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Cyclic Vector of the Weighted Mean Matrix Operator

Ebrahim Pazouki
Mazandaran Management and Planning Organization
P. O. Box 48177-87959, Sari, Iran.
Email: aha.pazoki@gmail.com

Bahmann Yousefi
Department of Mathematics,
Payame Noor University
P. O. Box 19395-3697, Tehran, Iran.
Email: b_yousefi@pnu.ac.ir

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Abstract. In this paper, we first prove cyclicity of bounded diagonal matrix on $H^p(\beta)$ and we provide some conditions for a weighted mean matrix operator to have eigenvectors. Then we study boundedness of matrix of eigenvectors, diagonalization and cyclicity of the weighted mean matrix operator on the weighted Hardy spaces.

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1. Introduction

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series and the set of such series is denoted by $H^p(\beta)$, so we call this space "weighted Hardy space". Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$.The study of weighted Hardy spaces lies at the interface of analytic function theory and operator theory. As a part of operator theory, research on weighted Hardy spaces is of fairly recent origin, dating back to valuable work of Allen Shields in the mid- 1970s. For some other sources, see [10-18].

2. Notations and Preliminaries

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive numbers, and let $A_n = \sum_{i=0}^{n} a_i\beta(i)^p$. The weighted mean matrix operator on $H^p(\beta)$ is an infinite matrix $A = [a_{nk}]_{n,k}$ with

$$a_{nk} = \begin{cases} \frac{a_k\beta(n)^p}{A_n} & 0 \le k \le n \\ 0 & k > n \end{cases}.$$

The definition was introduced by Pazouki [10]. The weighted mean matrix has been the focus of attention for several decades and many of its properties have been studied. Some of basic and useful works in this area are due to Browein et al.
Let X be a nontrivial complex Banach space of complex sequences. The spectrum $(\sigma(A))_X$ of a bounded operator A on X, is the set all of complex numbers λ such that the operator $A-\lambda I$ is not invertible on X. The resolvent $(\rho(A))_X$ of A is the complement of $(\sigma(A))_X$.
A complex number λ is an eigenvalue of the operator A, whenever there exists a nonzero complex sequence $\{\hat{f}(n)\}_{n=0}^{\infty}$ in X such that $A\{\hat{f}(n)\} = \lambda\{\hat{f}(n)\}$. This nonzero complex sequence $\{\hat{f}(n)\}_{n=0}^{\infty}$ is called an eigenvector corresponding to λ [6].
A vector x in a Banach space X is a cyclic vector of a bounded operator T on X if the closed linear span $\{T^n(x) : n \geq 0\}$ is all of X, i.e. $span\{T^n(x) : n \geq 0\} = X$. Some of basic and useful works on these topics are due to [1-13].

3. Cyclicity

Section 3 is intended to motivate our investigation of cyclicity and boundedness of diagonal matrix whose nonzero elements is diagonal entries of weighted mean matrix operator, diagonalization of weighted mean matrix operator on $H^p(\beta)$. This idea goes back to [2-5].

Proposition 3.1. *Let D be a diagonal matrix on $H^p(\beta)$. Then D is a bounded operator if and only if* $\sup\{|d_{nn}| : n \geq 0\} < \infty$.

Proof. The proof is trivial. □

Theorem 3.2. *Let $\lim\limits_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = 1$ and $\frac{a_i\beta(i)^p}{A_i} \neq \frac{a_j\beta(j)^p}{A_j}$ for all $i \neq j$. Then the bounded diagonal matrix operator $D = [d_{ij}]$ with $d_{ii} = \frac{a_i\beta(i)^p}{A_i}$ is cyclic on $H^p(\beta)$.*

