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Generalized $(\in, \in \vee q_k)$-Fuzzy Quasi-Ideals in Semigroups

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Abstract. In this article, we introduce the concept of $(\in, \in \vee q_k^\delta)$-fuzzy (generalized) bi-ideal, $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal, $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal in semigroups. We show that each $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal and each $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal but the converses are not true in general.

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1. INTRODUCTION

The concept of fuzzy set was first introduced by Zadeh in his seminal paper [30]. This inspirational paper has opened up new awareness and application in a wide range of scientific fields. After the Zadeh's fuzzy sets many researchers convoyed the researches on the generalizations of the of fuzzy sets ideas with huge applications in computer science, artificial intelligence, control engineering, expert, robotics, automat theory, finite state machine, graph theory, logics and many branches of pure and applied mathematics. The characterization of *m*-polar fuzzy graphs by level graphs has been studied by Akram and Shahzadi in [3]. In [17], Khan and Sumitra studied some common fixed point theorems for converse commuting and occasionally weakly compatible (owc) maps using implicit relations in fuzzy metric spaces. Rao and Parvathi studied general common fixed point theorems for two maps in fuzzy metric spaces [24]. In [1], Abbas et al. introduced the concepts of fuzzy upper and fuzzy lower contra-continuous, contra-irresolute and contra semi-continuous multifunctions. Rosenfeld was the first who used the notion of a fuzzy subset and introduced the concept of fuzzy groups [25]. Ahmad et al. [2], studied some new subclasses of AG-groupoids as rectangular AG-groupoid and produced a variety of examples and counter examples using the latest computational techniques of GAP, Mace4 and Prover9. Syed in [29], studied Unitary Irreducible Representation (UIR) Matrix Elements of Finite Rotations of SO(2; 1) Decomposed According to the Subgroup T_1. The theory of fuzzy semigroups was first initiated by Kuroki [18]. In [19], Kuroki initiated the notion fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals of a semigroup. The book by Mordeson et al. [21] deals with the theory of fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. Fuzziness offers a natural place to the field of formal languages. In [20], Mordeson and Malik deal with the application of the fuzzy approach to the automata and formal languages. Pu and Lia [23] introduced the notion of "belongs to" relation ($\in$). In [22], Murali initiated the notion of belongingness (q) of a fuzzy point to a fuzzy subset under an expected equality on a fuzzy subset. These two notions played a vital role in generating some different types of fuzzy subgroups. In [6, 7], Bhakat and Das provided the notion of (α, β)-fuzzy subgroups. The concept of (α, β)-fuzzy subgroups are further studied in [4, 5]. By using the "belong to" ($\in$) relation and "quasi coincident" with (q) relation between a fuzzy point and a fuzzy subgroup and introduced the concept of $(\in, \in \vee q)$-fuzzy subgroup. The concept of $(\in, \in \vee q)$-fuzzy subgroups is a possible generalization of Rosenfeld's fuzzy subgroups. The idea of $(\in, \in \vee q)$-fuzzy subrings/ideals are introduced in [8]. In [9], Davvaz introduced the concept of $(\in, \in \vee q)$-fuzzy sub near-rings/ideals in a near ring. Jun and Song in [11] proposed the study of (α, β)-fuzzy interior ideals in a semigroup. By using the concept of $(\in, \in \vee q)$-fuzzy ideals, Shabir et al. characterized regular semigroup [26]. In [15], Kazanci and Yamak considered $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup and prove some motivated results. Generalizing the idea of the quasi-coincident of a fuzzy point with a fuzzy subset, Jun introduced $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras respectively [12]. In [26], Shabir et al, introduced the concept of an $(\in, \in \vee q)$-fuzzy quasi-ideal of a semigroup. In [13] Jun initiated the concept of general types of quasi-coincident in BCK-algebra. The concept of $(\in, \in \vee q_k)$-fuzzy subsemigroup was initiated by Kang in [14]. In [27], Shabir et al, generalized the concept of an $(\in, \in \vee q)$-fuzzy bi(interior, quasi)-ideal of a semigroup and introduced the notion of an $(\in, \in \vee q_k)$-fuzzy bi(interior ideal, quasi)-ideal in a semigroup. In [28], the authors characterized regular semigroups by the properties of $(\in, \in \vee q_k)$-fuzzy ideals. In [10, 16], the authors characterized regular semigroup in term of $(\in, \in \vee q_k)$-fuzzy ideals. The concept of an $(\in, \in \vee q_k^\delta)$-fuzzy (generalized) bi-ideal/quasi-ideal of a semigroup is a generalization of the concepts studied in [11, 15, 26, 27]. If we take $\delta = 1$, then we get, an $(\in, \in \vee q_k)$-fuzzy bi-ideal and an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a semigroup. If we take $\delta = 1$ and $k = 0$, then we get, an $(\in, \in \vee q)$-fuzzy bi-ideal and an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a semigroup. Which means that these fuzzy ideals become a special case of an $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideal/quasi-ideal of a semigroup. Due to the motivation and inspiration of the concept, in this paper, we introduce the concept of $(\in, \in \vee q_k^\delta)$-fuzzy (generalized) bi-ideal, $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal and $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal in semigroups. We show that each $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal each $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal and each $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal but the converses are not true in general.

2. Preliminaries

Throughout this paper S will denote a semigroup unless otherwise specified. Let A be a non-empty subset of S. Then A is said to be a subsemigroup of S if $A^2 \subseteq A$. Let I be a non-empty subset of S. Then I is said to be a left(right) ideal of S if $SI \subseteq I(IS \subseteq I)$. Let I be a non-empty subset of S. Then I is said to be an ideal of S if it is both left and right ideal of S. Let B be a non-empty subset of S. Then B is said to be a generalized bi-ideal of S if $BSB \subseteq B$, B is said to be a bi-ideal of S if it is both a subsemigroup and a generalized bi-ideal of S. A non-empty subset Q of S is said to be a quasi-ideal of S if $QS \cap SQ \subseteq Q$.

A fuzzy subset λ of a universe X is a function from X into the closed interval $[0, 1]$, i.e, $\lambda : X \to [0, 1]$.

Let λ be a fuzzy set of a semigroup S and $t \in [0, 1]$, the set $U(\lambda; t) = \{a \in S | \lambda(a) \geq t\}$ is said to be a level subset of the fuzzy set λ.

Let A be a non-empty subset of S. The characteristic function of A denoted by λ_A and is defined by the mapping from S into $[0, 1]$:

$$\lambda_A(a) = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ if } a \notin A \end{cases}$$

Obviously λ_A is a fuzzy subset of S.

A fuzzy subset λ in a universe X of the form

$$\lambda(a) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a fuzzy point with support a and value t and is denoted by (a, t).

For a fuzzy subset λ in S, a fuzzy point (a, t) is said to

- be contained in λ, denoted by $(a, t) \in \lambda$, if $\lambda(a) \geq t$.
- be quasi-coincident with λ, denoted by $(a, t)\, q\lambda$, if $\lambda(a) + t > 1$.

For a fuzzy subset λ and fuzzy point (a, t) in a set S, we say that

- $(a, t) \in \wedge q\lambda$ if $(a, t) \in \lambda$ or $(a, t)\, q\lambda$.

