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An Asymptotic Method with Applications to Nonlinear Coupled Partial Differential Equations

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Abstract. In this paper the approximate solution of coupled burgers equations is obtained by asymptotic MOHAM. The obtained results are compared through tables and graphs. The convergence and residual is plotted.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: MOHAM, HPM, Exact, coupled burgers and coupled mKdV equations..

1. Introduction

The nonlinear problems (NP) arising in science and engineering are treated by various methods. Some software packages like HOMPACK90 and POLSYSPLP [1-3] are used. The exact solution of NP cannot be found easily as every NPDEs have infinitely many solutions. The analytical and exact solution of such problems are either not available in the literature or may be found by using transformation based on, invariance group analysis method [4], Lie infinitesimal criterion [5], the symbolic computation [6] and Backlund transformation [7]. All these methods reduced the complex equations into simple equations by using the transformation. In the literate most of the methods like Variational iterative method (VIM) [8], Adomian decomposition method (ADM) [9], Differential transform method (DTM) [10] and Homotopy perturbation method (HPM) [11] have been used for the solution of weakly NPDEs and few for strongly NPDEs. To tackle the strongly NPDEs the perturbation method were introduced [12-14]. These methods contain a small parameter which cannot be found easily. New analytic methods such as artificial parameters method [15], Homotopy analysis method (HAM) [16] and Homotopy perturbation method (HPM) [11] were introduced. These methods combined the homotopy with the perturbation techniques. Some applications of this method are given [17-27].T.Nabil et al.[28] has investigated MHD flow and heat transfer for a peristaltic motion of Carreau fluid embedded a Porous Medium.Baloch and Solangi [29] has worked on the simulation of rotational flows in cylindrical. Mahmood et all.[30] investigated the magnetohydrodynamic flow due to a stretching surface in rotating fluid.The other relevant work can be seen in [31-36]

2. Fundamental Mathematical Theory of Multistage OHAM

$$\begin{aligned} \mathcal{A}_1(\xi) + \delta_1 &= 0 \\ \mathcal{A}_2(\zeta) + \delta_2 &= 0, \quad x \in \Omega, \end{aligned} \tag{2. 1}$$

$$\begin{aligned} \beta_1(\frac{\partial \xi}{\partial x}) &= 0, \\ \beta_2(\frac{\partial \zeta}{\partial x}) &= 0, \quad x \in \Gamma. \end{aligned} \tag{2. 2}$$

Where $\mathcal{A}_1, \mathcal{A}_2$ are differential operators,ζ, ξ are unknown functions, Γ is the boundary of Ω and δ_1, δ_2 are known analytic functions can be divided into two parts

$$\begin{aligned} \mathcal{A}_1 &= \mathcal{L}_1 + \mathcal{N}_1, \\ \mathcal{A}_2 &= \mathcal{L}_2 + \mathcal{N}_2. \end{aligned} \tag{2. 3}$$

The homotopy $\mu(x, t; l) : \Omega \times [0, \ \ 1] \to \Re, \ \ \nu(x, t; l) : \Psi \times [0, \ \ 1] \to \Re$ Satisfying

$$\begin{aligned} H(\mu, l) &= (1 - l)\mathcal{L}_1 + \delta_1 - U_1(l)\mathcal{A}_1 + \delta_1 = 0, \\ H(\nu, l) &= (1 - l)\mathcal{L}_2 + \delta_2 - U_2(l)\mathcal{A}_2 + \delta_2 = 0. \end{aligned} \tag{2. 4}$$

When l varies from 0 to 1 ,the solution varies from $\xi_0(x, t), \zeta_0(x, t)$ to $\xi(x, t), \zeta(x, t)$. For $l = 0$

$$\begin{aligned} \mathcal{L}_1(\xi_0, \zeta_0) + s_1(x, t) = 0, \ \ \beta_1(\xi_0, \frac{\partial \xi_0}{\partial x}) = 0, \\ \mathcal{L}_2(\xi_0, \zeta_0) + s_2(x, t) = 0, \ \ \beta_2(\xi_0, \frac{\partial \xi_0}{\partial x}) = 0. \end{aligned} \tag{2. 5}$$

