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(n, k)-Multiple Factorials with Applications

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Abstract. In this paper, we define (n, k) triple factorial and extend the definition up to a finite number of multi-factorials of the said type. We express the Pochhammer's symbol and hypergeometric functions involving these factorials. Also, we express some elementary functions in the form of $(n, k)!_r$ satisfying the classical results.

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1. INTRODUCTION

The factorial notation $(!)$ was introduced by Christian Kramp in 1808 for positive integers and is frequently used to compute the binomial coefficients. The relationship between classical gamma function and ordinary factorial is $\Gamma(n) = (n-1)!, n \in \mathbb{N}$. Also, gamma function is defined for all real numbers except $n = 0, -1, -2, \cdots$ Afterwards, the German mathematician Leo Pochhammer defined the shifted (rising) factorial, which was named as

Pochhammer's symbol and is given by (see [1, 14])

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1)\,, n\in\mathbb{N} \\ 1, \quad n=0\,,\alpha\neq 0. \end{cases}$$

It follows that $(1)_n = n!$ and very simply, for $n, m \in \mathbb{N}$, we can derive the expression for the rising factorial of a negative integer as

$$(-n)_m = \begin{cases} \frac{(-1)^m n!}{(n-m)!}, 1\leq m\leq n \\ 0, \quad m\geq n+1. \end{cases}$$

Also, Kono [8], gave the definition of double, triple and multi factorials as

$$n!! = \begin{cases} n(n-2)(n-4)\cdots 6\cdot 4\cdot 2 & \text{if } n \text{ is even} \\ n(n-2)(n-4)\cdots 5\cdot 3\cdot 1 & \text{if } n \text{ is odd} \\ 1, \quad \text{if } n=0,-1 \quad ; \quad (-n)!!=\infty, n\in\mathbb{N}-\{0,1\}, \end{cases}$$

$$n!!! = \begin{cases} n(n-3)(n-6)\cdots 9\cdot 6\cdot 3 & \text{if } n \text{ is of the form } 3n \\ n(n-3)(n-6)\cdots 8\cdot 5\cdot 2 & \text{if } n \text{ is of the form } (3n-1) \\ n(n-3)(n-6)\cdots 7\cdot 4\cdot 1 & \text{if } n \text{ is of the form } (3n-2) \\ 1, \quad \text{if } n=0,-1,-2 \quad ; \quad (-n)!!!=\infty, n\in\mathbb{N}-\{0,1,2\} \end{cases} \tag{1.1}$$

and !!! $\cdots$!(r-times), denoted by $!_r$, is given by

$$n!_r = \begin{cases} n(n-r)(n-2r)\cdots 3r\cdot 2r\cdot r & \text{if } n \text{ is of the form } rm \text{ for some } m \\ n(n-r)(n-2r)\cdots(2r-1)\cdot(r-1) & \text{if } n \text{ is like } (rm-1) \text{ for some } m \\ n(n-r)(n-2r)\cdots(2r-2)\cdot(r-2) & \text{if } n \text{ is like } (rm-2) \text{ for some } m \\ \vdots \\ n(n-r)(n-2r)\cdots[(r-(r-1)], \text{if } n \text{ is like } (rm-(r-1)) \text{ for some } m \\ 1, \quad \text{if } n=0,-1,-2,\cdots,(r-1) \quad ; \quad (-rn)!_r=\infty,\, n\in\mathbb{N}. \end{cases} \tag{1.2}$$

Diaz and Pariguan [3] introduced the generalized gamma k-function as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k>0, x\in\mathbb{C}\setminus k\mathbb{Z}^-$$

and also gave the properties of said function. The Γ_k is one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k \to 1$. The Γ_k is based on the repeated appearance of the expression of the form

$$\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)\ldots(\alpha+(n-1)k). \tag{1.3}$$

The function of the variable α given by the statement (1.3), denoted by $(\alpha)_{n,k}$, is called the Pochhammer k-symbol. Thus, we have

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)(\alpha+3k)\ldots(\alpha+(n-1)k)\,, n\in\mathbb{N},\, k>0 \\ 1, \quad n=0\,,\alpha\neq 0. \end{cases}$$

We obtain the usual Pochhammer's symbol $(\alpha)_n$ by taking $k=1$ which is given by

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+(n-1))\,, n\in\mathbb{N}, \\ 1, \quad n=0\,, \alpha\neq 0. \end{cases} \tag{1.4}$$