Generalizing the concept of $(a,t)\,q\lambda$, Jun [12], defined $(a,t)\,q_k\lambda$, where $k \in [0,1)$ as $(a,t)\,q_k\lambda$ if $\lambda(a)+t+k>1$ and $(a,t) \in \vee q_k\lambda$ if $(a,t)\in\lambda$ or$(a,t)\,q_k\lambda$. Jun et al in [13], considered the general form of the symbol $(a,t)\,q_k\lambda$ and $(a,t) \in \vee q_k\lambda$ as follows: For a fuzzy point (a,t) and fuzzy subset λ in a set S, we say that

$i)$ $(a,t)\,q^{\delta}\lambda$ if $\lambda(a)+t>\delta$,
$ii)$ $(a,t)\,q_k^{\delta}\lambda$ if $\lambda(a)+t+k>\delta$,
$iii)$ $(a,t)\in\vee q_k^{\delta}\lambda$ if $(a,t)\in\lambda$ or $(a,t)\,q_k^{\delta}\lambda$;
$iv)$ $(a,t)\,\overline{\alpha}\lambda$ if $(a,t)\,\alpha\lambda$ does not hold, for $\alpha\in\{\in, q, \in\vee q, \in\vee q_k, q_k^{\delta}, \in\vee q_k^{\delta}\}$.
where $k\in[0,1)$ and $k<\delta$ in $[0,1]$. Obviously, $(a,t)\,q^{\delta}\lambda$ implies $(a,t)\,q_k^{\delta}\lambda$.

Definition 2.1. [14] A fuzzy subset λ in S is said to be an $(\in, \in\vee q_k^{\delta})$-fuzzy subsemigroup of S if for all $t_1,t_2\in(0,1]$ and $a,b\in S$,

$$(a,t_1)\in\lambda, (b,t_2)\in\lambda \Rightarrow (ab, t_1\wedge t_2)\in\vee q_k^{\delta}\lambda$$

Lemma 2.2. [14] *Suppose A is a subsemigroup of S and λ a fuzzy subset of S defined by*

$$\lambda(a)=\begin{cases}\varepsilon \text{ if } a\in A\\ 0 \text{ if } a\notin A\end{cases}$$

where $\varepsilon\geq\frac{1-k}{2}$. then,
$i)$ *λ is a $(q^{\delta}, \in\vee q_k^{\delta})$-fuzzy subsemigroup of S.*
$ii)$ *λ is an $(\in, \in\vee q_k^{\delta})$-fuzzy subsemigroup of S.*

Lemma 2.3. [14] *Suppose λ is a fuzzy subset of S, then λ is an $(\in,\in\vee q_k^{\delta})$-fuzzy subsemigroup of S, if and only if $\lambda(ab)\geq\lambda(a)\wedge\lambda(b)\wedge\frac{\delta-k}{2}$ for all $a,b\in S$ and $k\in[0,1)$.*

Lemma 2.4. [14] *Suppose λ is a fuzzy subset of S, then λ is an $(\in,\in\vee q_k^{\delta})$-fuzzy subsemigroup of S if and only if $U(\lambda;t)(\neq 0)$ is a subsemigroup of S for all $t\in(0,\frac{\delta-k}{2}]$.*

Definition 2.5. [27] A fuzzy subset λ in S is said to be an $(\in,\in\vee q_k)$-fuzzy left (right) ideal of S if the following condition holds:

$$(b,t)\in\lambda\Rightarrow(ab,t)\in\vee q_k\lambda \quad ((b,t)\in\lambda\Rightarrow(ba,t)\in\vee q_k\lambda)$$

Definition 2.6. [27] A fuzzy subset λ in S is said to be an $(\in,\in\vee q_k)$-fuzzy ideal of S if it is both an $(\in,\in\vee q_k)$-fuzzy left ideal and $(\in,\in\vee q_k)$-fuzzy right ideal of S.

Lemma 2.7. [27] *Suppose A is a left (right) of S and λ a fuzzy subset of S defined by*

$$\lambda(a)=\begin{cases}\varepsilon \text{ if } a\in A\\ 0 \text{ if } a\notin A\end{cases}$$

where $\varepsilon\geq\frac{1-k}{2}$, then,
$i)$ *λ is a $(q,\in\vee q_k)$-fuzzy left (right) ideal of S.*
$ii)$ *λ is an $(\in,\in\vee q_k)$-fuzzy left (right) ideal of S.*

Lemma 2.8. [27] *Suppose λ is a fuzzy subset of S, then λ is an $(\in,\in\vee q_k)$-fuzzy left (right) ideal of S if and only if $\lambda(ab)\geq\lambda(a)\wedge\frac{1-k}{2}$ $\left(\lambda(ab)\geq\lambda(b)\wedge\frac{1-k}{2}\right)$ for all $a,b\in S$ and $k\in[0,1)$.*

Lemma 2.9. [27] *Suppose λ is a fuzzy subset of S, then λ is an $(\in,\in\vee q_k)$-fuzzy left (right) ideal of S if and only if $U(\lambda;r)(\neq 0)$ is a left (right) ideal of S for all $t\in(0,\frac{1-k}{2}]$.*

3. $(\in, \in \vee q_k^{\delta})$-FUZZY BI-IDEAL

In this section we introduce the concept of $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideal in semigroups and prove some of its related properties.

Definition 3.1. A fuzzy subset λ in S is said to be an $(\in, \in \vee q_k^{\delta})$-fuzzy left (right) ideal of S if the following condition holds;

$$(b,t) \in \lambda \Rightarrow (ab,t) \in \vee q_k^{\delta}\lambda \ ((b,t) \in \lambda \Rightarrow (ba,t) \in \vee q_k^{\delta}\lambda)$$

Definition 3.2. A fuzzy subset λ in S is said to be an $(\in, \in \vee q_k^{\delta})$-fuzzy ideal of S if it is both an $(\in, \in \vee q_k^{\delta})$-fuzzy left ideal and an $(\in, \in \vee q_k^{\delta})$-fuzzy right ideal of S.

Definition 3.3. A fuzzy subset λ in S is said to be an $(\in, \in \vee q_k^{\delta})$-fuzzy generalized bi-ideal of S if for all $t_1, t_2 \in (0,1]$ and $a, b, c \in S$,

$$(a, t_1) \in \lambda, \ (c, t_2) \in \lambda \Rightarrow (abc, t_1 \wedge t_2) \in \vee q_k^{\delta}\lambda$$

Example 3.4. Let $S = \{1,2,3,4,5\}$ and "$\cdot$" be a binary operation defined on S in the following table:

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	3	1
5	1	1	1	2	3

Then, $(S, \cdot)$ is a semigroup. One can easily show that $\{1,5\}$,$\{1,4,5\}$, $\{1,3,4\}$ $\{2,4,5\}$ are generalized bi-ideals of S. Define $\lambda : S \to [0,1]$ by $\lambda(1) = 0.2$, $\lambda(2) = 0.8$, $\lambda(3) = \lambda(5) = 0$ and $\lambda(4) = 0.3$. Let $k = 0.3$ and $\delta = 0.4$. Then, λ is an $(\in, \in \vee q_{0.3}^{0.4})$-fuzzy generalized bi-ideal of S. But:

(1) : λ is not an $(\in, \in \vee q_{0.3})$-fuzzy generalized bi-ideal of S. Because, $(4, t_1) = (4, 0.1) \in \lambda$ and $(4, t_2) = (4, 0.2) \in \lambda$ but $(4 \cdot 4 \cdot 4, t_1 \wedge t_2) = (1, 0.1) \overline{\in \vee q_{0.2}}\lambda$, i.e.

