8^2\right)^{.10}\left(1-0.9^2\right)^{.20}\left(1-0.6^2\right)^{.30}}, \\ \sqrt{1-\left(1-0.5^2\right)^{.15}\left(1-0.6^2\right)^{.25}\left(1-0.4^2\right)^{.10}\left(1-0.4^2\right)^{.20}\left(1-0.6^2\right)^{.30}} \end{array}\right] \\ ,\left[\begin{array}{c} (0.6)^{.15}(0.6)^{.25}(0.6)^{.10}(0.8)^{.20}(0.6)^{.30}, \\ (0.5)^{.15}(0.4)^{.25}(0.3)^{.10}(0.3)^{.20}(0.5)^{.30} \end{array}\right] \end{array}\right)$$

$$IVPTFOWA\left(\tilde{p}_1,...,\tilde{p}_5\right) = \left(\left[0.39,0.41,0.36,0.32\right];\left[0.77,0.55\right],\left[0.63,0.40\right]\right).$$

Theorem 3.8. Let $\tilde{p}_{\pounds}\left(j=\pounds=1,2,...,\Phi\right)$ be a group of interval-valued Pythagorean trapezoidal fuzzy $(IVPTF)$ numbers and $\hbar=\left(\hbar_1,\hbar_2,...,\hbar_\Phi\right)^T$ is the weighting vector of $\tilde{p}_{\pounds}=\left(\pounds=1,2,...,\Phi\right)$, with $\hbar_{\pounds}\in[0,1]$ and $\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}=1$. Then, we have following properties;

(1) (Idempotent): If all $\tilde{p}_{\pounds}\left(j=\pounds=1,2,...,\Phi\right)$ are equal such that, $\tilde{p}_{\pounds}=\tilde{p}\ \forall\pounds$, then$IVPTFOWA_w\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right)=\tilde{p}$. (12)

(2) (Boundary):

$$\tilde{p}^-\leq IVPTFOWA\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right)\leq\tilde{p}^+,$$

for all $\hbar$, where$\tilde{p}^-=\min_{\pounds}\left(\tilde{p}_{\pounds}\right)$ and $\tilde{p}^+=\max_{\pounds}\left(\tilde{p}_{\pounds}\right)$.

(3) (Monotonicity): Let $\tilde{p}^*_{\pounds}\left(\pounds=1,2,...,\Phi\right)$ be a collection of interval-valued Pythagorean trapezoidal Fuzzy $(IVPTF)$ numbers. If $\tilde{p}_{\pounds}\leq\tilde{p}^*_{\pounds}\ \forall\ \pounds$, then,

$$IVPTFOWA_w\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right)\leq IVPTFOWA_w\left(\tilde{p}^*_1,\tilde{p}^*_2,...,\tilde{p}^*_\Phi\right)\forall\hbar. \quad (13)$$

Theorem 3.9. Let $\tilde{p}_{\pounds}\left(j=\pounds=1,2,...,\Phi\right)$ be a collection of $IVPTF$ numbers, and $\hbar=\left(\hbar_1,\hbar_2,...,\hbar_\Phi\right)^T$ is the weighting vector of $IVPTFOWA$ operator, with $\hbar_{\pounds}\in[0,1]$ and $\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}=1$.

(1) If $\hbar=(1,0,...,0)^T$, then

$$IVPTFOWA_w\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right)=\max_{\pounds}\left(\tilde{p}_{\pounds}\right).$$

(2) If $\hbar=(0,0,...,1)^T$, then

$$IVPTFOWA_w\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right)=\min_{\pounds}\left(\tilde{p}_{\pounds}\right).$$

(3) If $\hbar=1$, $w_i=0$, and $i\neq\pounds$, then

$$IVPTFOWA_w\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_\Phi\right)=\tilde{p}_{\sigma(\pounds)},$$

here $\tilde{p}_{\sigma(\pounds)}$ is the jth largest of $\tilde{p}_i\left(i=1,2,...,\Phi\right)$. We shall define interval-valued Pythagorean trapezoidal fuzzy hybrid averaging $(IVPTFHA)$ operator in the next Theorem.

Definition 3.10. Let $\tilde{p}_{\pounds}\left(j=\pounds=1,2,...,\Phi\right)$ be a collection of $IVPTF$ numbers. An interval-valued Pythagorean trapezoidal fuzzy hybrid averaging $(IVPTFHA)$ operator of dimension Φ is a mapping $IVPTFHA : \Omega^{\Phi} - > \Omega$, that has an associated vector $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$ such that $\hbar_£ > 0$ and $\sum_{£=1}^{\Phi} \hbar_£ = 1$. Furthermore,

$$IVPTFHA_{w,w}\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right) = (\hbar_1 \dot{\tilde{p}}_{\sigma_{(1)}} \oplus \hbar_1 \dot{\tilde{p}}_{\sigma_{(2)}} ... \oplus \hbar_1 \dot{\tilde{p}}_{\sigma_{(\Phi)}}), \tag{14}$$

where $\dot{\tilde{p}}_{\sigma_{(£)}}$ is the $£$ th largest of the weighted interval-valued Pythagorean trapezoidal fuzzy numbers such that, $\dot{\tilde{p}}_£ \left(\dot{\tilde{p}}_£ = \Phi \hbar_£ \dot{\tilde{p}}_£, £ = 1, 2, ..., \Phi \right)$, and $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$ be the weighting vector of $\tilde{p}_£$ $(£ = 1, 2, ..., \Phi)$, and $\hbar_£ > 0$ and $\sum_{£=1}^{\Phi} \hbar_£ = 1$ and Φ is the balancing coefficient.