Proof. Define the decreasing positive sequence $\alpha_n = \min\{|\frac{a_i\beta(i)^p}{A_i} - \frac{a_j\beta(j)^p}{A_j}| : 0 \leq i \neq j \leq n\}$. Since $A_i = \sum_{j=0}^{i} a_j\beta(j)^p$, so we have $0 \leq \frac{a_i\beta(i)^p}{A_i} \leq 1$, this gives us

$$|\frac{a_i\beta(i)^p}{A_i} - \frac{a_j\beta(j)^p}{A_j}| < 1,$$

for all $0 \leq i \neq j \leq n$, so $0 < \alpha_n < 1$. Consider the polynomials $P_{nk}(z) = \prod_{i=0,i\neq k}^{n} \frac{z-\lambda_i}{\lambda_k-\lambda_i}$, where $\lambda_i = \frac{a_i\beta(i)^p}{A_i}$ for all $0 \leq i \neq k \leq n$. Thus $P_{nk}(\lambda_k) = 1$ and $P_{nk}(\lambda_i) = 0$ for $0 \leq i \neq k \leq n$. For all $r > n$ we have

$$\begin{aligned} |P_{nk}(\lambda_r)| &= \prod_{i=0,i\neq k}^{n} |\frac{\lambda_r - \lambda_i}{\lambda_k - \lambda_i}| \\ &\leq \frac{1}{(\alpha_n)^n} \leq \frac{1}{(\alpha_r)^r}. \end{aligned}$$

Note that $P_{nk}(D) = [d_{rs}]$ where $d_{kk} = 1$, $d_{rr} = P_{nk}(\lambda_r)$ when $r = s > n$ and $d_{rs} = 0$ otherwise. If

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in H^p(\beta),$$

then

$$P_{nk}(D)f(z) = \hat{f}(k)z^k + \sum_{r>n}^{\infty} P_{nk}(\lambda_r)\hat{f}(r)z^r.$$

Suppose that

$$f_1(z) = \sum_{n=0}^{\infty} \hat{f}_1(n)z^n \in H^p(\beta),$$

where $0 < |\hat{f}_1(r)|^p < \frac{(\alpha_r)^r}{2^r \beta(r)^p}$ and $r \geq 0$. To show that

$$M = span\{D^k(f_1) : k \geq 0\} = H^p(\beta),$$

it is suffices to prove that for every $k \geq 0$, $f_k \in M$. If there exists m such that $f_m \notin M$, then there is a linear functional L_1 on $H^p(\beta)$ Such that $L_1 = 0$ on M and $L_1(f_m) = 1$. On the other hand by the Lemma 2 in Yousefi [13], there exists

$$g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n \in H^q(\beta^{\frac{p}{q}}), \quad \frac{1}{p} + \frac{1}{q} = 1$$

such that

$$L_1(f(z)) = \sum_{n=0}^{\infty} \hat{f}(n)\overline{(\hat{g}(n))}\beta(n)^p,$$

so there is a natural number $n \geq m$ such that $(\sum_{r>n}^{\infty} \frac{1}{2^r})^{\frac{1}{p}} < \epsilon$. Thus

$$P_{nm}(D)f_1(z) = \hat{f}_1(m)z^m + \sum_{r>n}^{\infty} P_{nm}(\lambda_r)\hat{f}_1(r)z^r,$$

and so

$$\begin{aligned}
|L_1[P_{nm}(D)f_1(z) - \hat{f}_1(m)z^m]| &\leq \sum_{r>n}^{\infty} |P_{nm}(\lambda_r)||\hat{f}_1(r)||\hat{g}(r)|\beta(r)^p \\
&\leq (\sum_{r>n}^{\infty} |P_{nm}(\lambda_r)\hat{f}_1(r)|^p\beta(r)^p)^{\frac{1}{p}}(\sum_{r>n}^{\infty} |\hat{g}(r)|^q\beta(r)^{q(p-1)})^{\frac{1}{q}} \\
&\leq (\sum_{r>n}^{\infty} \frac{1}{2^r})^{\frac{1}{p}}\|g\|,
\end{aligned}$$

which tends to zero as $n \to \infty$. Thus $L_1(\hat{f}_1(m)z^m) = 0$ that is a contradiction. So $f_1(z)$ is a cyclic vector for D and the proof is complete. □

Theorem 3.3. *Let* $\lim_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = 1$, $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$. *Then we have the following statements*
i) For every $j \geq 0$, $c_j = \frac{a_j\beta(j)^p}{A_j}$ *is an eigenvalue of the weighted mean matrix operator* A.
ii) We have $f(z) = \sum_{k=0}^{\infty} \beta(k)^p z^k \in H^p(\beta)$ *is an eigenvector corresponding to the eigenvalue* $c_0 = 1$.