$$\begin{aligned} \lambda(4 \cdot 4 \cdot 4) + t_1 \wedge t_2 + k &= \lambda(1) + 0.1 \wedge 0.2 + 0.3 \\ &= 0.2 + 0.1 + 0.3 \\ &= 0.6 \ngtr 1 \end{aligned}$$

(2) : λ is not an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. Because, $(4, t_1) = (4, 0.1) \in \lambda$ and $(4, t_2) = (4, 0.2) \in \lambda$ but $(4 \cdot 4 \cdot 4, t_1 \wedge t_2) = (1, 0.1) \overline{\in \vee q}\lambda$, i.e.

$$\begin{aligned} \lambda(4 \cdot 4 \cdot 4) + t_1 \wedge t_2 + 0.5 &= \lambda(1) + 0.1 \wedge 0.2 + 0.5 \\ &= 0.2 + 0.1 + 0.5 \\ &= 0.8 \ngtr 1 \end{aligned}$$

Definition 3.5. A fuzzy subset λ of S is said to be an $(\in, \in \vee q_k^{\delta})$-fuzzy bi-ideal of S if it is both $(\in, \in \vee q_k^{\delta})$-fuzzy subsemigroup and $(\in, \in \vee q_k^{\delta})$-fuzzy generalized bi-ideal of S.

Theorem 3.6. *Suppose λ is a fuzzy subset of S. Then λ is an $(\in, \in \vee q_k^{\delta})$-fuzzy bi-ideal of S, if and only if*

(i) $\lambda(ab) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta - k}{2}$

(ii) $\lambda(abc) \geq \lambda(a) \wedge \lambda(c) \wedge \frac{\delta-k}{2}$
for all $a, b, c \in S$ *where* $k \in [0,1)$ *and* $k < \delta$ *in* $[0,1]$.

Proof. Let λ be an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. Assume that $\lambda(ab) < \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2}\}$ for some $a, b \in S$. If $\lambda(a) \wedge \lambda(b) \geq \frac{\delta-k}{2}$, then $\lambda(ab) < \frac{\delta-k}{2}$. Hence $\left(a, \frac{\delta-k}{2}\right) \in \lambda$ and $\left(b, \frac{\delta-k}{2}\right) \in \lambda$, but $\left(ab, \frac{\delta-k}{2}\right) \overline{\in} \lambda$. Moreover, $\lambda(ab) + \frac{\delta-k}{2} < \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, and so $\left(ab, \frac{\delta-k}{2}\right) \overline{q_k^\delta} \lambda$. Thus, $\left(ab, \frac{\delta-k}{2}\right) \overline{\in \vee q_k^\delta} \lambda$. This is a contradiction. If $\lambda(a) \wedge \lambda(b) < \frac{\delta-k}{2}$, then $\lambda(ab) < \lambda(a) \wedge \lambda(b)$. Hence there exists $t \in (0,1]$ such that $\lambda(ab) < t \leq \lambda(a) \wedge \lambda(b)$. It follows that $(a,t) \in \lambda$ and $(b,t) \in \lambda$, but $(ab,t) \overline{\in} \lambda$. Also, $\lambda(ab) + t < \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, i.e., $(ab,t) \overline{q_k^\delta} \lambda$. Thus $(ab,t) \overline{\in \vee q_k^\delta} \lambda$, a contradiction. Hence, (i) is valid.

Now suppose that $\lambda(axb) < \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2}$ for some $a, x, b \in S$. So we have the following two cases:

$Case\,(a)$. $\lambda(a) \wedge \lambda(b) \geq \frac{\delta-k}{2}$, $Case\,(b)$. $\lambda(a) \wedge \lambda(b) < \frac{\delta-k}{2}$.

$Case\,(a)$ implies that $\left(a, \frac{\delta-k}{2}\right) \in \lambda$, $\left(b, \frac{\delta-k}{2}\right) \in \lambda \Rightarrow \lambda(axb) < \frac{\delta-k}{2}$ that is $\left(axb, \frac{\delta-k}{2}\right) \overline{\in} \lambda$, and $\lambda(axb) + \frac{\delta-k}{2} < \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$ it implies that $\left(axb, \frac{\delta-k}{2}\right) \overline{q_k^\delta} \lambda$. Hence $\left(ab, \frac{\delta-k}{2}\right) \overline{\in \vee q_k^\delta} \lambda$, a contradiction.

For $Case\,(b)$ we have $\lambda(axb) < \lambda(a) \wedge \lambda(b)$ and so $\lambda(axb) < t \leq \lambda(x) \wedge \lambda(y)$ for some $t \in (0,1]$. Thus $(a,t) \in \lambda$ and $(b,t) \in \lambda$ but $(axb,t) \overline{\in} \lambda$. Also $\lambda(axb) + t < t + t < \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, that is $(axb,t)\, \overline{q_k^\delta} \lambda$. Hence $(axb,t) \overline{\in \vee q_k^\delta} \lambda$, a contradiction. Therefore (ii) is valid.

Conversely, suppose that λ satisfies conditions (i) and (ii). Let $a, b \in S$ and $t_1, t_2 \in (0,1]$ be such that $(a,t_1) \in \lambda$ and $(b,t_2) \in \lambda$. Then $\lambda(a) \geq t_1$ and $\lambda(b) \geq t_2$. Using (i), we have $\lambda(ab) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2} \geq t_1 \wedge t_2 \wedge \frac{\delta-k}{2}$. If $t_1 \wedge t_2 > \frac{\delta-k}{2}$, then $\lambda(ab) \geq \frac{\delta-k}{2}$ and so $\lambda(ab) + t_1 \wedge t_2 > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$ that is $(ab, t_1, t_2) q_k^\delta \lambda$. If $t_1 \wedge t_2 \leq \frac{\delta-k}{2}$ then $\lambda(ab) \geq \frac{\delta-k}{2}$, that is $(ab, t_1, t_2) \in \lambda$. Hence $(ab, t_1, t_2) \in \vee q_k^\delta \lambda$. Let $a, x, b \in S$ and $t_1, t_2 \in (0,1]$ be such that $(a,t_1) \in \lambda$ and $(b,t_2) \in \lambda$. Then $\lambda(a) \geq t_1$ and $\lambda(b) \geq t_2$. It follows from (ii) that $\lambda(axb) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2} \geq t_1 \wedge t_2 \wedge \frac{\delta-k}{2}$. If $t_1 \wedge t_2 > \frac{\delta-k}{2}$, then $\lambda(axb) \geq \frac{\delta-k}{2}$ and so $\lambda(axb) + t_1 \wedge t_2 > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, i.e., $(axb, t_1 \wedge t_2) q_k^\delta \lambda$. If $t_1 \wedge t_2 \leq \frac{\delta-k}{2}$, then $\lambda(axb) \geq t_1 \wedge t_2$, that is, $(axb, t_1 \wedge t_2) \in \lambda$. Thus $(axb, t_1 \wedge t_2) \in \vee q_k^\delta \lambda$. Therefore λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. □

Example 3.7. Consider a semigroup defined in Example 3.4. If we define a fuzzy subset λ on S by $\lambda(1) = 0.2$, $\lambda(2) = 0.8$, $\lambda(3) = 0.3$, $\lambda(4) = 0.4$ and $\lambda(5) = 0.5$, and take $k = 0.4$ and $\delta = 0.5$. Then, λ is an $(\in, \in \vee q_{0.4}^{0.5})$-fuzzy bi-ideal of S.