. The authors [2, 4] have discussed some properties involving special functions. Also, the researchers [5-6, 9-12] have worked on the generalized gamma, beta and k-function and discussed the following properties:

$$\Gamma_k(\alpha k) = k^{\alpha-1}\Gamma(\alpha),\ k>0, \alpha\in\mathbb{R},$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!,\ k>0,\ n\in\mathbb{N},$$

$$\Gamma_k((2n+1)\frac{k}{2}) = k^{\frac{2n-1}{2}}\frac{(2n)!\sqrt{\pi}}{2^n n!},\ k>0,\ n\in\mathbb{N},$$

$$\Gamma_k(k) = 1,$$

and

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}).$$

Recently, Mubeen and Rehman [13] defined the factorial function in terms of k-symbol, called (n,k)-factorial as

$$(n,k)! = nk(nk-k)(nk-2k)(nk-3k)\cdots 3k\cdot 2k\cdot k\,, n\in\mathbb{N}, k>0$$

and simplifying the right hand side of the above equation, we get a link between $(n,k)!$ and classical gamma function. Thus we have

$$(n,k)! = k^n n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1 = k^n n! = k^n\Gamma(n+1). \tag{1.5}$$

Using the above definition of $(n,k)!$, they introduced the following properties of the said factorial

$$(nk,k)! = k^n (nk)!,$$

$$(n+a,k)! = k^n (n+a)!\,,\ a\ \in\mathbb{R},\ n\ \in\mathbb{N},$$

$$[(n+b)k,k]! = k^n[(n+b)k]!\,,\ b\in\mathbb{R},\ n\ \in\mathbb{N},$$

and

$$(0,k)! = 1 \quad , \quad (-n,k)! = \infty\,,\ n\in\mathbb{N}\,, k>0$$

and also gave some results involving the gamma k-function in terms of (n,k)-factorial as

$$\Gamma_k(x) = \lim_{n\to\infty}\frac{(n,k)!(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k>0, x\in\mathbb{C}\setminus k\mathbb{Z}^-,$$

$$\Gamma_k(nk) = (n-1,k)!,\ k>0,\ n\in\mathbb{N},$$

$$\Gamma_k((2n+1)\frac{k}{2}) = \frac{(2n,k)!}{2^n(n,k)!}\sqrt{\frac{\pi}{k}},\ k>0,\ n\in\mathbb{N},$$

and

$$(n,k)! = \Gamma_k(nk+k) = k^n\Gamma(n+1),\ k>0,\ n\in\mathbb{N}.$$

Remarks: Taking $k=1$, we see that $(n,1) = n!$ and all the above results can be converted into their classical representations.

Mubeen and Rehman [13] also gave the definition of $(n,k)!!\cdot$ If n is even and $n \in \mathbb{N},\ k > 0$, then

$$(2n,k)!! = 2nk(2nk-2k)(2nk-4k)\cdots 4k\cdot 2k$$

$$= k^n 2n(2n-2)\cdots 4\cdot 2 = k^n(2n)!!$$

and if n is odd (say) of the form $2n-1$, $k>0$, then

$$(2n-1,k)!! = (2nk-k)(2nk-3k)\cdots 3k\cdot k$$

$$= k^n(2n-1)\cdots 3\cdot 1 = k^n(2n-1)!!$$

and

$$(n,k)!! = 1\,,\text{for } n = 0,-1 \qquad ; \qquad (-2n,k)!! = \infty\,,\ n\in\mathbb{N}.$$

They proved the following properties involving (n,k)-double factorial, classical gamma function $\Gamma(x)$ and gamma k-function $\Gamma_k(x)$

$$(n,k)!! \times (n-1,k)!! = (n,k)!,$$

$$(2n,k)!! = k^n 2^n n! = k^n(2n)!!,$$

$$(2n-1,k)!! = k^n 2^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} = 2^n \frac{\Gamma_k(nk+\frac{k}{2})}{\Gamma_k(\frac{k}{2})},$$

and

$$[-(2n+1),k]!! = (-1)^{-n}\frac{2}{(2n-1,k)!!}.$$

2. Main Results

Here, we introduce $(n,k)!!!$ and extend the definition up to a finite number of higher order factorials. Also, we prove some results involving gamma k-function and classical gamma function using the definitions of (n,k)-triple or higher order such factorials.