We choose auxiliary functions in the form

$$\begin{aligned} U_1(l) = lD_{11} + l^2 D_{12} + l^3 D_{13} + ... + l^m D_{1m}, \\ U_2(l) = lD_{21} + l^2 D_{22} + l^3 D_{23} + ... + l^m D_{2m}. \end{aligned} \tag{2. 6}$$

To get the approximate solutions, we expand $\mu(D_{1i})$,$\nu(D_{2i})$ by Taylor's series

$$\begin{aligned} \mu(D_{1i}) &= \xi_0 + \sum_{k \geq 1} \xi_k(D_{1i}) l_k, \\ \nu(D_{1i}) &= \zeta_0 + \sum_{k \geq 1} \zeta_k(D_{2i)} l_k, \end{aligned} \tag{2. 7}$$

where $k = l = n = i = 1, 2, 3, ..$ Now substituting Eq. $(2.6) - (2.7)$ into Eq. (2.4) and equating the coefficient of like powers of l , we obtain Zeroth order system (2.5), the first and second order systems $(2.7) - (2.9)$ respectively and the general form (2.10):

$$\begin{aligned} \mathcal{L}_1(\xi_1) - \mathcal{L}_1\xi_0, \zeta_0) = D_{11}(\mathcal{L}_1(\xi_0, \zeta_0) + \mathcal{N}_1(\xi_0, \zeta_0))\beta_1(\xi_1, \frac{\partial \xi_1}{\partial x}) = 0, \\ \mathcal{L}_2(\zeta_1) - \mathcal{L}_2\xi_0, \zeta_0) = D_{21}(\mathcal{L}_2(\xi_0, \zeta_0) + \mathcal{N}_2(\xi_0, \zeta_0)), \beta_1(\zeta_1, \frac{\partial \zeta_1}{\partial x}) = 0, \end{aligned} \tag{2. 8}$$

$$\begin{aligned} \mathcal{L}_1(\xi_2) - \mathcal{L}_1(\xi_1 = D_{11}(\mathcal{L}_1(\zeta_1, \xi_1) + \mathcal{N}_1(\zeta_1, \xi_1)) + D_{12}(\mathcal{L}_2(\zeta_0, \xi_0) + \mathcal{N}_2(\zeta_0, \xi_0)), \\ \mathcal{L}_2(\zeta_2) - \mathcal{L}_1\zeta_2 = D_{21}(\mathcal{L}_1(\zeta_1, \xi_1) + \mathcal{N}_2(\zeta_1, \xi_1)) + D_{22}(\mathcal{L}_2(\zeta_0, \xi_0) + \mathcal{N}_2(\zeta_0, \xi_0)), \\ \beta_1(\xi_1, \frac{\partial \xi_1}{\partial x}) = 0, \quad \beta_2(\zeta_2, \frac{\partial \zeta_2}{\partial x}) = 0, \end{aligned} \tag{2. 9}$$

$$\begin{aligned} \mathcal{L}_1(\xi_k) - \mathcal{L}_1(\xi_{k-1}, \zeta_{k-1}) = \sum_{i=1}^{k} D_{1i}(\mathcal{L}_1(\xi_{k-1}, \zeta_{k-1}) + \mathcal{N}_1(\xi_{k-1}, \zeta_{k-1}), \\ \beta_1(\xi_k, \frac{\partial \xi_k}{\partial x}) = 0, \\ \mathcal{L}_2(\zeta_k) - \mathcal{L}_2(\xi_{k-1}, \zeta_{k-1}) = \sum_{i=1}^{k} D_{2i}(\mathcal{L}_2(\xi_{k-1}, \zeta_{k-1}) + \mathcal{N}_2(\xi_{k-1}, \zeta_{k-1}), \\ \beta_2(\zeta_k, \frac{\partial \zeta_k}{\partial x}) = 0. \end{aligned} \tag{2. 10}$$