Also, λ is not a fuzzy, an $(\in, \in \vee q)$ and $(\in, \in \vee q_{0.4})$-fuzzy bi-ideal of S. Because,

$$\lambda(3 \cdot 3 \cdot 3) = \lambda(1) = 0.2 \ngeq 0.3 = \lambda(3) \wedge \lambda(3),$$

$$\lambda(3 \cdot 3 \cdot 3) = \lambda(1) = 0.2 \ngeq 0.5 = \lambda(3) \wedge \lambda(3) \wedge 0.5,$$

$$\lambda(3 \cdot 3 \cdot 3) = \lambda(1) = 0.2 \ngeq 0.3 = \lambda(3) \wedge \lambda(3) \wedge 0.3$$

respectively.

In 3.4 λ is not an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. Because, $\lambda(4.4) = \lambda(3) = 0.0 \ngeq 0.4 = \lambda(4) \wedge \lambda(4)$. This means that an $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of S is not an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S.

Theorem 3.8. *Let λ be a fuzzy subset of S. Then λ is an $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of S if and only if $U(\lambda; t)(\neq \emptyset)$ is generalized bi-ideal of S for all $t \in (0, \frac{\delta-k}{2}]$.*

Lemma 3.9. *If $\{\lambda_i\}_{i\in\Lambda}$ is a family of $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideals of S, then $\bigcap\limits_{i\in\Lambda} \lambda_i$ is an $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of S.*

Proof. Suppose $\{\lambda_i\}_{i\in\Lambda}$ is a family of $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideals of S and $a, b, c \in S$, then, $\bigcap\limits_{i\in\Lambda} \lambda_i (abc) = \bigcap\limits_{i\in\Lambda} (\lambda_i (abc))$. (Since each λ_i is an $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of S, so from Theorem 3.6, we have $\lambda_i (abc) \geq \lambda_i (a) \wedge \lambda_i (c) \wedge \frac{\delta-k}{2}$.) Thus,

$$\begin{aligned}
\left(\bigcap_{i\in\Lambda} \lambda_i\right)(abc) &= \bigcap_{i\in\Lambda}(\lambda_i (abc)) \\
&\geq \bigcap_{i\in\Lambda}\left(\lambda_i (a) \wedge \lambda_i (c) \wedge \frac{\delta-k}{2}\right) \\
&= \left(\bigcap_{i\in\Lambda}(\lambda_i (a) \wedge \lambda_i (c))\right) \wedge \frac{\delta-k}{2} \\
&= \left(\bigcap_{i\in\Lambda} \lambda_i\right)(a) \wedge \left(\bigcap_{i\in\Lambda} \lambda_i\right)(c) \wedge \frac{\delta-k}{2}
\end{aligned}$$

Hence $\bigcap\limits_{i\in\Lambda} \lambda_i$ is an $(\in, \in \vee q_k^\delta)$-fuzzy generalized bi-ideal of S. □

Theorem 3.10. *Suppose λ is a fuzzy subset of S, then λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S if and only if $U(\lambda; t)(\neq \emptyset)$ is bi-ideal of S for all $t \in (0, \frac{\delta-k}{2}]$.*

Proof. Proof of the Theorem follows from Lemma 2.4 and Theorem 3.8 □

Lemma 3.11. *If $\{\lambda_i\}_{i\in\Lambda}$ is a family of $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S, then $\bigcap\limits_{i\in\Lambda} \lambda_i$ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S.*

Remark 3.12. Suppose $\{\lambda_i\}_{i\in I}$ is a family of $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. Is $\bigcup\limits_{i\in I} \lambda_i$ an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S? Where $\bigcup\limits_{i\in I} \lambda_i = \sup_{i\in I} \lambda_i$. For this we have the following example.

Example 3.13. Consider a semigroup $S = \{1, 2, 3, 4\}$ with the following multiplication table:

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	4	1
3	1	1	1	1
4	1	1	1	1

Suppose λ and μ are two fuzzy subsets in S, define by

$\lambda(1) = 0.5$, $\lambda(2) = 0.8$, $\lambda(3) = \lambda(4) = 0.1$ and $\mu(1) = 0.3$, $\mu(2) = \mu(4) = 0.1$, $\mu(3) = 0.4$. If we take $k = 0.2$ and $\delta = 0.7$, Hence for $t \in (0, 0.2]$, we have

$$U(\lambda; t) = \begin{cases} S \text{ if } t \in (0, 0.1] \\ \{1, 2\} \text{ if } t \in (0.1, 0.2] \end{cases}$$

and

$$U(\mu; t) = \begin{cases} S \text{ if } t \in (0, 0.1] \\ \{1, 3\} \text{ if } t \in (0.1, 0.2] \end{cases}$$

Thus λ and μ are $(\in, \in \vee q_{0.2}^{0.7})$-fuzzy bi-ideals of S by Theorem 3.10. The union $\lambda \cup \mu$ of λ and μ is given by $(\lambda \cup \mu)(1) = 0.5$, $(\lambda \cup \mu)(2) = 0.8$, $(\lambda \cup \mu)(3) = 0.4$, and $(\lambda \cup \mu)(4) = 0.1$, Hence

$$U(\lambda \cup \mu, t) = \begin{cases} S \text{ if } t \in (0, 0.1] \\ \{1, 2, 3\} \text{ if } t \in (0.1, 0.2] \end{cases}$$

Since $\{1, 2, 3\}$ does not satisfy the first condition of bi-ideal of S. i.e, Therefore, by Theorem 3.10, $\lambda \cup \mu$ is not an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideals of S.

In the following theorem we show that the union of any family of an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideals of S is again an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S.

Theorem 3.14. *Suppose* $\{\lambda_i\}_{i\in I}$ *is a family of* $(\in, \in \vee q_k^\delta)$*-fuzzy bi-ideal of* S*, such that* $\lambda_i \subseteq \lambda_j$ *or* $\lambda_j \subseteq \lambda_i$ *for all* $i, j \in I$*, then* $\bigcup_{i\in I} \lambda_i$ *is an* $(\in, \in \vee q_k^\delta)$*-fuzzy bi-ideal of* S*. Where* $\bigcup_{i\in I} \lambda_i = \sup_{i\in I} \lambda_i$.