Theorem 3.11. Let $\tilde{p}_£$ $(j = £ = 1, 2, ..., \Phi)$ be a collection of $IVPTF$ numbers, then their aggregated value by using the $IVPTFHA$ operator is also $IVPTF$ number such that,

$$IVPTFHA_{w,w}(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi) = \left(\begin{array}{c} \left[\sum_{£=1}^{\Phi} \hbar_£ \dot{\tilde{p}}_{\sigma(£)}, \sum_{£=1}^{\Phi} \hbar_£ \dot{\tilde{q}}_{\sigma(£)}, \sum_{£=1}^{\Phi} \hbar_£ \dot{\tilde{r}}_{\sigma(£)}, \sum_{£=1}^{\Phi} \hbar_£ \dot{\tilde{s}}_{\sigma(£)} \right]; \\ \left[\sqrt{1 - \prod_{£=1}^{\Phi} \left(1 - \underline{\dot{\Psi}}_{\sigma(£)}^2\right)^{\hbar_£}}, \sqrt{1 - \prod_{£=1}^{\Phi} \left(1 - \bar{\dot{\Psi}}_{\sigma(£)}^2\right)^{\hbar_£}} \right], \\ \left[\prod_{£=1}^{\Phi} \underline{\dot{\Upsilon}}_{\sigma(£)}^{\hbar_£}, \prod_{£=1}^{\Phi} \bar{\dot{\Upsilon}}_{\sigma(£)}^{\hbar_£} \right] \end{array} \right). \tag{15}$$

Theorem 3.12. The $IVPTFWA$ operator is a special case of the $IVPTFHA$ operator.

Proof. Let $\hbar = \left(\frac{1}{\Phi}, \frac{1}{\Phi}, ..., \frac{1}{\Phi}\right)^T$, then

$$\begin{aligned} IVPTFHA_{w,w}\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right) &= \left(\hbar_1 \dot{\tilde{p}}_{\sigma_{(1)}} \oplus \hbar_2 \dot{\tilde{p}}_{\sigma_{(2)}} ... \oplus \hbar_\Phi \dot{\tilde{p}}_{\sigma_{(\Phi)}}\right) \\ &= \left(\frac{1}{\Phi} \dot{\tilde{p}}_{\sigma_{(1)}} \oplus \frac{1}{\Phi} \dot{\tilde{p}}_{\sigma_{(2)}} ... \oplus \frac{1}{\Phi} \dot{\tilde{p}}_{\sigma_{(\Phi)}}\right) \\ &= (\hbar_1 \tilde{p} \oplus \hbar_2 \tilde{p} ... \oplus \hbar_\Phi \tilde{p}\ ) \\ &= IPTFWA_w\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right). \end{aligned}$$

Theorem 3.13. The $IVPTFOWA$ operator is a special case of the $IVPTFHA$ operator. □

Proof. Let $\hbar = \left(\frac{1}{\Phi}, \frac{1}{\Phi}, ..., \frac{1}{\Phi}\right)^T$, then $\left(\dot{\tilde{p}}_£ = \tilde{p}_£, £ = 1, 2, ..., \Phi\right)$

$$\begin{aligned} IVPTFHA_{w,w}\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right) &= \left(\hbar_1 \dot{\tilde{p}}_{\sigma_{(1)}} \oplus \hbar_2 \dot{\tilde{p}}_{\sigma_{(2)}} ... \oplus \hbar_\Phi \dot{\tilde{p}}_{\sigma_{(\Phi)}}\right) \\ &= \left(\hbar_1 \tilde{p}_{\sigma_{(1)}} \oplus \hbar_2 \tilde{p}_{\sigma_{(2)}} ... \oplus \hbar_\Phi \tilde{p}_{\sigma_{(\Phi)}}\right) \\ &= IVPTFOWA_w\left(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi\right). \end{aligned}$$

□

4. An Application of Interval-Valued Pythagorean Trapezoidal Fuzzy Numbers With $MAGDM$ Problems

To solve the multiple attribute group decision making $(MAGDM)$ problem we use $IVPTFWA$ as well as $IVPTFHA$ operators with interval-valued Pythagorean trapezoidal fuzzy information. Let $B = \{B_1, B_2, ..., B_m\}$ be set a of alternatives and $C = \{C_1, C_2, ..., C_n\}$ be set of attributes. Let $P = \{P_1, P_2, ..., p_£\}$, is weighting vector and sum of $p_£$ is equal to one such that $\sum_{£=1}^{\Phi} P_£ = 1$. Let the set of decision makers is denoted by $Q = \{Q_1, Q_2, ..., Q_t\}$ whose weighting vector is $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$ such that, $\hbar_k \in [0,1]$ and $\sum_{k=1}^{t} \hbar_k = 1$.