The residual is given as

$$\begin{aligned} R_1(D_{1i}) = \mathcal{L}_1(\mu^r D_{1i}) + \delta_1 + \mathcal{N}_1(\mu^r D_{1i}), \\ R_2(D_{2i}) = \mathcal{L}_2(\nu^r D_{2i}) + \delta_2 + \mathcal{N}_2(\mu^r D_{2i}). \end{aligned} \tag{2. 11}$$

One can apply the Method of Least Squares as under

$$\begin{aligned} \kappa_1(D_{1i}) = \mathcal{L}_2(\nu^r D_{1i}) + \int_0^t \int_\Omega R_1^1(D_{1i}) dx dt \\ \kappa_2(D_{2i}) = \mathcal{L}_2(\nu^r D_{2i}) + \int_0^t \int_\Omega R_1^2(D_{2i}) dx dt \end{aligned} \tag{2. 12}$$

and

$$\frac{\partial \kappa_1}{\partial D_{1i}} = 0, \frac{\partial \kappa_1}{\partial D_{12}} = 0, ...\frac{\partial \kappa_1}{\partial D_{1m}} = 0, \ \frac{\partial \kappa_2}{\partial D_{2i}} = 0, \frac{\partial \kappa_2}{\partial D_{22}} = 0, ...\frac{\partial \kappa_2}{\partial D_{2m}} = 0. \tag{2. 13}$$

Therefore, the approximate analytic solution will be

$$\begin{aligned} \xi_k = \{\xi_1, \ \ t_0 \leq x \leq t_l \\ .... \\ \xi_N, \ \ t_{N-1} \leq t \leq T\} \\ \zeta_k = \{\zeta_1, \ \ t_0 \leq x \leq t_l \\ .... \\ \zeta_N, \ \ t_{N-1} \leq t \leq T\} \end{aligned} \tag{2. 14}$$

3. APPLICATION OF $MOHAM$ TO NONLINEAR COUPLED BURGER'S EQUATIONS

Consider the nonlinear coupled Burger's equations with initial conditions [26]

$$\begin{aligned}\frac{\partial\xi}{\partial t} &= \frac{1}{2}\frac{\partial^3\xi}{\partial x^3} - 3\xi^2\frac{\partial^2\xi}{\partial x^2} + \frac{3}{2}\frac{\partial\xi}{\partial x}(\xi\zeta) - 3\lambda\frac{\partial\xi}{\partial x},\\ \frac{\partial\zeta}{\partial t} &= -\frac{\partial^3\zeta}{\partial x^3} - 3\zeta\frac{\partial\zeta}{\partial x} - 3\frac{\partial^2\zeta}{\partial x^2} + 3\frac{\partial\zeta}{\partial x}\frac{\partial\xi}{\partial x} + 3\xi^2\frac{\partial\zeta}{\partial x} + 3\lambda\frac{\partial\zeta}{\partial x},\end{aligned} \tag{3. 15}$$

with initial conditions

$$\begin{aligned}\zeta(x,0) &= \frac{b}{2k} + ktanh(kx),\\ \varsigma(x,0) &= \frac{\lambda}{2}(1+\frac{k}{b}) + ktanh(kx).\end{aligned} \tag{3. 16}$$

The exact solutions of Eq. (3.1) are [26]

$$\begin{aligned}\zeta(x,t) &= \frac{b}{2k} + ktanh(k\tau),\\ \varsigma(x,t) &= \frac{\lambda}{2}(1+\frac{k}{b}) + ktanh(k\tau).\end{aligned} \tag{3. 17}$$

Where $\tau = x + \frac{1}{4}(-4k^2 - 6\lambda + \frac{6k\lambda}{b} + \frac{3b^2}{k^2})$. Applying the technique discussed in 2

$$\begin{aligned}(1-l)\frac{\partial\xi}{\partial t} - U_1(l)\frac{1}{2}\frac{\partial^3\xi}{\partial x^3} - 3\xi^2\frac{\partial^2\xi}{\partial x^2} + \frac{3}{2}\frac{\partial\xi}{\partial x}(\xi\zeta) - 3\lambda\frac{\partial\xi}{\partial x} &= 0,\\ (1-l)\frac{\partial\zeta}{\partial t} - U_1(l)\frac{\partial^3\zeta}{\partial x^3} - 3\zeta\frac{\partial\zeta}{\partial x} - 3\frac{\partial^2\zeta}{\partial x^2} + 3\frac{\partial\zeta}{\partial x}\frac{\partial\xi}{\partial x} + 3\xi^2\frac{\partial\zeta}{\partial x} + 3\lambda\frac{\partial\zeta}{\partial x} &= 0.\end{aligned} \tag{3. 18}$$