Proof. Suppose $a, b \in S$, since each λ_i is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. So we have,

$$\begin{aligned} \bigcup_{i\in I} \lambda_i(ab) &= \sup_{i\in I} \lambda_i(ab) \\ &\geq \sup_{i\in I} \left\{ \lambda_i(a) \wedge \lambda_i(b) \wedge \frac{\delta - k}{2} \right\} \text{ (By Theorem 3.6)} \\ &= \sup_{i\in I} \left\{ \{\lambda_i(a) \wedge \lambda_i(b)\} \wedge \frac{\delta - k}{2} \right\} \\ &= \left(\bigcup_{i\in I} \lambda_i \right)(a) \wedge \left(\bigcup_{i\in I} \lambda_i \right)(b) \wedge \frac{\delta - k}{2}. \end{aligned}$$

Clearly

$$\sup_{i\in I} \left\{ \lambda_i(a) \wedge \lambda_i(b) \wedge \frac{\delta - k}{2} \right\} \leq \left(\bigcup_{i\in I} \lambda_i \right)(a) \wedge \left(\bigcup_{i\in I} \lambda_i \right)(b) \wedge \frac{\delta - k}{2}.$$

Suppose

$$\sup_{i\in I} \left\{ \lambda_i(a) \wedge \lambda_i(b) \wedge \frac{\delta - k}{2} \right\} \neq \left(\bigcup_{i\in I} \lambda_i \right)(a) \wedge \left(\bigcup_{i\in I} \lambda_i \right)(b) \wedge \frac{\delta - k}{2}.$$

Then there exists t such that

$$\sup_{i\in I}\left\{\lambda_i(a)\wedge\lambda_i(b)\wedge\frac{\delta-k}{2}\right\}<t<\left(\bigcup_{i\in I}\lambda_i\right)(a)\wedge\left(\bigcup_{i\in I}\lambda_i\right)(b)\wedge\frac{\delta-k}{2}.$$

Since $\lambda_i\subseteq\lambda_j$ or $\lambda_j\subseteq\lambda_i$ for all $i,j\in I$, there exists $h\in I$ such that

$$t<\lambda_h(a)\wedge\lambda_h(b)\wedge\frac{\delta-k}{2}.$$

On the other hand

$$\lambda_h(a)\wedge\lambda_h(b)\wedge\frac{\delta-k}{2}<t$$

for all $i\in I$. Which is a contradiction. Hence

$$\sup_{i\in I}\left\{\lambda_i(a)\wedge\lambda_i(b)\wedge\frac{\delta-k}{2}\right\}=\left(\bigcup_{i\in I}\lambda_i\right)(a)\wedge\left(\bigcup_{i\in I}\lambda_i\right)(b)\wedge\frac{\delta-k}{2}.$$

Suppose $a,b,c\in S$, then we have

$$\begin{aligned}\bigcup_{i\in I}\lambda_i(abc) &= \sup_{i\in I}\lambda_i(abc)\\ &\geq \sup_{i\in I}\left\{\lambda_i(a)\wedge\lambda_i(b)\wedge\frac{\delta-k}{2}\right\}\text{(By Theorem 3.6)}\\ &= \sup_{i\in I}\left\{\{\lambda_i(a)\wedge\lambda_i(b)\},\frac{\delta-k}{2}\right\}\\ &= \left(\bigcup_{i\in I}\lambda_i\right)(a)\wedge\left(\bigcup_{i\in I}\lambda_i\right)(b)\wedge\frac{\delta-k}{2}.\end{aligned}$$

Therefore $\bigcup_{i\in I}\lambda_i$ is an $(\in,\in\vee q_k^\delta)$-fuzzy bi-ideal of S. □

Theorem 3.15. *Suppose B is a non-empty subset of S, then B is a bi-ideal of S if and only if χ_B is an $(\in,\in\vee q_k^\delta)$-fuzzy bi-ideal of S.*

Proof. Suppose B is a bi-ideal of S, then χ_B is a fuzzy bi-ideal of S and so is an $(\in,\in\vee q_k^\delta)$-fuzzy bi-ideal of S.

Conversely, suppose that χ_B is an $(\in,\in\vee q_k^\delta)$-fuzzy bi-ideal of S, then for all $a,b\in B$, we have

$$\chi_B(ab)\geq\chi_B(a)\wedge\chi_B(b)\wedge\frac{\delta-k}{2}=\frac{\delta-k}{2}\text{(By Theorem 3.6)}$$

and so $ab\in B$. Now for every $a,b,c\in B$, we have

$$\chi_B(abc)\geq\chi_B(a)\wedge\chi_B(b)\wedge\frac{\delta-k}{2}=\frac{\delta-k}{2}$$

and so $abc\in S$. Therefore B is a bi-ideal of S. □

Theorem 3.16. *Suppose B is a bi-ideal of S, then for every $t\in(0,\frac{\delta-k}{2}]$ there exists an $(\in,\in\vee q_k^\delta)$-fuzzy bi-ideal λ of S such that $U(\lambda;t)=B$.*

Proof. Suppose λ is a fuzzy ideal of S defined by

$$\lambda(a) = \begin{cases} t \text{ if } a \in B \\ 0 \text{ if } a \notin B \end{cases}$$

for all $a \in S$, where $t \in (0, \frac{\delta-k}{2}]$. Obviously, $U(\lambda; t) = B$. Assume that $\lambda(ab) < \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2}$ for $a, b \in S$. It follows that $\lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2} = t$ and $\lambda(ab) = 0$. Hence $\lambda(a) = t = \lambda(b)$ and so $a, b \in B$. Since B is a bi-ideal of S, it implies that $ab \in B$. Which is a contradiction. Therefore $\lambda(ab) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2}$ for all $a, b \in S$. In similar way we have

$$\lambda(abc) \geq \lambda(a) \wedge \lambda(c) \wedge \frac{\delta - k}{2}$$

for all $a, b, c \in S$. Hence λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. □

Theorem 3.17. *Let $\phi : S \to S'$ be a homomorphism from a semigroups S onto a semigroup S' and λ and μ are $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S and S' respectively. Then*

(i) $\phi^{-1}(\mu)$ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S.

(ii) If λ satisfies the sup property, that is for any subset H of S there exists x_0 such that $\lambda(x_0) = \bigvee_{x \in H} \lambda(x)$, then $\phi(\lambda)$ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S.

Proof. (i) : Let $a, b \in S$ and $t_1, t_2 \in (0, 1]$ be such that $(a, t_1) \in \phi^{-1}(\mu)$ and $(b, t_2) \in \phi^{-1}(\mu)$. Then $(\phi(a), t_1) \geq \mu$ and $(\phi(b), t_2) \geq \mu$. Since μ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. Thus, $(\phi(ab), t_1 \wedge t_2) = (\phi(a)\phi(b), t_1 \wedge t_2) \in \vee q_k^\delta \mu$.It implies that $(ab, t_1 \wedge t_2) \in \vee q_k^\delta \phi^{-1}(\mu)$. Now let $a, x, b \in S$ and $t_1, t_2 \in (0, 1]$ be such that $(a, t_1) \in \phi^{-1}(\mu)$ and $(b, t_2) \in \phi^{-1}(\mu)$. It follows that $(\phi(a), t_1) \geq \mu$ and $(\phi(b), t_2) \geq \mu$. Since μ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S'. It implies that $(\phi(axb), t_1 \wedge t_2) = (\phi(a)\phi(x)\phi(b), t_1 \wedge t_2) \in \vee q_k^\delta \mu$ it implies that $(axb, t_1 \wedge t_2) \in \vee q_k^\delta \phi^{-1}(\mu)$. Hence $\phi^{-1}(\mu)$ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S.