Definition (2.1): For $n\in\mathbb{N},\ k>0$, (n,k)-triple factorial is defined by

$$(n,k)!!! = \begin{cases} nk(nk-3k)(nk-6k)\cdots 9k\cdot 6k\cdot 3k & \text{if 3 divides } n \\ nk(nk-3k)(nk-6k)\cdots 8k\cdot 5k\cdot 2k & \text{if 3 divides } n+1 \\ nk(nk-3k)(nk-6k)\cdots 7k\cdot 4k\cdot k & \text{if 3 divides } n+2 \\ 1, \quad \text{if } n = 0,-1,-2 \quad ; \quad (-3n)!!! = \infty, n\in\mathbb{N}. \end{cases}$$

Using the relation (1.1), the above definition gives the results as

$$(3n,k)!!! = k^n(3n-3)(3n-6)\cdots 9\cdot 6\cdot 3 = k^n(3n)!!! = k^n 3^n n!, \tag{1.6}$$

$$(3n-1,k)!!! = k^n(3n-1)(3n-4)(3n-7)\cdots 8\cdot 5\cdot 2 = k^n(3n-1)!!! \tag{1.7}$$

and

$$(3n-2,k)!!! = k^n(3n-2)(3n-5)(3n-8)\cdots 7\cdot 4\cdot 1 = k^n(3n-2)!!!\cdot \tag{1.8}$$

From the above definition, we observe that

$$(1,k)!!! = k1!!! = k, \qquad (2,k)!!! = k2!!! = 2k, \qquad (3,k)!!! = k3!!! = 3k,$$

$$(4,k)!!! = 4k.k, \qquad (5,k)!!! = 5k.2k, \qquad (6,k)!!! = 6k.3k,$$

$$(7,k)!!! = 7k.4k.k, \qquad (8,k)!!! = 8k.5k.2k, \qquad (9,k)!!! = 9k.6k.3k \ \ldots$$

Lemma (2.2): If n denotes the natural number and $\Gamma(z)$ is the classical gamma function, then the following results hold (see [8]).

(1) $n!!! \times (n-1)!!! \times (n-2)!!! = n!$

(2) $(3n)!!! = 3^n \frac{\Gamma(n+\frac{3}{3})}{\Gamma(\frac{3}{3})} = 3^n \Gamma(n+1) = 3^n n!$

(3) $(3n-1)!!! = 3^n \frac{\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})}$

(4) $(3n-2)!!! = 3^n \frac{\Gamma(n+\frac{1}{3})}{\Gamma(\frac{1}{3})}$

(5) $[-(3n+1)]!!! \times [-(3n+2)]!!! = \frac{3}{(3n-1)!!! \times (3n-2)!!!}$

Theorem (2.3): For $n, \in \mathbb{N},\ k > 0$, classical gamma function $\Gamma(x)$ and gamma k-function $\Gamma_k(x)$, the following expressions involving (n,k)-triple factorial hold

(i): $(n,k)!!! \times (n-1,k)!!! \times (n-2,k)!!! = (n,k)!$

(ii): $(3n-1,k)!!! = k^n 3^n \frac{\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})} = 3^n \frac{\Gamma_k(nk+\frac{2k}{3})}{\Gamma_k(\frac{2k}{3})}$

(iii): $(3n-2,k)!!! = k^n 3^n \frac{\Gamma(n+\frac{1}{3})}{\Gamma(\frac{1}{3})} = 3^n \frac{\Gamma_k(nk+\frac{k}{3})}{\Gamma_k(\frac{k}{3})}$

(iv): $[-(3n+1),k]!!! \times [-(3n+2),k]!!! = \frac{3}{(3n-1,k)!!! \times (3n-2,k)!!!}$

(v): $(3n,k)!!! = k^n 3^n n! = k^n (3n)!!!\cdot$

Proof: We give the proofs of the above properties by using the definition of $(n,k)!!!$ along with the lemma (2.2).

(i). Multiplying the equations (1.6), (1.7) and (1.8), we see that

$$(3n,k)!!! \times (3n-1,k)!!! \times (3n-2,k)!!! = k^n(3n)!!! \times k^n(3n-1)!!! \times k^n(3n-2)!!!$$

and by the lemma (2.2(1)), we get

$$(3n,k)!!! \times (3n-1,k)!!! \times (3n-2,k)!!! = k^{3n}(3n)!\cdot$$

Replacing $3n$ by n and using the relation (1.5), we get the required proof.

(ii). From the equation (1.7), we have $(3n-1,k)!!! = k^n(3n-1)!!!$ and use of the lemma (2.2(3)) gives

$$(3n-1,k)!!! = k^n 3^n \frac{\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})}$$

and application of the relation $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$ provides the second part of the required result.