Let

$$\tilde{Z}^k = \left(\tilde{z}_{i£}^k\right)_{m\times n} = \left[p_{i£}^k, q_{i£}^k, r_{i£}^k, s_{i£}^k\right]; [\Psi_{i£}^k, \bar{\Psi}_{i£}^k], [\Upsilon_{i£}^k, \bar{\Upsilon}_{i£}^k])_{m\times n},$$

be the interval-valued Pythagorean trapezoidal fuzzy decision matrix. Then,

$$[\Psi_{i£}^k, \bar{\Psi}_{i£}^k] \subset [0,1], \text{and } [\Upsilon_{i£}^k, \bar{\Upsilon}_{i£}^k] \subset [0,1] \ \ \Psi_{i£}^{2k} + \bar{\Psi}_{i£}^{2k} \leq 1 \text{ and } \Upsilon_{i£}^{2k} + \bar{\Upsilon}_{i£}^{2k} \leq 1,$$
$$(£ = 1, 2, ..., \Phi, i = 1, 2, ..., m, k = 1, 2, ..., t).$$

In the following steps, we solve $MAGDM$ problems by applying $IVPTF$ information by using the following steps;

Algorithm

Step 1. In this step, we construct the interval-valued Pythagorean trapezoidal fuzzy decision matrix

Step 2. In this step, we apply the attribute weight on the $IVPTFWA$ operator such that

$$\left(\tilde{z}_i^k\right) = IVPTFWA(z_{i1}^k, z_{i2}^k, ..., z_{i\Phi}^k), \ (i = 1, 2, ..., m, k = 1, 2, ..., t), \quad (16)$$

is the individual overall preference interval-valued Pythagorean trapezoidal fuzzy values $\left(\tilde{z}_i^k\right)$ of the alternative B_i.

Step 3. In this step, we determine the ordered of interval-valued Pythagorean trapezoidal fuzzy decision matrix $\left(\tilde{z}_i^k\right)$ of the alternative B_i, by applying operational law 3 and score function by using E.q.(3).

Step 4. We utilize, $IVPTFHA$ operator to derive the collective overall preference values of $IVPTF$ values $\tilde{z}_i \, (i = 1, 2, ..., m)$ of the alternative B_i;

$$(\tilde{z}_i) = ([p_i, q_i, r_i, s_i]; [\Psi_i, \bar{\Psi}_i], [\Upsilon_i, \bar{\Upsilon}_i]) = IVPTFHA_{w,w}(z_{i£}^1, z_{i£}^2, ..., z_{i£}^t), \quad (17)$$

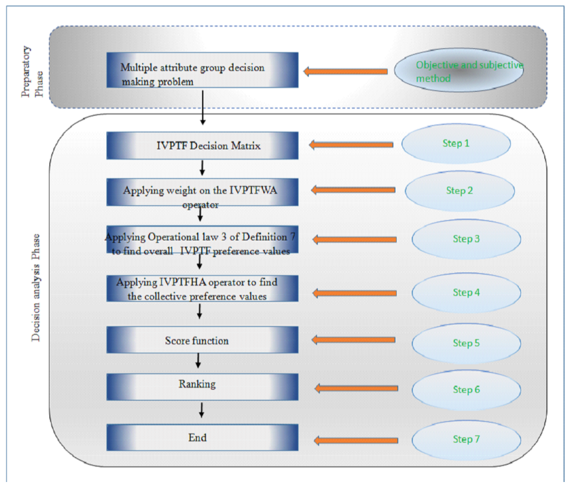
here $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T)$ is the weighting vector of decision makers. with $\hbar_k \in [0,1]$ and $\sum_{k=1}^{t} \hbar_k = 1$, $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_t)^T$ is the associated weight vector of the $IVPTFHA$ operator with $\Gamma_k \in [0,1]$ and $\sum_{k=1}^{t} \Gamma_k = 1$.

Step 5. In this step, we use score function to aggregate values of each alternative.

Step 6. In this step, we determine the rank of alternative B_i and select the best option according to descending order.

Step 7. End.

Flow chart of proposed algorithm



5. NUMERICAL EXAMPLE

Global environmental concern is a certainty and consideration on the black manufacture in several industries. A car company wanted to choose the most suitable black supplier having the key factor in its industrial process. Subsequently pre-evaluation, four suppliers $Bi(i = 1, 2, 3, 4)$ have persisted as alternatives for further evaluation. There are four criteria to be supposed such that C_1 creation worth C_2 equipment competence C_3 contamination control C_4 atmosphere supervision. Suppose $p = (0.4, 0.3, 0.2, 0.1)^T$ is the weighting vector. The company arranged four group decision maker's form four fidelity branches q_1 is from the engineering branch q_2 is from the acquiring branch q_3 is from the quality assessment branch q_4 is from the fabrication branch having weight $\hbar = (0.20, 0.30, 0.35, 0.15)^T$. They constructed the decision matrix $Z^{(k)} = \left(z_{ij}^{(k)}\right)_{4\times 4}$ $(k = 1, 2, 3, 4)$ as follows:

Step1: The decision maker's give his decision in the following tables.