(ii) : Let $a, b \in S'$ and $t_1, t_2 \in (0, 1]$ be such that $(a, t_1) \in \phi(\lambda)$ and $(b, t_2) \in \phi(\lambda)$. Then $\phi(\lambda)(a) > t_1$ and $\phi(\lambda)(b) > t_2$. Since λ has the sup property so there exists $x \in \phi^{-1}(a)$ and $y \in \phi^{-1}(b)$ such that $\lambda(x) = \bigvee \left\{\lambda(p) : p \in \phi^{-1}(a)\right\}$ and $\lambda(y) = \inf\left\{\lambda(q) : q \in \phi^{-1}(b)\right\}$. Then $(x, t_1) \in \lambda$ and $(y, t_2) \in \lambda$. Since λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. We have $(xy, t_1 \wedge t_2) \in \vee q_k^\delta \lambda$. Now $xy \in \phi^{-1}(x)$ and so $(\phi(\lambda))(xy) \geq \lambda(xy)$. Thus, $(\phi(\lambda))(xy) \geq t_1 \wedge t_2$ or $(\phi(\lambda))(xy) + t_1 \wedge t_2 + k + 1 > \delta$ which means that $(xy, t_1 \wedge t_2) \in \vee q_k^\delta \phi(\lambda)$. Similarly taking $x, y, z \in S'$ and $t_1, t_2 \in (0, 1]$ such that $(x, t_1) \in \phi(\lambda)$ and $(z, t_2) \in \phi(\lambda)$, we get $(xyz, t_1 \wedge t_2) \in \vee q_k^\delta \phi(\lambda)$. □

Definition 3.18. A fuzzy subset λ in S is said to be an $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal of S if

(i) $(a, t_1) \in \lambda$, $(b, t_2) \in \lambda \Rightarrow (ab, t_1 \wedge t_2) \in \vee q_k^\delta \lambda$

(ii) $(a, t_1) \in \lambda$, $(b, t_2) \in \lambda, (c, t_3) \in \lambda \Rightarrow (ax(bc), t_1 \wedge t_2 \wedge t_3) \in \vee q_k^\delta \lambda$

for all $a, x, b, c \in S$.

Theorem 3.19. *Suppose λ is a fuzzy subset of S. Then λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S, if and only if*

(i) $\lambda(ab) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2}$

(ii) $\lambda(ax(bc)) \geq \lambda(a) \wedge \lambda(b) \wedge \lambda(c) \wedge \frac{\delta-k}{2}$

for all $a, b, c \in S$ *where* $k \in [0, 1)$ *and* $k < \delta$ *in* $[0, 1]$.

Proof. Proof of the Theorem follows from Theorem 3.6. □

The following result shows that every $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal in a semigroup S is an $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal of S.

Theorem 3.20. *Every* $(\in, \in \vee q_k^\delta)$*-fuzzy bi-ideal of a semigroup* S *is an* $(\in, \in \vee q_k^\delta)$*-fuzzy* $(1, 2)$*-ideal of* S.

Proof. Let λ be an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S and $x, a, b, c \in S$. Then,

$$\begin{aligned} \lambda(ax(bc)) &= \lambda((axb)c) \\ &\geq \lambda(axb) \wedge \lambda(c) \wedge \frac{\delta - k}{2} \text{ (By Theorem 3.6)} \\ &\geq \left[\lambda(a) \wedge \lambda(b) \wedge \frac{\delta - k}{2}\right] \wedge \lambda(c) \wedge \frac{\delta - k}{2} \\ &\quad \text{(Since } \lambda \text{ is an } \left(\in, \in \vee q_k^\delta\right) \text{-fuzzy bi-ideal of } S) \\ &= \lambda(a) \wedge \lambda(b) \wedge \lambda(c) \wedge \frac{\delta - k}{2}. \end{aligned}$$

Hence λ is an $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal of S. □

The converse of the above Theorem is not true in general. For this we have the following example.

Example 3.21. Let $S = \{1, 2, 3, 4\}$ and "$\cdot$"be a binary operation defined on S in the following table:

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	2	3	2
3	1	1	1	1
4	1	1	2	4

Define a fuzzy subset λ in S by $\lambda(1) = 0.5$, $\lambda(2) = 0.1 = \lambda(3)$, $\lambda(4) = 0.2$. Then λ is an $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal of S, where $\delta = 0.7$ and $k = 0.3$. λ is not an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S as it does not satisfy the second condition of the Definition 3.5. i.e $\lambda(4 \cdot 3 \cdot 4) = \lambda(2) = 0.1 \ngeq 0.2 = \lambda(4) \wedge \lambda(4) \wedge \frac{\delta - k}{2}$.

Theorem 3.22. *Suppose* S *is a regular semigroup and* λ *a fuzzy subset of* S*, then every* $(\in, \in \vee q_k^\delta)$*-fuzzy* $(1, 2)$*-ideal of semigroup* S *is an* $(\in, \in \vee q_k^\delta)$*-fuzzy bi-ideal of* S.

Proof. Suppose S is regular and λ an $(\in, \in \vee q_k^\delta)$-fuzzy $(1, 2)$-ideal of S. Let $a, x, b \in S$. Since S is regular we have $ax \in (aSa)S \subseteq aSa$, which implies that $ax = asa$ for some

$s \in S$. Thus

$$\begin{aligned} \lambda(axb) &= \lambda((asa)b) \\ &= \lambda(as(ab)) \\ &\geq \lambda(a) \wedge \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2} \text{ (By Theorem 3.19)} \\ &= \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2} \end{aligned}$$

Hence λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. □

Theorem 3.23. *Suppose λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. If S is a completely regular and $\lambda(a) < \frac{\delta-k}{2}$ for all $a \in S$, then $\lambda(a) = \lambda(a^2)$ for all $a \in S$.*

Proof. Suppose $a \in S$. Then there exists $a \in S$ such that $a = a^2xa^2$. Hence

$$\begin{aligned} \lambda(a) &= \lambda\left(a^2xa^2\right) \\ &\geq \lambda\left(a^2\right) \wedge \lambda\left(a^2\right) \wedge \frac{\delta-k}{2} \text{ (By Theorem 3.6)} \\ &= \lambda\left(a^2\right) \wedge \frac{\delta-k}{2} \\ &\geq \left[\lambda(a) \wedge \lambda(a) \wedge \frac{\delta-k}{2}\right] \wedge \frac{\delta-k}{2} \\ &= \lambda(a) \wedge \frac{\delta-k}{2} = \lambda(a). \end{aligned}$$

It follows that $\lambda(a) = \lambda\left(a^2\right)$. □

If we put $\delta = 1$ in Theorem 3.23, then we get the following corollary.

Corollary 3.24. *Suppose λ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. If S is a completely regular and $\lambda(a) < \frac{1-k}{2}$ for all $a \in S$, then $\lambda(a) = \lambda\left(a^2\right)$ for all $a \in S$.*

Theorem 3.25. *Suppose λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. If S is completely regular, then $\lambda(a) \wedge \frac{\delta-k}{2} = \lambda\left(a^2\right) \wedge \frac{\delta-k}{2}$ for all $a \in S$.*

Proof. Suppose $a \in S$, then there exists $x \in S$ such that $a = a^2xa^2$. Hence

$$\begin{aligned} \lambda(a) \wedge \frac{\delta-k}{2} &= \lambda\left(a^2xa^2\right) \wedge \frac{\delta-k}{2} \\ &\geq \lambda\left(a^2\right) \wedge \lambda\left(a^2\right) \wedge \frac{\delta-k}{2} \text{ (By Theorem 3.6)} \\ &= \lambda\left(a^2\right) \wedge \frac{\delta-k}{2} \\ &\geq \left[\lambda(a) \wedge \lambda(a) \wedge \frac{\delta-k}{2}\right] \wedge \frac{\delta-k}{2} \\ &= \lambda(a) \wedge \frac{\delta-k}{2}. \end{aligned}$$

It implies that $\lambda(a) \wedge \frac{\delta-k}{2} = \lambda\left(a^2\right) \wedge \frac{\delta-k}{2}$ for all $a \in S$. □

4. $(\in, \in \vee q_k^\delta)$-FUZZY QUASI-IDEAL

In this section we introduce the concept of an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of a semigroup S and study some of its properties. Based on Theorem 3.6, we define $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of a semigroup S as follows.