We consider

$$\begin{aligned}\xi &= \xi_0 + l\xi_1 + l^2\xi_2,\\ \zeta &= \zeta_0 + l\zeta_1 + l^2\zeta_2,\\ U_1(l) &= lC_{11} + l^2C_{12},\\ U_2(l) &= lC_{21} + l^2C_{22}.\end{aligned} \tag{3. 19}$$

$$\begin{aligned}\frac{\partial\xi_0}{\partial t} &= 0\\ \frac{\partial\zeta_0}{\partial t} &= 0,\end{aligned} \tag{3. 20}$$

with initial conditions

$$\begin{aligned}\xi_0(x,0) &= \frac{b}{2k} + ktanh(kx),\\ \zeta_0(x,0) &= \frac{\lambda}{2}(1+\frac{k}{b}) + ktanh(kx).\end{aligned} \tag{3. 21}$$

Its solution

$$\begin{aligned}\xi_0(x,0) &= 5 + 0.333333333tanh(0.333333333333x),\\ \zeta_0(x,0) &= 0.06666666667 + tanh(0.3333333x),\end{aligned} \tag{3. 22}$$

$$\frac{\partial \xi_1}{\partial t} = -C_{11}\frac{1}{2}\frac{\partial^3 \xi_0}{\partial x^3} - 3C_{11}\xi_0^2\frac{\partial^2 \xi_0}{\partial x^2} + \frac{3}{2}C_{11}\frac{\partial \xi_0}{\partial x}(\xi_0\zeta_0) - 3C_{11}\lambda\frac{\partial \xi_0}{\partial x}, \tag{3. 23}$$

$$\frac{\partial \zeta_1}{\partial t} = -C_{21}\frac{\partial^3 \zeta_0}{\partial x^3} - 3C_{21}\zeta_0\frac{\partial \zeta_0}{\partial x} - 3C_{21}\frac{\partial^2 \zeta_0}{\partial x^2} + 3C_{21}\frac{\partial \zeta_0}{\partial x}\frac{\partial \xi_0}{\partial x} + 3C_{21}\xi_0^2\frac{\partial \zeta_0}{\partial x} + 3C_{21}\lambda\frac{\partial \zeta_0}{\partial x}, \tag{3. 24}$$

$$\xi_1(x,0), \zeta_1(x,0) = 0. \tag{3. 25}$$

Its solution

$$\begin{aligned}
&\xi_1(x,t,C_{11}) = tC_{11}[(8.033333sech^2(0.3333333x) + 0.013456790123sech^4(0.3333333x))\\
&+ (1.088888sech^2(0.3333333x)tanh(0.33333333x) + 0.01234567901\\
&sech^2(0.3333333x)tanh^2(0.33333333x)]),\\
&\xi_1(x,t,C_{11}) = tC_{11}[[(1.6733333sech^2(0.3333333x) - 0.012345678901sech^4(0.3333333x))\\
&(0.217777sech^4(0.3333333x)tanh(0.33333333x) + 0.01728395061\\
&sech^2(0.3333333x)tanh^2(0.33333333x)]
\end{aligned} \tag{3. 26}$$

Adding Eqs. (3.9) and (3.11), we obtain

$$\xi_1(x,0), \zeta_1(x,0) = 0. \tag{3. 27}$$

Its solution

$$\begin{aligned}
&\xi_1(x,t,C_{11}) = 5 + 0.333333333333tanh(0.333333333333x) + tC_{11}[(8.033333\\
&sech^2(0.3333333x) + 0.013456790123sech^4(0.3333333x)) + (1.088888\\
&sech^2(0.3333333x)tanh(0.33333333x)\\
&+ 0.01234567901sech^2(0.3333333x)tanh^2(0.33333333x)]),\\
&\xi_1(x,t,C_{11}) = 0.066666666667 + tanh(0.333333333333x)\\
&- tC_{11}[[(1.6733333sech^2(0.3333333x) - 0.012345678901sech^4(0.3333333x))\\
&(0.217777sech^4(0.3333333x)tanh(0.33333333x) + 0.01728395061\\
&sech^2(0.3333333x)tanh^2(0.33333333x)].
\end{aligned} \tag{3. 28}$$