Proof. Let j be a natural number and $\lambda = \frac{a_j\beta(j)^p}{A_j}$. Then we show that there exists $f_\lambda(z) = \sum_{j=0}^{\infty} \hat{f}_\lambda(j)z^j \in H^p(\beta)$, such that $A(f_\lambda) = \lambda(f_\lambda)$. Define $\hat{f}_\lambda(i) = 0$ for $0 \leq i < j$, and $\hat{f}_\lambda(j) = a > 0$. From $(\widehat{A(f_\lambda)})(j+1) = c_j\hat{f}_\lambda(j+1)$, we have

$$
\begin{aligned}
(c_j - c_{j+1})\hat{f}_\lambda(j+1) &= \frac{a_j\beta(j+1)^p}{A_{j+1}}\hat{f}_\lambda(j) \\
&= (\frac{\beta(j+1)}{\beta(j)})^p\frac{a_j\beta(j)^p}{A_{j+1}}\hat{f}_\lambda(j) \\
&= (\frac{\beta(j+1)}{\beta(j)})^p c_j(1-c_{j+1})\hat{f}_\lambda(j),
\end{aligned}
$$

thus

$$
\hat{f}_\lambda(j+1) = (\frac{\beta(j+1)}{\beta(j)})^p\frac{(1-c_{j+1})}{(1-\frac{c_{j+1}}{c_j})}\hat{f}_\lambda(j).
$$

Let

$$
\hat{f}_\lambda(j+k) = (\frac{\beta(j+k)}{\beta(j)})^p(\prod_{i=1}^{k}\frac{(1-c_{j+i})}{(1-\frac{c_{j+i}}{c_j})})\hat{f}_\lambda(j),
$$

thus

$$
(\widehat{A(f_\lambda)})(j+k+1) = c_j\hat{f}_\lambda(j+k+1),
$$

it implies that

$$
\begin{aligned}
(c_j - c_{j+k+1})\hat{f}_\lambda(j+k+1) &= \frac{a_j\beta(j+k+1)^p}{A_{j+k+1}}\hat{f}_\lambda(j) \\
&+ \sum_{i=1}^{k}(\frac{\beta(j+i)}{\beta(j)})^p\frac{a_{j+i}\beta(j+k+1)^p}{A_{j+k+1}}(\prod_{l=1}^{i}\frac{(1-c_{j+l})}{(1-\frac{c_{j+l}}{c_j})})\hat{f}_\lambda(j) \\
&= (\frac{\beta(j+k+1)}{\beta(j)})^p(\frac{a_j\beta(j)^p}{A_{j+k+1}} \\
&+ \sum_{i=1}^{k}\frac{a_{j+i}\beta(j+i)^p}{A_{j+k+1}}\prod_{l=1}^{i}\frac{(1-c_{j+l})}{(1-\frac{c_{j+l}}{c_j})})\hat{f}_\lambda(j),
\end{aligned}
$$

therefore

$$
\begin{aligned}
(c_j - c_{j+k+1})\hat{f}_\lambda(j+k+1) &= (\frac{\beta(j+k+1)}{\beta(j)})^p(c_j \prod_{n=0}^{k} \frac{A_{j+n}}{A_{j+n+1}} \\
&+ \sum_{i=1}^{k} c_{j+i} \prod_{n=0}^{k-i} \frac{A_{j+i+n}}{A_{j+i+n+1}} \prod_{l=1}^{i} \frac{(1-c_{j+l})}{(1-\frac{c_{j+l}}{c_j})})\hat{f}_\lambda(j) \\
&= (\frac{\beta(j+k+1)}{\beta(j)})^p(c_j \prod_{n=0}^{k}(1-c_{j+n+1}) \\
&+ \sum_{i=1}^{k} c_{j+i} \prod_{n=0}^{k-i}(1-c_{j+i+n+1}) \prod_{l=1}^{i} \frac{(1-c_{j+l})}{(1-\frac{c_{j+l}}{c_j})})\hat{f}_\lambda(j) \\
&= (\frac{\beta(j+k+1)}{\beta(j)})^p c_j (\prod_{i=1}^{k+1}(1-c_{j+i})) \\
&(1+\sum_{i=1}^{k} \frac{c_{j+l}c_j^{i-1}}{\prod_{l=1}^{i}(c_j - c_{j+l})})\hat{f}_\lambda(j) \\
&= (\frac{\beta(j+k+1)}{\beta(j)})^p c_j^{k+1} \frac{\prod_{i=1}^{k+1}(1-c_{j+i})}{\prod_{i=1}^{k}(c_i - c_{j+i})} \hat{f}_\lambda(j) \\
&= \frac{\beta(j+k+1)^p}{\beta(j)^p}(\prod_{i=1}^{k} \frac{(1-c_{j+i})}{(1-\frac{c_{j+i}}{c_j})})c_j(1-c_{j+k+1}))\hat{f}_\lambda(j).
\end{aligned}
$$