Decision matrix of expert-1

$Z^{(1)} =$

	C_1
B_1	$([0.3, 0.4, 0.4, 0.3]\,; [0.5, 0.6], [0.4, 0.7])$
B_2	$([0.4, 0.3, 0.6, 0.3]\,; [0.4, 0.6], [0.6, 0.5])$
B_3	$([0.5, 0.3, 0.6, 0.4]\,; [0.5, 0.7], [0.8, 0.5])$
B_4	$([0.9, 0.6, 0.4, 0.1]\,; [0.3, 0.8], [0.6, 0.5])$
	C_2
B_1	$([0.7, 0.5, 0.6, 0.3]\,; [0.4, 0.6], [0.5, 0.6])$
B_2	$([0.3, 0.1, 0.2, 0.4]\,; [0.3, 0.8], [0.6, 0.5])$
B_3	$([0.2, 0.1, 0.3, 0.5]\,; [0.6, 0.5], [0.3, 0.8])$
B_4	$([0.4, 0.3, 0.4, 0.2]\,; [0.3, 0.8], [0.6, 0.5])$
	C_3
B_1	$([0.3, 0.4, 0.5, 0.6]\,; [0.8, .4], [0.5, 0.7])$
B_2	$([0.4, 0.5, 0.7, 0.2]\,; [0.6, 0.7], [0.7, 0.6])$
B_3	$([0.3, 0.1, 0.2, 0.3]\,; [0.3, 0.8], [0.8, 0.6])$
B_4	$([0.4, 0.3, 0.4, 0.6]\,; [0.4, 0.7], [0.6, 0.4])$
	C_4
B_1	$([0.4, 0.5, 0.2, 0.3]\,; [0.6, 0.5], [0.4, 0.7])$
B_2	$([0.4, 0.3, 0.2, 0.1]\,; [0.4, 0.6], [0.7, 0.5])$
B_3	$([0.6, 0.8, 0.9, 0.2]\,; [0.3, 0.9], [0.8, 0.4])$
B_4	$([0.4, 0.5, 0.4, 0.3]\,; [0.5, 0.5], [0.8, 0.6])$

Decision matrix of expert-2

$Z^{(2)} =$

	C_1
B_1	$([0.4, 0.5, 0.7, 0.5]; [0.3, 0.9], [0.8, 0.3])$
B_2	$([0.3, 0.4, 0.4, 0.6]; [0.3, 0.7], [0.8, 0.3])$
B_3	$([0.5, 0.4, 0.2, 0.3]; [0.4, 0.7]; [0.8, 0.5])$
B_4	$([0.4, 0.6, 0.3, 0.4]; [0.8, 0.6]; [0.5, 0.7])$
	C_2
B_1	$([0.4; 0.3, 0.6, 0.7]; [0.4, 0.6]; [0.6, 0.6])$
B_2	$([0.3, 0.4, 0.5, 0.6]; [0.9, 0.3]; [0.3, 0.8])$
B_3	$([0.4, 0.5, 0.7, 0.8]; [0.6, 0.7]; [0.4, 0.7])$
B_4	$([0.3, 0.4, 0.5, 0.7]; [0.7, 0.6]; [0.5, 0.8])$
	C_1
B_1	$([0.8, 0.2, 0.3, 0.4]; [0.5, 0.5]; [0.6, 0.7])$
B_2	$([0.4, 0.3, 0.2, 0.1]; [0.6, 0.5]; [0.5, 0.6])$
B_3	$([0.3, 0.4, 0.5, 0.6]; [0.5, 0.5]; [0.6, 0.7])$
B_4	$([0.4, 0.5, 0.6, 0.7]; [0.4, 0.8]; [0.7, 0.3])$
	C_2
B_1	$([0.3, 0.4, 0.5, 0.6]; [0.4, 0.8]; [0.8, 0.5])$
B_2	$([0.4, 0.5, 0.3, 0.1]; [0.3, 0.9]; [0.8, 0.3])$
B_3	$([0.5, 0.5, 0.4, 0.6]; [0.3, 0.8]; [0.6, 0.4])$
B_4	$([0.4, 0.3, 0.1, 0.3]; [0.8, 0.6]; [0.4, 0.6])$