Definition 4.1. Suppose λ is a fuzzy subset of a S, then λ is said to be an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S, if it satisfies the following condition:

$$\lambda(a) \geq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta - k}{2}.$$

where w is a fuzzy subset of S mapping every element of S on 1.

Theorem 4.2. *Suppose λ is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S, then the set $\lambda_0 = \{a \in \lambda | \lambda(a) > 0\}$ is a quasi-ideal of S.*

Proof. To show that λ is a quasi-ideal of S, we need only to to show that $S\lambda_0 \cap \lambda_0 S \subseteq \lambda_0$. Let $a \in S\lambda_0 \cap \lambda_0 S$ implies that $a \in S\lambda_0$ and $a \in \lambda_0 S$. So $a = rx$ and $a = ys$ for $r, s \in S$ and $x, y \in S$. Thus, $\lambda(x) > 0$ and $\lambda(y) > 0$. Now $\lambda(a) \geq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta - k}{2}$. Since

$$\begin{aligned} (w \circ \lambda)(a) &= \bigvee_{a=pq} \{w(p) \wedge \lambda(q)\} \\ &\geq w(r) \wedge \lambda(x) \text{ because } a = rx \\ &= \lambda(x). \end{aligned}$$

Hence $(w \circ \lambda)(a) \geq \lambda(x)$. Similarly we can show that $(w \circ \lambda)(a) \geq \lambda(b)$. Thus

$$\begin{aligned} \lambda(a) &\geq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta - k}{2} \\ &\geq \lambda(x) \wedge \lambda(y) \wedge \frac{\delta - k}{2} \\ &> 0. \text{ because } \lambda(x) > 0 \text{ and } \lambda(y) > 0 \end{aligned}$$

Thus $a \in \lambda_0$. Hence λ_0 is a quasi-ideal of S. □

Example 4.3. Suppose $S = \{1, 2, 3, 4\}$ and " $\cdot$ " a binary operation defined on S in the following table:

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	2	4	1
3	1	1	1	1
4	1	4	1	1

Then, $(S, \cdot)$ is a semigroup. Define $\lambda : S \to [0, 1]$ of S by $\lambda(1) = 0.9$, $\lambda(2) = 0.8$, $\lambda(3) = 0.7$, $\lambda(4) = 0.4$. If we take, $\delta = 0.4$ and $k = 0.2$, then λ is an $(\in, \in \vee q_{0.2}^{0.4})$-fuzzy quasi-ideal of S.

Theorem 4.4. *A fuzzy subset λ in S is an $\left(\in, \in \vee q_k^\delta\right)$-fuzzy quasi-ideal of S if and only if $U(\lambda; t)\,(\neq \emptyset)$ is a quasi-ideal of S for all $t \in (0, \frac{\delta - k}{2}]$.*

Proof. Suppose λ is an $\left(\in, \in \vee q_k^\delta\right)$-fuzzy quasi-ideal of S and $a \in (U(\lambda;t)\,S] \cap (SU(\lambda;t)]$ for some $t \in (0, \frac{\delta-k}{2}]$, then $a \in (U(\lambda;t)\,S]$ and $a \in (SU(\lambda;t)]$. This implies that $a = xy$, and $a = wz$, where $x, z \in S$ and $y, w \in U(\lambda;t)$

$$\begin{aligned}
\lambda(a) &\geq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta-k}{2} \\
&= \left[\bigvee_{a=pq} \{\lambda(p) \wedge w(q)\}\right] \wedge \left[\bigvee_{a=p_1q_1} \{w(p_1) \wedge \lambda(q_1)\}\right] \wedge \frac{\delta-k}{2} \\
&= \{\lambda(w) \wedge w(z)\} \wedge \{w(x) \wedge \lambda(y)\} \wedge \frac{\delta-k}{2} \text{ since } a = xy, \text{and } a = wz, \\
&= \{\lambda(w) \wedge 1\} \wedge \{1 \wedge \lambda(y)\} \wedge \frac{\delta-k}{2} \\
&= \lambda(w) \wedge \lambda(y) \wedge \frac{\delta-k}{2} \\
&= t \wedge t \wedge \frac{\delta-k}{2} = t.
\end{aligned}$$

Thus $a \in U(\lambda;t)$ and hence $(U(\lambda;t)\,S] \cap (SU(\lambda;t)] \subseteq U(\lambda;t)$. Thus $U(\lambda;t)$ is a quasi-ideal of S.

Conversely, assume that for $t \in (0, \frac{\delta-k}{2}]$, the set $U(\lambda;t)\,(\neq \emptyset)$ is a quasi-ideal of S. Let there exists $a \in S$ such that

$$\lambda(a) \ngeq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta-k}{2}.$$

and that $\lambda(a) < t_1 \leq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta-k}{2}$ for some $t_1 \in (0,1)$, Then $(\lambda \circ w)(a) \geq t_1$ and $(w \circ \lambda)(a) \geq t_1$ but $\lambda(a) < t_1$ which implies that $a \notin U(\lambda;t_1)$. which is a contradiction. Thus

$$\lambda(a) \geq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta-k}{2}.$$

Hence λ is an $\left(\in, \in \vee q_k^\delta\right)$-fuzzy quasi-ideal of S. □

Lemma 4.5. *If $\{\lambda_i\}_{i\in\Lambda}$ is a family of $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideals of S, then $\bigcap_{i\in\Lambda} \lambda_i$ is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S.*

Proof. Straightforward. □

Theorem 4.6. *Suppose Q is a non-empty subset of S, then Q is a quasi-ideal of S if and only if C_Q, the characteristic function of Q is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S.*

Proof. Suppose Q is a quasi-ideal of S and C_Q is the characteristic function of Q. Let, $a \in S$; if $a \notin Q$. Then $a \notin SQ$ or $a \notin QS$. Thus $(w \circ C_Q)(a) = 0$, $(C_Q \circ w)(a) = 0$ and so $(C_Q \circ w)(a) \wedge (w \circ C_Q)(a) \wedge \frac{\delta-k}{2} = 0 = C_Q$. If $a \in C_Q$, then $(C_Q)(a) = 1 \geq (C_Q \circ w)(a) \wedge (w \circ C_Q)(a) \wedge \frac{\delta-k}{2}$. Hence C_Q is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S. Conversely, assume that C_Q is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S. Let $a \in QS \cap SQ$.