Thus $f_\lambda(z) = \sum_{j=0}^{\infty} \hat{f}_\lambda(j)z^j$, is a formal power series such that $A(f_\lambda) = \lambda(f_\lambda)$. For every $j \in \mathbb{N}$, put $\epsilon = 1 - \frac{2}{1+\frac{1}{c_j}} > 0$, so there exists natural number N_1 such that

$$
c_n > 1 - \epsilon = \frac{2c_j}{1+c_j}, \quad \frac{\beta(n+1)^p}{\beta(n)^p} < 1
$$

for all $n \geq N_1$. Let $m > 0$ such that $j + m > N_1$, thus

$$
\frac{c_{j+m+1}}{c_j} - 1 > \frac{1-c_j}{1+c_j}, \quad 1 - c_{j+m+1} < \frac{1-c_j}{1+c_j}.
$$

Consider

$$
\begin{aligned}
\frac{|\hat{f}_\lambda(j+m+1)|^p \beta(j+m+1)^p}{|\hat{f}_\lambda(j+m)|^p \beta(j+m)^p} &= \frac{\beta(j+m+1)^{2p}}{\beta(j+m)^{2p}}(\frac{1-c_{j+m+1}}{\frac{c_{j+m+1}}{c_j}-1})^p \\
&< (\frac{1-c_{j+m+1}}{\frac{c_{j+m+1}}{c_j}-1})^p \\
&< 1.
\end{aligned}
$$

It follows that the formal power series $\sum_{k=1}^{\infty} \hat{f}_\lambda(k)z^k$ is in $H^p(\beta)$. Therefore (i) holds. Since $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} = r < 1$, so there exists a natural number N_2 such that $\frac{\beta(n+1)}{\beta(n)} - r <$

$\frac{1-r}{2}$ for every $n \geq N_2$. Thus for every $n \geq N_2 + 1$ we get $\beta(n) < (\frac{1+r}{2})^{n-N_2}$, i.e. $\sum_{k=0}^{\infty} \beta(k) < \infty$, it's clear that $f_0(z) = \sum_{k=0}^{\infty} \beta(k)^p z^k \in H^p(\beta)$, thus

$$\begin{aligned}(\widehat{A(f_0)})(n) &= \sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} \beta(k)^p \\ &= \beta(n)^p,\end{aligned}$$

so (ii) holds. Now the proof is complete. □

Lemma 3.4. *Let $B = [b_{nk}]$ be an infinite matrix, such that $b_{nn} = 1$, $b_{nk} \neq 0$ when $k < n$ and $b_{nk} = 0$ for $k > n$. Then B is invertible and $B^{-1} = [c_{nk}]$ with $c_{nn} = 1$, $c_{nk} = -\sum_{i=k}^{n-1} b_{ni} c_{ik}$ as $k < n$ and $b_{nk} = 0$ for $k > n$.*

Proposition 3.5. *Let $\lim\limits_{n\to\infty} \frac{a_n \beta(n)^p}{A_n} = 1$ and $\lim\limits_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$. Then the weighted mean matrix operator A is diagonazible.*

Proof. First we construct the infinite matrix $P = [p_{nk}]$ of eigenvectors of the weighted mean matrix operator A by Theorem 3.3 with

$$p_{nk} = \begin{cases} \beta(n)^p & k = 0 \\ \frac{\beta(n)^p}{\beta(k)^p} \prod_{i=1}^{n-k} \frac{1-c_{k+i}}{1-(\frac{c_{k+i}}{c_k})} & 1 \leq k < n \\ 1 & k = n \\ 0 & k > n \end{cases}.$$