For the computation of the constants C_{11}, C_{12} , using (3.12) in (3.1) and applying the method of least square, we get

$$\begin{aligned}
&\xi(x,t) = [3 + 1.00043 \times 10^{-10} sech^4(0.3333333x) + 0.3333333 tanh(0.333333x),\\
&0 \leq x \leq 0.5\\
&+ 1.13412 \times 10^{-10} sech^2(0.3333333x) + (0.12626 + tanh(0.333333x)\\
&(80 + tanh(0.333333x)))2 + 0.13400 \times 10^{-10} + 10 sech^4(0.3333333x)\\
&+ 0.3333333 tanh(0.333333x), \quad 0 \leq x \leq 0.5\\
&+ 0.13431 \times 10^{-10} sech^4(0.3333333x) + (0.12626 + tanh(0.333333x)\\
&(0.0737 + tanh(0.333333x)))],\\
&\zeta(x,t) = 0.0666667 - 1.20113 \times 10^{-9} sech^4(0.3333333x)\\
&+ tanh(0.3333333333x) + sech^2(0.33333333x)2.0134 \times 10^{-10} + 2 \times 10^{-8}\\
&+ 2 \times 10^{-9} tanh(0.333333x)] tanh(0.3333333x), \quad 0 \leq x \leq 0.5\\
&0.0000667 - 0.30513 \times 10^{-9} sech^4(0.3333333x) + tanh(0.3333333333x) + sech^2\\
&(0.33333333x)[0.0310 \times 10^{-7} + 0.10863 \times 10^{-9} tanh(0.333333x)]\\
&tanh(0.3333333x), \quad 0 \leq x \leq 0.5].
\end{aligned} \tag{3. 29}$$

4. RESULTS AND DISCUSSIONS AND CONCLUSION

In Table 1, the MOHAM results for the coupled burgers equations are compared with exact and HPM solutions for different values of and at . In Table 2-3 the absolute errors of the coupled burgers equations corresponding to exact solutions are given. Figs. 4-6 show the 2D comparison of the approximate solutions with exact solutions. The convergence of MOHAM is presented in Figs. 7-8 and the residuals are given in Figs.9-10. MOHAM is simple in applicability, contain less computational work and fastly convergent through auxiliary constants.

5. CONCLUSION

The convergence of MOHAM is presented in Figs. 7-8 and the residuals are given in Figs.9-10. MOHAM is simple in applicability, contain less computational work and fastly convergent through auxiliary constants.

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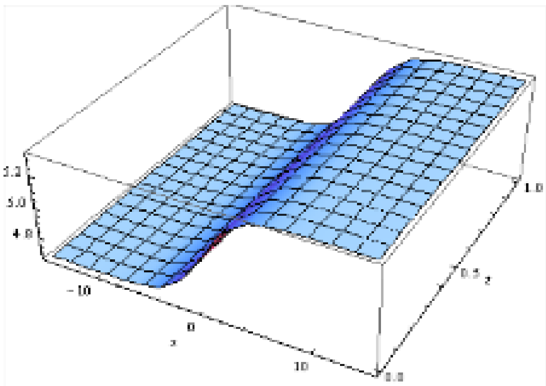
TABLE 1. Absolute error of MOHAM solution $\zeta(x,t)$ corresponding to the Exact Solution