Then their exists $y, z \in S$ and $b, c \in Q$ such that $a = by$ and $a = zc$. Thus

$$\begin{aligned}(C_Q \circ w)(a) &= \bigvee_{a=pq} \{C_Q(p) \wedge w(q)\} \\ &= C_Q(a) \wedge w(a) \\ &= C_Q(by) \wedge w(by) \\ &\geq C_Q(b) \wedge w(y) \\ &= 1 \wedge 1 \\ &= 1\end{aligned}$$

So $(C_Q \circ w)(a) = 1$. Similarly, $(w \circ C_Q)(a) = 1$. Hence

$$(C_Q)(a) \geq (C_Q \circ w)(a) \wedge (w \circ C_Q)(a) \wedge \frac{\delta - k}{2}.$$

Thus $(C_Q)(a) = 1$, which implies that $a \in Q$. Hence $QS \cap SQ \subseteq Q$. That is Q is a quasi-ideal of S. □

Theorem 4.7. *Every $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S.*

Proof. Let $a \in S$ and λ be an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S; then

$$\begin{aligned}(w \circ \lambda)(a) &= \bigvee_{a=xy} \{w(x) \wedge \lambda(y)\} \\ &= \bigvee_{a=xy} \{1 \wedge \lambda(y)\} \\ &= \bigvee_{a=xy} \lambda(y)\end{aligned}$$

Now

$$\begin{aligned}(w \circ \lambda)(a) \wedge \frac{\delta - k}{2} &= \bigvee_{a=xy} \{w(x) \wedge \lambda(y)\} \wedge \frac{\delta - k}{2} \\ &= \bigvee_{a=xy} \{1 \wedge \lambda(y)\} \wedge \frac{\delta - k}{2} \\ &= \bigvee_{a=xy} \left\{\lambda(x) \wedge \frac{\delta - k}{2}\right\} \\ &\leq \bigvee_{a=xy} \lambda(xy) \text{ (Since } \lambda \text{ is an } (\in, \in \vee q_k^\delta)\text{-fuzzy left ideal of } S) \\ &= \lambda(a).\end{aligned}$$

Thus

$$(w \circ \lambda)(a) \wedge \frac{\delta - k}{2} \leq \lambda(a).$$

Hence

$$\lambda(a) \geq (w \circ \lambda)(a) \geq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge \frac{\delta - k}{2}.$$

Thus λ is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S. □

Remark 4.8. The converse of the above theorem is not true in general, that is an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S need not be an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S. For this we have the following example.

Example 4.9. Consider a semigroup $S = \{a, b, c, d\}$ with a binary operation " $\cdot$ " defined in the following table:

$\cdot$	a	b	c	d
a	a	a	a	d
b	a	a	a	a
c	b	a	a	a
d	d	a	a	a

Define a fuzzy subset λ on S by $\lambda(a) = 0.2 = \lambda(b), \lambda(c) = 0.4, \lambda(d) = 0$. If we take $k = 0.4$ and $\delta = 0.8$ then λ is an is an $(\in, \in \vee q_{0.4}^{0.8})$-fuzzy quasi-ideal of S. But it is not an $(\in, \in \vee q_{0.4}^{0.8})$-fuzzy left (right) ideal of S. Because $\lambda(da) = \lambda(d) = 0 \ngeq \lambda(a) \wedge \frac{\delta-k}{2}$ and $\lambda(ad) = \lambda(d) = 0 \ngeq \lambda(a) \wedge \frac{\delta-k}{2}$. Also λ is not an $(\in, \in \vee q_{0.4})$-fuzzy quasi-ideal of S. Because $(\lambda \circ w)(a) = 0.3 = (w \circ \lambda)(a)$, but $\lambda(a) = 0.2 \ngeq (\lambda \circ w)(a) \wedge (w \circ \lambda)(a) \wedge 0.3$. It is also not a fuzzy quasi-ideal nor an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.

Theorem 4.10. *Every $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S.*

Proof. Let λ be an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S. Now

$$\begin{aligned}
\lambda(ab) &\geq (\lambda \circ w)(ab) \wedge (w \circ \lambda)(ab) \wedge \frac{\delta - k}{2} \\
&= \left[\bigvee_{ab=xy} \{\lambda(x) \wedge w(y)\}\right] \wedge \left[\bigvee_{ab=xy} \{w(x) \wedge \lambda(y)\}\right] \wedge \frac{\delta - k}{2} \\
&\geq [\lambda(a) \wedge w(b)] \wedge [w(a) \wedge \lambda(b)] \wedge \frac{\delta - k}{2} \\
&= [\lambda(a) \wedge 1] \wedge [1 \wedge \lambda(b)] \wedge \frac{\delta - k}{2} \\
&= \lambda(a) \wedge \lambda(b) \wedge \frac{\delta - k}{2}.
\end{aligned}$$

So

$$\lambda(ab) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta - k}{2}.$$

Also

$$\begin{aligned}\lambda(axb) &\geq (\lambda \circ w)(axb) \wedge (w \circ \lambda)(axb) \wedge \frac{\delta - k}{2} \\ &= \left[\bigvee_{axb=yz} \{\lambda(y) \wedge w(z)\}\right] \wedge \left[\bigvee_{axb=yz} \{w(y) \wedge \lambda(z)\}\right] \wedge \frac{\delta - k}{2} \\ &\geq [\lambda(a) \wedge w(xb)] \wedge [w(ax) \wedge \lambda(b)] \wedge \frac{\delta - k}{2} \\ &= [\lambda(a) \wedge 1] \wedge [1 \wedge \lambda(b)] \wedge \frac{\delta - k}{2} \\ &= \lambda(a) \wedge \lambda(b) \wedge \frac{\delta - k}{2}.\end{aligned}$$

So

$$\lambda(axb) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta - k}{2}.$$

Hence λ is an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal of S. □

Remark 4.11. The converse of the above theorem is not true in general. That is $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal need not be an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of S. For this we have the following example.

Example 4.12. Consider a semigroup $S = \{1, 2, 3, 4\}$ defined in the following table as:

$\cdot$	1	2	3	4
1	3	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	2	1

Define a fuzzy subset λ by $\lambda(1) = 0.8, \lambda(2) = \lambda(4) = 0, \lambda(3) = 0.3$, if we take $k = 0.2$ and $\delta = 0.8$, then λ is an $(\in, \in \vee q_{0.2}^{0.8})$-fuzzy bi-ideal of S. But it is not an $(\in, \in \vee q_{0.2}^{0.8})$-fuzzy quasi-ideal of S. As $(\lambda \circ w)(2) = 0.3 = (w \circ \lambda)(2)$, but $\lambda(2) = 0 \ngeq (\lambda \circ w)(2) \wedge (w \circ \lambda)(2) \wedge \frac{\delta - k}{2}$. It also should be noted that λ is not a fuzzy bi-ideal, not an $(\in, \in \vee q)$-fuzzy bi-ideal and not an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

5. Conclusion

The concept of an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal/quasi-ideal of a semigroup is a generalization of the concepts studied in [11, 15, 26, 27]. If we take $\delta = 1$, then we get, an $(\in, \in \vee q_k)$-fuzzy bi-ideal/quasi-ideal of a semigroup. If we take $\delta = 1$ and $k = 0$, then we get, an $(\in, \in \vee q)$-fuzzy bi-ideal/quasi-ideal of a semigroup. Which means that these fuzzy ideals become a special case of an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal/quasi-ideal of a semigroup. Due to the motivation and inspiration of the concepts in this paper we studied the concept of an $(\in, \in \vee q_k^\delta)$-fuzzy bi-ideal, an $(\in, \in \vee q_k^\delta)$-fuzzy quasi-ideal of a semigroup. In the domain of modern mathematics the use of algebraic structures in computer science, control theory and fuzzy automata theory constantly increase the attention of researchers. Algebraic structures mostly semigroups play a vital role in such applied branches. Further, the fuzzification of several subsystems of semigroups is used in numerous models including uncertainties.

$1)$: In future we can apply the present concept to other algebraic structures, i.e Ring, Hemiring, Nearring etc.

$2)$: We will define $(\in, \in \vee q_k^\delta)$-fuzzy soft (generalized) bi-ideals/quasi-ideals.

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