Under the assumptions of Lemma 3.4, the matrix P is invertible and $P^{-1}AP = D$ where $D = [d_{ij}]$ is an infinite diagonal matrix operator with $d_{ii} = \frac{a_i \beta(i)^p}{A_i}$. It is sufficient to show that $AP = PD$. Consider $n \geq 0$ thus for $k = 0$, we have

$$\begin{aligned}(AP)_{n0} &= \sum_{j=0}^{n} a_{nj} p_{j0} \\ &= \sum_{j=0}^{n} \frac{a_j \beta(n)^p}{A_n} \beta(j)^p \\ &= \beta(n)^p \\ &= p_{n0} d_{00} \\ &= \sum_{j=0}^{n} p_{j0} d_{j0}.\end{aligned}$$

If $0 < k < n$, then

$$\begin{aligned}
(AP)_{nk} &= \sum_{j=k}^{n} a_{nj} p_{jk} \\
&= \frac{a_k \beta(n)^p}{A_n} + \frac{a_{k+1}\beta(n)^p}{A_n} \frac{\beta(k+1)^p}{\beta(k)^p} \frac{1 - c_{k+1}}{1 - (\frac{c_{k+1}}{c_k})} + ... \\
&+ \frac{a_{k+2}\beta(n)^p}{A_n} \frac{\beta(k+2)^p}{\beta(k)^p} \prod_{i=1}^{2} \frac{1 - c_{k+i}}{1 - (\frac{c_{k+i}}{c_k})} \\
&= \frac{a_n \beta(n)^p}{A_n} \frac{\beta(n)^p}{\beta(k)^p} \prod_{i=1}^{n-k} \frac{1 - c_{k+i}}{1 - (\frac{c_{k+i}}{c_k})} \\
&= \frac{\beta(n)^p}{\beta(k)^p} (c_k \frac{A_k}{A_n} + c_{k+1} \frac{A_{k+1}}{A_n} \frac{1 - c_{k+1}}{1 - (\frac{c_{k+1}}{c_k})} + ... + c_n \prod_{i=1}^{n-k} \frac{1 - c_{k+i}}{1 - (\frac{c_{k+i}}{c_k})}) \\
&= \frac{\beta(n)^p}{\beta(k)^p} c_k \prod_{i=1}^{n-k} \frac{1 - c_{k+i}}{1 - (\frac{c_{k+i}}{c_k})} \\
&= p_{nk} d_{kk} \\
&= \sum_{j=k}^{n} p_{nj} d_{jk} \\
&= (PD)_{nk}.
\end{aligned}$$

It is clear that $(AP)_{nn} = (PD)_{nn}$ and $(AP)_{kn} = (PD)_{kn} = 0$ where $k > n$. This completes the proof. □

Lemma 3.6. *Let $p > 1$, $z_n \geq 0$ and $\{e_{nk}\}_{n,k=0}^{\infty}\}$ be two positive sequence where $e_{nk} \neq 0$ as $0 \leq k \leq n$ and $e_{nk} = 0$ otherwise. Then the following statements hold.*
i) $(\sum_{k=1}^{n} a_k)^p \leq p \sum_{k=1}^{n} a_k (\sum_{v=1}^{k} a_v)^{p-1}$,
ii) $\sum_{n\geq 0} \sum_{k=0}^{n} e_{nk} = \sum_{k\geq 0} \sum_{n=k} e_{nk}$.

Proof. Let $\lambda_r = a_1 + ... + a_r$, $1 \leq r \leq n$. Then

$$\begin{aligned}
\lambda_n^p &= p \int_0^{\lambda_n} x^{p-1} dx \\
&= p((\int_0^{\lambda_1} + \sum_{k=1}^{n-1} \int_{\lambda_k}^{\lambda_{k+1}}) x^{p-1} dx) \\
&\leq p(\lambda_1 \lambda_1^{p-1} + (\lambda_2 - \lambda_1)\lambda_2^{p-1} + ... + (\lambda_n - \lambda_{n-1})\lambda_n^{p-1} \\
&= p(a_1 a_1^{p-1} + a_2(a_1 + a_2)^{p-1} + ... + a_n(\sum_{i=0}^{n} a_i)^{p-1}.
\end{aligned}$$