Decision matrix of expert-3

$$
Z^{(3)} = \left[\begin{array}{|c|l|}
\hline
 & C_1 \\ \hline
B_1 & ([0.4, 0.5, 0.6, 0.5]\,;[0.4, 0.8], [0.6, 0.4]) \\ \hline
B_2 & ([0.9, 0.6, 0.3, 0.4]\,;[0.4, 0.8], [0.6, 0.4]) \\ \hline
B_3 & ([0.4, 0.5, 0.1, 0.2]\,;[0.4, 0.8], [0.8, 0.6]) \\ \hline
B_4 & ([0.3, 0.4, 0.3, 0.5]\,;[0.4, 0.7], [0.8, 0.3]) \\ \hline
 & C_2 \\ \hline
B_1 & ([0.3, 0.4, 0.5, 0.4]\,;[0.3, 0.7], [0.7, 0.6]) \\ \hline
B_2 & ([0.4, 0.5, 0.4, 0.1]\,;[0.3, 0.7], [0.8, 0.4]) \\ \hline
B_3 & ([0.4, 0.5, 0.4, 0.1]\,;[0.3, 0.7], [0.8, 0.4]) \\ \hline
B_4 & ([0.8, 0.4, 0.3, 0.4]\,;[0.4, 0.8], [0.7, 0.5]) \\ \hline
 & C_1 \\ \hline
B_1 & ([0.6, 0.2, 0.3, 0.4]\,;[0.5, 0.6], [0.6, 0.5]) \\ \hline
B_2 & ([0.4, 0.5, 0.6, 0.7]\,;[0.5, 0.8], [0.8, 0.4]) \\ \hline
B_3 & ([0.4, 0.6, 0.2, 0.3]\,;[0.5, 0.7], [0.8, 0.4]) \\ \hline
B_4 & ([0.4, 0.6, 0.3, 0.4]\,;[0.4, 0.8], [0.8, 0.5]) \\ \hline
 & C_2 \\ \hline
B_1 & ([0.4, 0.5, 0.6, 0.9]\,;[0.2, 0.8], [0.7, 0.4]) \\ \hline
B_2 & ([0.3, 0.4, 0.5, 0.9]\,;[0.5, 0.6], [0.8, 0.5]) \\ \hline
B_3 & ([0.6, 0.7, 0.8, 0.6]\,;[0.4, 0.9], [0.8, 0.3]) \\ \hline
B_4 & ([0.4, 0.5, 0.6, 0.5]\,;[0.6, 0.5], [0.7, 0.8]) \\ \hline
\end{array} \right]
$$

Decision matrix of expert-4

$$
Z^{(4)} = \left[\begin{array}{|c|l|}
\hline
 & C_1 \\ \hline
B_1 & ([0.9, 0.3, 0.1, 0.2]\,;[0.5, 0.6], [0.8, 0.4]) \\ \hline
B_2 & ([0.4, 0.2, 0.2, 0.5]\,;[0.3, 0.8], [0.8, 0.4]) \\ \hline
B_3 & ([0.3, 0.4, 0.5, 0.7]\,;[0.4, 0.7], [0.7, 0.4]) \\ \hline
B_4 & ([0.4, 0.9, 0.6, 0.3]\,;[0.5, 0.5], [0.6, 0.7]) \\ \hline
 & C_2 \\ \hline
B_1 & ([0.3, 0.4, 0.5, 0.6]\,;[0.5, 0.8], [0.7, 0.4]) \\ \hline
B_2 & ([0.4, 0.5, 0.6, 0.4]\,;[0.6, 0.5], [0.8, 0.8]) \\ \hline
B_3 & ([0.4, 0.6, 0.3, 0.5]\,;[0.7, 0.8], [0.6, 0.5]) \\ \hline
B_4 & ([0.6, 0.3, 0.1, 0.4]\,;[0.5, 0.6], [0.7, 0.5]) \\ \hline
 & C_1 \\ \hline
B_1 & ([0.1, 0.4, 0.5, 0.3]\,;[0.4, 0.8], [0.8, 0.5]) \\ \hline
B_2 & ([0.4, 0.7, 0.8, 0.9]\,;[0.4, 0.7], [0.5, 0.6]) \\ \hline
B_3 & ([0.6, 0.8, 0.3, 0.2]\,;[0.3, 0.8], [0.8, 0.4]) \\ \hline
B_4 & ([0.3, 0.4, 0.7, 0.2]\,;[0.7, 0.5], [0.5, 0.8]) \\ \hline
 & C_2 \\ \hline
B_1 & ([0.6, 0.2, 0.5, 0.1]\,;[0.5, 0.7], [0.7, 0.6]) \\ \hline
B_2 & ([0.4, 0.6, 0.3, 0.2]\,;[0.8, 0.3], [0.2, 0.9]) \\ \hline
B_3 & ([0.5, 0.8, 0.7, 0.3]\,;[0.4, 0.7], [0.8, 0.3]) \\ \hline
B_4 & ([0.3, 0.5, 0.6, 0.9]\,;[0.4, 0.7], [0.8, 0.5]) \\ \hline
\end{array} \right]
$$

Step 2. In this step, we apply the decision information given in the interval-valued Pythagorean trapezoidal fuzzy decision matrix, $Z^{(k)}$ $(i = k = 1, 2, 3, 4)$ and the $IVPTFWA$ operator to find the individual overall preference $IVPTF$ values $\tilde{z}_i^k$ of the alternative B_i

$\tilde{z}_1^{(1)} =$	$([0.43, 0.44, 0.46, 0.36]\,; [0.59, 0.57], [0.44, 0.46])$
$\tilde{z}_1^{(2)} =$	$([0.37, 0.31, 0.46, 0.29]\,; [0.43, 0.71], [0.62, 0.51])$
$\tilde{z}_1^{(3)} =$	$([0.38, 0.25, 0.46, 0.39]\,; [0.49, 0.72], [0.59, 0.58])$
$\tilde{z}_1^{(4)} =$	$([0.96, 0.44, 0.40, 0.25]\,; [0.36, 0.76], [0.61, 0.48])$