x	$t=0.2$	$t=0.1$	$t=0.01$	$t=0.001$	$t=0.0001$
-15	1.18888×10^{-5}	1.18888×10^{-5}	6.68012×10^{-5}	1.02594×10^{-5}	1.02857×10^{-6},
-12	1.39349×10^{-4}	1.39349×10^{-4}	4.93661×10^{-4}	7.57713×10^{-4}	7.59653×10^{-5}
-9	1.01076×10^{-3}	1.01076×10^{-3}	3.65102×10^{-4}	5.57916×10^{-3}	5.59337×10^{-4}
-6	6.89963×10^{-3}	6.89963×10^{-3}	2.71230×10^{-3}	4.01754×10^{-2}	4.02739×10^{-3}
-3	2.78361×10^{-2}	2.78361×10^{-2}	1.97907×10^{-2}	5.56286×10^{-2}	2.47923×10^{-3}
0	1.11107×10^{-2}	1.11107×10^{-2}	7.63359×10^{-1}	2.47463×10^{-1}	6.66665×10^{-2}
3	2.47383×10^{-2}	2.47383×10^{-2}	4.26791×10^{-2}	6.66663×10^{-2}	3.12030×10^{-3}
6	4.88596×10^{-2}	4.88596×10^{-2}	7.62229×10^{-2}	3.12482×10^{-2}	5.39250×10^{-3}
9	6.97165×10^{-3}	6.97165×10^{-3}	1.07375×10^{-3}	5.40207×10^{-3}	7.56104×10^{-4}
12	9.50825×10^{-4}	9.50825×10^{-4}	1.46114×10^{-4}	7.57481×10^{-4}	1.02824×10^{-5}
15	1.28738×10^{-5}	1.28738×10^{-5}	1.97890×10^{-5}	1.39503×10^{-5}	1.39248×10^{-6}

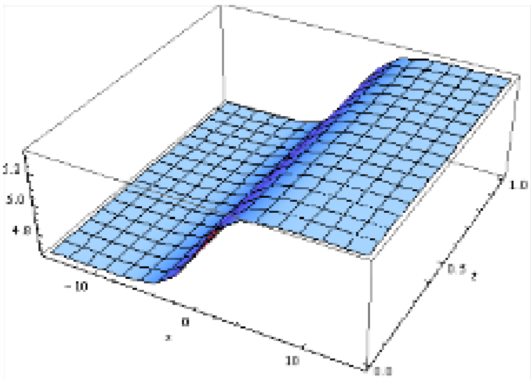


(a) 3D Approximate solution.

FIGURE 1. 3D Approximate solution of $\xi(x,t)$ at $t=0.1$.

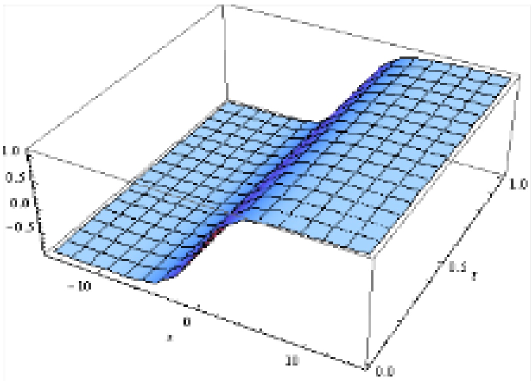
TABLE 2. Absolute error of MOHAM solution $\xi(x,t)$ corresponding to the Exact Solution

x	$t=0.2$	$t=0.1$	$t=0.01$	$t=0.001$	$t=0.0001$
-15	6.10573×10^{-5}	6.10573×10^{-5}	1.56495×10^{-5}	2.85825×10^{-5}	2.86617×10^{-6},
-12	4.50488×10^{-4}	4.50488×10^{-4}	1.15656×10^{-4}	2.11093×10^{-4}	2.11675×10^{-5}
-9	3.29250×10^{-3}	3.29250×10^{-3}	8.55714×10^{-4}	1.55395×10^{-3}	1.55822×10^{-4}
-6	2.24608×10^{-3}	2.24608×10^{-3}	6.37501×10^{-3}	1.11717×10^{-2}	1.11201×10^{-3}
-3	9.56286×10^{-2}	9.56286×10^{-2}	4.72517×10^{-2}	6.81789×10^{-2}	6.83167×10^{-3}
0	4.98356×10^{-2}	4.98356×10^{-2}	1.90703×10^{-1}	1.80846×10^{-1}	1.80843×10^{-2}
3	4.98356×10^{-2}	5.76831×10^{-2}	1.11505×10^{-2}	8.54788×10^{-2}	8.53431×10^{-3}
6	1.19566×10^{-2}	1.19566×10^{-2}	2.01656×10^{-2}	1.48555×10^{-2}	1.48268×10^{-3}
9	1.71650×10^{-3}	1.71650×10^{-3}	2.84625×10^{-3}	2.08494×10^{-3}	2.08081×10^{-4}
12	2.34172×10^{-4}	2.34172×10^{-4}	3.87416×10^{-4}	2.83573×10^{-4}	2.83009×10^{-5}
15	3.17261×10^{-5}	3.17261×10^{-5}	5.24719×10^{-5}	3.84033×10^{-5}	3.83269×10^{-6}