It is clear that$\sum_{n\geq 0}(\sum_{k=0}^{n} e_{nk}) = \sum_{k\geq 0}(\sum_{n\geq k} e_{nk})$. Thus the proof is complete. □

Proposition 3.7. *Let $B = [b_{nk}]$ ba an infinite lower matrix on $H^p(\beta)$ such that*

$$M_1 = \sup_{k\geq 0}(\sum_{n\geq k}^{\infty}|b_{nk}|\frac{\beta(n)^p}{\beta(k)^p}) < \infty,$$

and $|b_{ni}| \leq |b_{ki}|$ for all $0 \leq i \leq k \leq n$. Then B is a bounded operator on $H^p(\beta)$.

Proof. Let $f(z) = \sum_{n=0}^{\infty}\hat{f}(n)z^n \in H^p(\beta)$, then

$$(Bf)(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}b_{nk}\hat{f}(k))z^n.$$

By the above lemma we have

$$\begin{aligned}
\|B(f)\|^p &= \sum_{n\geq 0}|\sum_{k=0}^{n}b_{nk}\hat{f}(k)|^p\beta(n)^p \\
&\leq \sum_{n\geq 0}(\sum_{k=0}^{n}|b_{nk}||\hat{f}(k)|\beta(n))^p \\
&\leq p\sum_{n\geq 0}\sum_{k=0}^{n}|b_{nk}||\hat{f}(k)|\beta(n)^p(\sum_{i=0}^{k}|b_{ni}||\hat{f}(i)|)^{p-1} \\
&= p\sum_{n\geq 0}\sum_{k=0}^{n}|b_{nk}||\hat{f}(k)|\beta(n)^p|B\hat{(f)}(k)|^{p-1} \\
&\leq p\sum_{k\geq 0}|\hat{f}(k)|\beta(k)^p|B\hat{(f)}(k)|^{p-1}(\sum_{n=k}^{\infty}|b_{nk}|\frac{\beta(n)^p}{\beta(k)^p}) \\
&\leq pM_1\|B(f)\|^{\frac{p}{q}}\|f\|.
\end{aligned}$$

So B is bounded on $H^p(\beta)$. Now the proof is complete. □

4. CONCLUSION

The aim of this paper is cyclicity of weighted mean matrix operator on $H^p(\beta)$.

Theorem 4.1. *If $c_n = \frac{a_n\beta(n)^p}{A_n}$ is an increasing sequence with $\lim_{n\to\infty}\frac{a_n\beta(n)^p}{A_n} = 1$ and if $\lim_{n\to\infty}\frac{\beta(n+1)}{\beta(n)} < 1$. Then the weighted mean matrix operator A is cyclic.*

Proof. Let $k \in \aleph$ be fix. Then the function $f(x) = \frac{1-x}{\frac{x}{c_k}-1}$ is uniformly continuous and decreasing on $[c_k + \delta_0, 1]$ where $0 < \delta_0$. We have $\lim_{n\to\infty} f(c_n) = 0$ as $\lim_{n\to\infty} c_n = 1$. Without loss of generality we can assume that $\frac{f(c_{n+1})}{f(c_n)} < 1$ for $n > k$. Consider the matrix

P for which it's columns are eigenvector of A. If $0 \leq i \leq k \leq n$, then

$$\begin{aligned} |p_{ni}| &= \frac{\beta(n)^p}{\beta(i)^p} \prod_{j=1}^{n-i} \frac{1-c_{j+i}}{(\frac{c_{j+i}}{c_j})-1} \\ &\leq \frac{\beta(k)^p}{\beta(i)^p} \prod_{j=1}^{k-i} \frac{1-c_{j+i}}{(\frac{c_{j+i}}{c_j})-1} \\ &\leq p \sum_{n\geq 0} \sum_{k=0}^{n} |b_{nk}||\hat{f}(k)|\beta(n)^p (\sum_{i=0}^{k} |b_{ni}||\hat{f}(i)|)^{p-1} \\ &= |p_{ki}|. \end{aligned}$$