$\tilde{z}_2^{(1)} =$	$([0.47, 0.37, 0.57, 0.55]\,; [0.40, 0.78], [0.69, 0.46])$
$\tilde{z}_2^{(2)} =$	$([0.33, 0.39, 0.38, 0.45]\,; [0.69, 0.64], [0.54, 0.48])$
$\tilde{z}_2^{(3)} =$	$([0.43, 0.44, 0.43, 0.54]\,; [0.49, 0.69], [0.59, 0.57])$
$\tilde{z}_2^{(4)} =$	$([0.37, 0.49, 0.40, 0.54]\,; [0.73, 0.67], [0.52, 0.60])$

$\tilde{z}_3^{(1)} =$	$([0.41, 0.41, 0.51, 0.49]\,; [0.39, 0.74], [0.63, 0.47])$
$\tilde{z}_3^{(2)} =$	$([0.62, 0.53, 0.41, 0.42]\,; [0.41, 0.76], [0.71, 0.40])$
$\tilde{z}_3^{(3)} =$	$([0.51, 0.57, 0.25, 0.26]\,; [0.61, 0.72], [0.69, 0.56])$
$\tilde{z}_3^{(4)} =$	$([0.48, 0.45, 0.33, 0.45]\,; [0.44, 0.75], [0.75, 0.42])$

$\tilde{z}_4^{(1)} =$	$([0.53, 0.34, 0.34, 0.33]\,; [0.49, 0.73], [0.75, 0.43])$
$\tilde{z}_4^{(2)} =$	$([0.40, 0.43, 0.45, 0.52]\,; [0.52, 0.71], [0.79, 0.57])$
$\tilde{z}_4^{(3)} =$	$([0.41, 0.57, 0.42, 0.50]\,; [0.52, 0.78], [0.69, 0.41])$
$\tilde{z}_4^{(4)} =$	$([0.43, 0.58, 0.47, 0.37]\,; [0.55, 0.62], [0.63, 0.62])$

Step 3. We utilize the known weight vector by using operational law 3 and Definition 2.8. We find the score function to ordered the overall preference interval-valued Pythagorean trapezoidal fuzzy values such that,

$\tilde{z}_1^{(1)} =$	$([0.52, 0.35, 0.64, 0.54]\,; [0.67, 0.50], [0.66, 0.35])$
$\tilde{z}_1^{(2)} =$	$([0.44, 0.37, 0.55, 0.34]\,; [0.46, 0.14], [0.75, 0.44])$
$\tilde{z}_1^{(3)} =$	$([0.31, 0.35, 0.36, 0.28]\,; [0.53, 0.51], [0.51, 0.71])$
$\tilde{z}_1^{(4)} =$	$([0.57, 0.26, 0.24, 0.15]\,; [0.28, 0.74], [0.63, 0.64])$

$\tilde{z}_2^{(1)} =$	$([0.37, 0.29, 0.45, 0.44]\,; [0.36, 0.74], [0.72, 0.53])$
$\tilde{z}_2^{(2)} =$	$([0.51, 0.68, 0.56, 0.75]\,; [0.80, 0.40], [0.75, 0.48])$
$\tilde{z}_2^{(3)} =$	$([0.56, 0.44, 0.68, 0.66]\,; [0.43, 0.64], [0.82, 0.39])$
$\tilde{z}_2^{(4)} =$	$([0.60, 0.61, 0.60, 0.75]\,; [0.56, 0.47], [0.67, 0.45])$

$\tilde{z}_3^{(1)} =$	$([0.28, 0.27, 0.19, 0.27]\,; [0.33, 0.84], [0.62, 0.19])$
$\tilde{z}_3^{(2)} =$	$([0.71, 0.79, 0.35, 0.36]\,; [0.69, 0.59], [0.77, 0.44])$
$\tilde{z}_3^{(3)} =$	$([0.74, 0.63, 0.49, 0.50]\,; [0.44, 0.66], [0.75, 0.33])$
$\tilde{z}_3^{(4)} =$	$([0.32, 0.32, 0.40, 0.39]\,; [0.35, 0.69], [0.68, 0.54])$

$\tilde{z}_4^{(1)} =$	$([0.42, 0.27, 0.27, 0.26] ; [0.44, 0.79], [0.67, 0.50])$
$\tilde{z}_4^{(2)} =$	$([0.48, 0.56, 0.54, 0.62] ; [0.56, 0.75], [0.75, 0.50])$
$\tilde{z}_4^{(3)} =$	$([0.25, 0.34, 0.28, 0.22] ; [0.44, 0.75], [0.50, 0.75])$
$\tilde{z}_4^{(4)} =$	$([0.57, 0.79, 0.58, 0.07] ; [0.59, 0.59], [0.85, 0.28])$