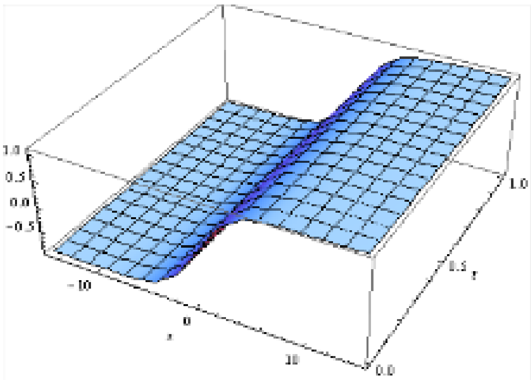
(a) 3D exact solution .

FIGURE 2. 3D Exact solution of $\xi(x,t)$ at $t=0.1$.



(a) 3D Approximate solution .

FIGURE 3. 3D Approximate solution of $\xi(x,t)$ at $t = 0.1$.



(a) 3D exact solution .

FIGURE 4. 3D Exact solution of $\xi(x,t)$ at $t = 0.1$.

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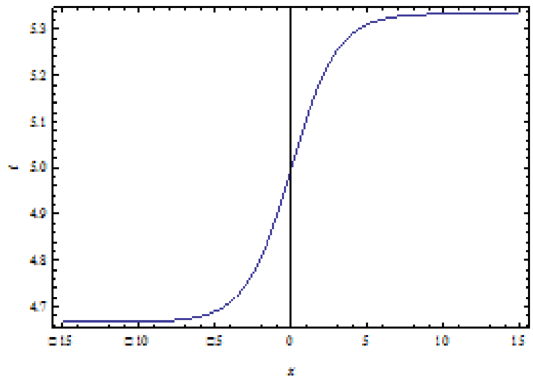
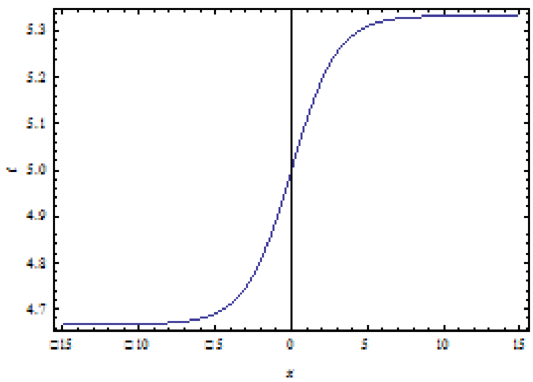


(a) 3D Approximate solution .

FIGURE 5. 3D Approximate solution of $\xi(x,t)$ at $t = 0.1$.



(a) 3D exact solution .

FIGURE 6. 3D Exact solution of $\xi(x,t)$ at $t = 0.1$.