Since

$$\frac{p_{(n+1)k}}{p_{nk}} = \frac{\beta(n+1)^p}{\beta(n)^p} \frac{1-c_{n+1}}{\frac{c_{n+1}}{c_k}-1}$$

for all $n \geq k$, it follows that $\sum_{n\geq k} |p_{nk}| < \infty$. Under the assumptions of Proposition 3.7, the matrix P is bounded. By Proposition 3.5 we get $A = PDP^{-1}$, and hence bounded. Theorem 3.2 implies that there exists $f_1(z) \in H^p(\beta)$ such that

$$span\{D^k(f_1(z)); k \geq 0\} = H^p(\beta).$$

If $f(z) \in H^p(\beta)$ is any formal power series, then there is a $f'(z) \in H^p(\beta)$ with $P(f'(z)) = f_(z)$ and a sequence $h_n(z) \in span\{D^k(f_1(z)); k \geq 0\}$ with $h_n(z) \to f'(z)$ as $n \to \infty$. Since P^{-1} is surjective, it follows that there exists $f_2(z) \in H^p(\beta)$ with $P^{-1}(f_2(z)) = f_1(z)$ and $P(D^n(P^{-1}(f_2(z)) \to P(f'(z))$. Thus $A^n(f_2(z)) \to P(f'(z)) = f(z)$ as $n \to \infty$. Now the proof is complete. □

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References

[1] F. Bayart and E. Matheron, *Dynamics of linear operators*, Cambridge, 2009.

[2] A. Bucur, *On the fine spectra of some averaging operators*, General Mathmematics. **9**, (2001) 23-35.

[3] F. P. Cass and B. E. Rhoades, *Mercerian theorems via spectral theory*, Pacific J. Math. **73**, (1977) 63-71.

[4] C. Coskun, *A special Norlund mean and its eigenvalues*, commum. Fac. Sci. Univ. Ank. Series A1, **52**, (2003) 27-31.

[5] C. Coskun, *The spectra and fine spectra for P-Cesaro operators*, Tr. J. of Mathematices, **21**, (1997) 207-212.

[6] I. Deters and S. M. Seubrt, *Cyclic vectors of diagonal operatros on the space of functins analyic on a disk*, J. Math. Anal. Appl. **334**, (2007) 1209-1219.

[7] B. E. Rhoades and M. Yildirim, *Spectra for factoriable matrices on l^p*, Integral Equations and Operator Theory, **55**, (2005) 111-126.

[8] B. E. Rhoades, *The fine spectra for weigthed mean operators in $B(l^p)$*, Integral Equations and Operator Theory, **12**, (1989) 82-98.

[9] B. E. Rhoades, *The fine spectra for weighted mean operators*, Pacific J. Math. **104**, (1983) 219-230.

[10] E. Pazouki, *Boundedness, compactness and cyclicity on weighted Hardy spaces*, Ph.D. Thesis, Payame Noor University, 2013.

[11] E. Pazouki and B. Yousefi, *The spectra and eigenvectors for the weieghted Mean Matrix operator*, Arab J. of Math. **21**, (2015) 179-193.

[12] E. Pazouki and B. Yousefi, *Compactness of the weighted mean operator matrix on weighted Hardy spaces*, International Journal of Applied Mathematics, **25**, (2012) 495-502.

[13] E. Pazouki and B. Yousefi, *Weighted mean operator matrix on sequence spaces*, International Journal of Applied Mathematics, **25**, (2012) 515-524.

[14] B.Yousefi, *On the space* $l^p(\beta)$, Rendiconti Del Cirecolo Matematico Di Palermo, Serie II, Tomo XLIX, (2000), 115-120.

[15] B. Yousefi, *Strictly cyclic algebra of operators acting on Banach spaces* $H^p(\beta)$, Czechoslovak Mathematical Journal,**54**, (2004)261-266.

[16] B. Yousefi and A. Farrokhinia, *On the hereditarily hypercyclic vectors*, Journal of the Korean Mathematical Society,**43**, (2006) 1219-1229.

[17] B. Yousefi and E. Pazokinejad, *Cyclicity on the weighted Hardy spaces*, International Journal of Nonlinear Science, **11**, (2011) 459-463.

[18] B. Yousefi and E. Pazokinejad, *Boundedness and compactness of the mean operator matrix on the weighted Hardy spaces,* International Scholarly Research Network, Math. Analysis, Vol. 2012, Article ID945741, 13 pages.