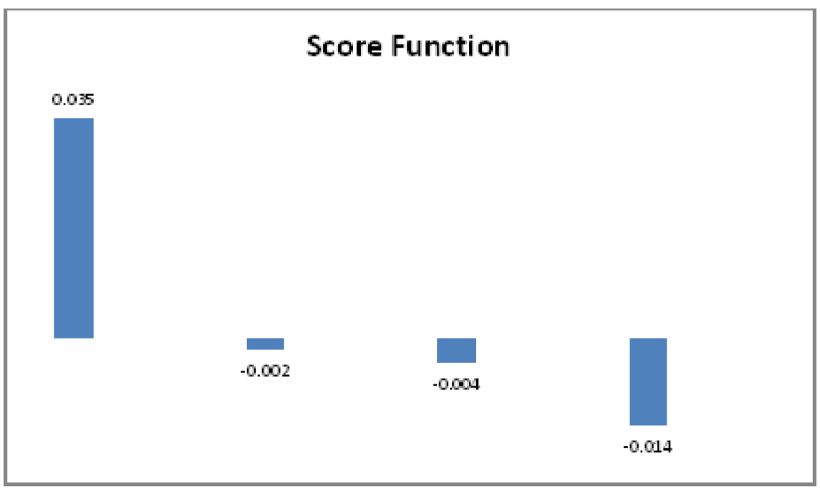
Step 4. We utilize $IVPTFHA$ operator to derive the collective overall preference interval-valued Pythagorean trapezoidal fuzzy values $\tilde{z}_i$. Suppose that, $(\hbar = 0.20, 0.30, 0.35, 0.15)$ and $\Gamma = (0.155, 0.345, 0.345, 0.155)$.

$\tilde{z}_1 =$	$([0.36, 0.28, 0.36, 0.38] ; [0.44, 0.75], [0.66, 0.34])$
$\tilde{z}_2 =$	$([0.56, 0.65, 0.48, 0.53] ; [0.70, 0.54], [0.75, 0.46])$
$\tilde{z}_3 =$	$([0.53, 0.47, 0.57, 0.47] ; [0.46, 0.65], [0.68, 0.39])$
$\tilde{z}_4 =$	$([0.49, 0.48, 0.47, 0.52] ; [0.49, 0.63], [0.69, 0.47])$

Step 5. In this step, we calculate the score function $s(\tilde{z}_i)$ of the collective overall preference values B_i. If there is no difference between two or more than two scores function then we have must to find out the accuracy degrees of the collective overall preference values.

$$s(\tilde{z}_1) = 0.035, s(\tilde{z}_2) = -0.002, s(\tilde{z}_3) = -0.004, s(\tilde{z}_4) = -0.014.$$

Figure 2.



Step 6**.** Now we arrange the scores of all alternatives in the form of descending order and select that alternative which has the highest score, function. Since $B_1 \geq B_4 \geq B_2 \geq B_1$. Thus the most wanted alternative is B_1.

Step 7**.** End.

6. Conclusions

In this paper we introduced the idea of interval-valued Pythagorean trapezoidal fuzzy weighted averaging $(IVPTFWA)$ operator, interval-valued Pythagorean trapezoidal fuzzy ordered weighted averaging $(IVPTFOWA)$ operator and interval-valued Pythagorean trapezoidal fuzzy hybrid averaging $(IVPTFHA)$ operator. We have defined some appropriate properties such as monotonicity, idempotency and bounded of $IVPTFOWA$ and $IVPTFHA$ operators. We also developed $IVPTFHA$ operator, which is a generalization of the $IVPTFWA$ and the $IVPTFOWA$ operators. At the end of this paper we have constructed numerical example of $IVPTFWA$ and $IVPTFHA$ operators to multiple attribute group decision making problems with interval-valued Pythagorean trapezoidal fuzzy information. In future we can extend this work.