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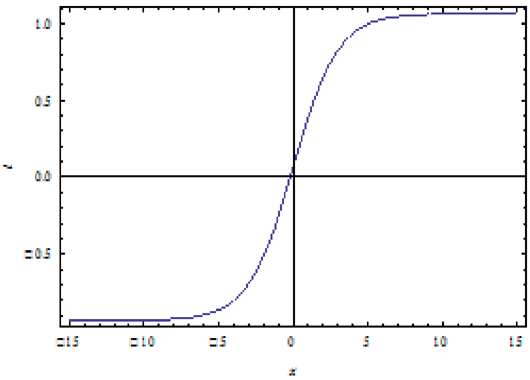
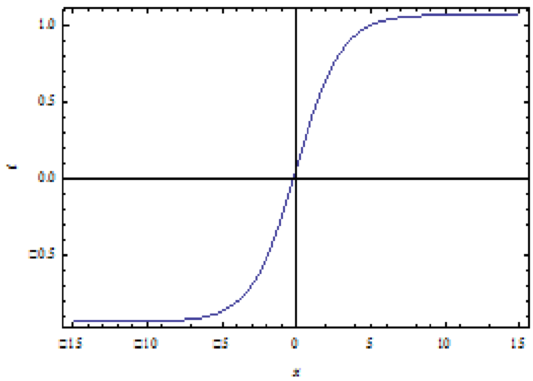


(a) 3D Approximate solution .

FIGURE 7. 3D Approximate solution of $\xi(x, t)$ at $t = 0.1$.



(a) 3D exact solution .

FIGURE 8. 3D Exact solution of $\xi(x, t)$ at $t = 0.1$.

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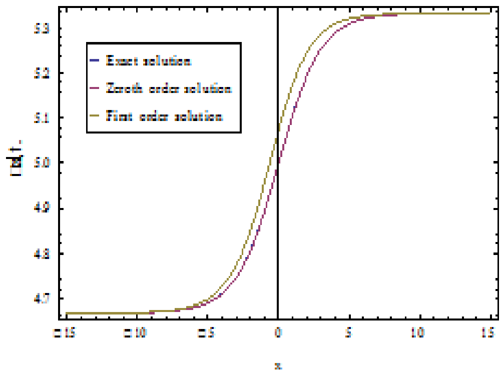
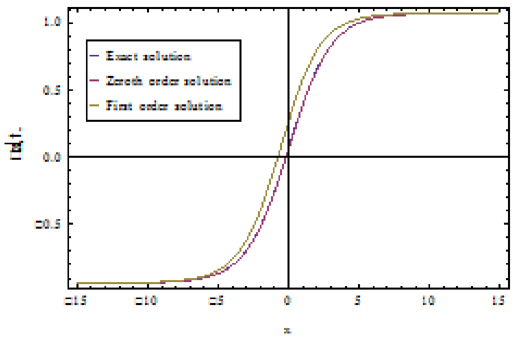


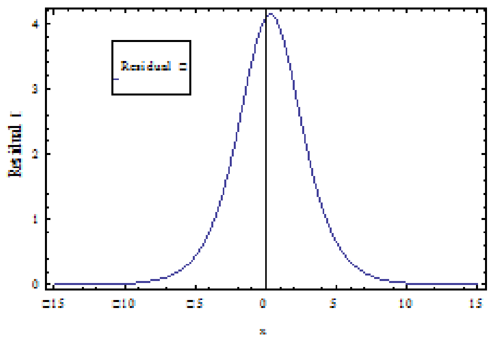
(a) 3D Approximate solution .

FIGURE 9. 3D Approximate solution of $\xi(x,t)$ at $t = 0.1$.



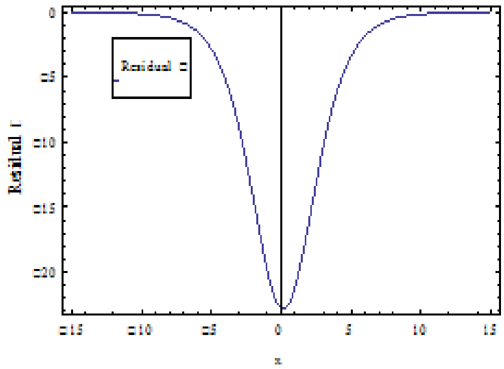
(a) 3D exact solution .

FIGURE 10. 3D Exact solution of $\xi(x,t)$ at $t = 0.1$.



(a) 3D Approximate solution .

FIGURE 11. 3D Approximate solution of $\xi(x,t)$ at $t = 0.1$.



(a) 3D exact solution .

FIGURE 12. 3D Exact solution of $\xi(x,t)$ at $t = 0.1$.

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