REFERENCES

[1] K. Atanassov, Intuitionstic fuzzy sets, Fuzzy Sets Syst. **20**, (1986) 87–96.
[2] K. Atanassov, G. Pasi, R. R. Yager, Intuitionistic fuzzy interpretations of multi-criteria multi-person and multi-measurement tool decision making, Int. J. Syst. Sci. **36**, (2005) 859–868.
[3] K. Atanassov, Remarks on the intuitionistic fuzzy sets— III, Fuzzy Sets Syst. (1995) 401–402.
[4] K. Atanassov, An equality between intuitionistic fuzzy sets, Fuzzy Sets Syst. **79**, (1996) 257–258.
[5] M. Alamgir Khan and Sumitra, Common fixed point Theorems for converse commuting and OWC maps in fuzzy metric spaces, Punjab Univ. J. Math. **44**, (2016) 57-63.
[6] S. E. Abbas, M. A. Hebeshi and I. M. Taha, On upper and lower contra-continuous fuzzy multifunctions, Punjab Univ. J. Math. **47**, (2017) 105-117.
[7] M. Akram and G. Shahzadi, Certain caracterization of m-polar fuzzy graphs by level graphs, Punjab Univ. J. Math. **49**, (2017) 1-12.
[8] C. Cornelis, G. Deschrijver and E. E. Kerre, Implication in intuitionistic fuzzy and interval-valued fuzzy set theory construction, classification, application, Int. J. Approx. Reason. **35**, (2004) 55–95.
[9] Z. P. Chen, W. Yang, A new multiple attribute group decision making method in intuitionistic fuzzy setting, Appl. Math. Model. **35**, (2011) 4424–4437.
[10] G. Deschrijver, E. E. Kerre, A generalization of operators on intuitionistic fuzzy sets, IEE Trans Fuzzy Syst. **8**, (2002) 19-27.
[11] D. Dalalah et al, M. Hayajneh and F. Batieha, A fuzzy multi-criteria decision making model for supplier selection, Expert Systems with Applications. **38**, (2011) 8384–8391.
[12] D. Dalalah et al, A mutually dependent multi-criteria decision making, Fuzzy Information and Engineering. **2**, (2012) 195-216.
[13] F. Herrera, E. Herrera-Viedma and F. Chiclana, Multi-person decision-making based on multiplicative preference relations, Eur. J. Oper. Res. **129**, (2001) 372–385.
[14] Y. C. Jiang, Y. Tang and Q. M. Chen, An adjustable approach to intuitionistic fuzzy soft sets based decision making, Appl. Math. Model. **35**, (2011) 824–836.
[15] Ye. Jun, Multicriteria fuzzy decision-making method using entropy weights-based correlation coefficients of interval-valued intuitionistic fuzzy sets, Applied Mathematical Modelling. **34**, (2010) 3864–3870.
[16] D. F. Li, Closeness coeffcient based nonlinear programming method form interval-valued intuitionistic fuzzy multiattribute decision making with incomplete preference information, Appl. Soft Comput. **11**, (2011) 3042-3418.
[17] T. Mahmood, F. Mehmood and Q. Khan, Some generalized aggregation operators for cubic hesitant fuzzy sets and their applications to multi criteria decision making, Punjab Univ. J. Math. **49**, (2017) 31-49.
[18] J. M. Merigó, A. M. Gil-Lafuente, Fuzzy induced generalized aggregation operators and its application in multi-person decision making, Expert Syst. Appl. **38**, (2011) 9761–9772.
[19] J. H. Park, Y. Park, Y. C. Kwun and X. G. Tan, Extension of the topsis method for decision making problems under interval-valued intuitionistic fuzzy environment, Appl. Math. Model. **35**, (2011) 2544–2556.
[20] K. Rahman, Pythagorean fuzzy ordered weighted geometric aggregation operator and their application to multiple attribute group decision making, Journal of Applied Environmental and Biological Sciences. **7**, (2017) 67–83.
[21] M. H. Shu, C. H. Cheng and J. R. Chan, Using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly, Microelectron. Reliab. **46**, (2006) 2139–2148.
[22] M. Shakeel, Induced averaging aggregation operators with interval Pythagorean trapezoidal fuzzy numbers and their application to group decision making, The Nucleus. **2**, (2017) 140-153.

[23] M. Sajjad Ali Khan, K. Rahman, A. Fahm and M. Shakeel, Generalized $(\epsilon, \epsilon \vee qk)$-fuzzy quasi-ideals in semigroups, Punjab Univ. J. Math. **50**, (2018) 35-53.
[24] J. Q. Wang, Overview on fuzzy multi-criteria decision making approach, Control Decision. **23**, (2008) 601–606.
[25] J. Q. Wang and Z. Zhang, Multi-criteria decision-making method with incomplete certain information based on intuitionistic fuzzy number, Control Decision. **24**, (2009) 226–230.
[26] G. W. Wei, Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making, Appl. Soft Comput. **10**, (2010) 423–431.
[27] G. W. Wei, Some arithmetic aggregation operators with intuitionistic trapezoidal fuzzy numbers and their application to group decision making, J. Comput. **5**, (2010) 345–351.
[28] Z. S. Xu, An overview of methods for determining owa weights, Int. J. Intell. Syst. **20**, (2005) 843–865.
[29] Z. S. Xu and R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, International Journal General System. **35**, (2006) 417–433.
[30] Z. S. Xu, intuitionistic preference relations and their application in group decision making, Information Sciences. **177**, (2007) 2363-2379.
[31] Z. S. Xu, Intuitionistic fuzzy aggregation operators, IEEE Trans. Fuzzy Syst. **15**, (2007) 1179–1187.
[32] Z. S. Xu, and J. Chen, An approach to group decision making based on interval-valued intuitionistic judgment matrices, System Engineering-Theory and Practice. **27**, (2007a) 126–133.
[33] Z. S. Xu, and J. Chen, On geometric aggregation over interval-valued intuitionistic fuzzy information, Fourth international conference on fuzzy systems and knowledge discovery. **2**, (2007) 466–471.
[34] Z. S. Xu, A method based on distance measure for interval-valued intuitionistic fuzzy group decision making, Inform. Sci. **180**, (2010) 181–190.
[35] J. Ye, Multicriteria fuzzy decision-making method using entropy weights-based correlation coeffcients of interval-valued intuitionistic fuzzy sets, Appl. Math. Model. **34**, (2010) 3864–3870.
[36] R. R.Yager, Pythagorean fuzzy subsets, In Proc. Joint IFSA World Congress and NAFIPS Annual Meeting ,Edmonton,Canada. (2013) 57-61.
[37] R. R. Yager, Pythagorean membership grades in multicriteria decision making, IEEE Trans fuzzy system. **22**, (2014) 958-965.
[38] L. A. Zadeh, Fuzzy sets, Information. Control. **8**, (1965) 338–353.
[39] H. Y. Zhang, W. X. Zhang and W. Z. Wu, On characterization of generalized interval-valued fuzzy rough sets on two universes of discourse, Int. J. Approx. Reason. **51**, (2009) 56–70.
[40] Xu. Zhang and Z. S. Xu, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, International Journal of Intelligent Systems. **29**, (2014) 1061-1078.