(iii). Similar result from the relations (1.8), $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$ and lemma (2.2(4)).

(iv). From the equations (1.7), (1.8) and the theorem (2.3(i)), we observe that

$$(3n-1,k)!!! \times (3n-2,k)!!! = (3n-1,k)!!! \times (3n-2,k)!!! \times \frac{(3n,k)!!!}{(3n,k)!!!} = \frac{(3n,k)!}{(3n,k)!!!}$$

which implies that

$$(3n-1,k)!!! \times (3n-2,k)!!! = \frac{k^{3n}(3n)!}{k^n 3^n n!} = \frac{k^{3n}\Gamma(1+3n)}{k^n 3^n \Gamma(1+n)}\cdot$$

and replacement of n by $-n$ gives

$$\Big(-(3n+1),k\Big)!!!\times\Big(-(3n+2),k\Big)!!!=\frac{k^{-2n}\Gamma(1-3n)}{3^{-n}\Gamma(1-n)}=\frac{3^n\Gamma\Big(-(3n-1)\Big)}{k^{2n}\Gamma\Big(-(n-1)\Big)}\cdot$$

By the singular point formula, $\frac{\Gamma\Big(-(3n-1)\Big)}{\Gamma\Big(-(n-1)\Big)}=(-1)^{(n-1)-(3n-1)}\frac{(n-1)!}{(3n-1)!}=\frac{(n-1)!}{(3n-1)!}$ (see [7]), the above equation becomes

$$\Big(-(3n+1),k\Big)!!!\times\Big(-(3n+2),k\Big)!!!$$
$$=\frac{3^n(n-1)!}{k^{2n}(3n-1)!}=\frac{3^n3n(n-1)!}{k^{2n}3n(3n-1)!}=\frac{3\cdot 3^nn!}{k^{2n}(3n)!}.$$

To convert into (n,k)-factorials, we multiply the numerator and denominator by k^n on R.H.S. and proceed as

$$\frac{3\cdot 3^nn!}{k^{2n}(3n)!}=\frac{3\cdot 3^nk^nn!}{k^{3n}(3n)!}=\frac{3\cdot 3^n(n,k)!}{(3n,k)!}$$

and application of the theorem (2.3(i)) and the relation (1.6) provides

$$=\frac{3\cdot 3^n(n,k)!}{(3n,k)!!!\times(3n-1,k)!!!\times(3n-2,k)!!!}$$
$$=\frac{3\cdot 3^n(n,k)!}{3^n(n,k)!\times(3n-1,k)!!!\times(3n-2,k)!!!}$$

which is equivalent to the desired proof.

(v). Obvious proof from the definition of $(n,k)!!!\cdot$

Remarks: From the parts (ii), (iii) and (v), using $n=0$, we have $(-1,k)!!!=(-2,k)!!!=(0,k)!!!=1$ and replacing n by $-n$ in (v), $(-3n)!!!=\infty$. Also, for $k=1$, we get the classical results [7].

Definition (2.4): For $k>0$ and n,r are natural numbers with $r\leq n$, if $!!!\cdots!$ r-times is denoted by $!_r$, then we define $(n,k)!_r$ as

$$(n,k)!_r=\begin{cases} nk(nk-rk)(nk-2rk)\cdots 3rk\cdot 2rk\cdot rk & \text{if } r \text{ divides } n\\ nk(nk-rk)\cdots(2rk-k)\cdot(rk-k) & \text{if } r \text{ divides } n+1\\ nk(nk-rk)\cdots(3rk-2k)\cdot(rk-2k) & \text{if } r \text{ divides } n+2\\ \vdots & \\ nk(nk-rk)(nk-2rk)\cdots[(rk-(rk-k)], & \text{if } r \text{ divides } n+r\\ 1, \quad \text{if } n=0,-1,-2,\cdots,(r-1) \quad ; \quad (-rn,k)!_r=\infty,\ n\in\mathbb{N}. & \end{cases}$$

Remarks: Using the relation (1.2) and definition (2.4), we observe that the following results hold

$$(rn,k)!_r=k^nn(n-r)(n-2r)\cdots 3r\cdot 2r\cdot r=k^n(rn)!_r \tag{1.9}$$

$$(rn-1,k)!_r = k^n n(n-r)(n-2r)\cdots(3r-1)\cdot(2r-1)\cdot(r-1) = k^n(rn-1)!_r \quad (1.10)$$
$$(rn-2,k)!_r = k^n n(n-r)(n-2r)\cdots(3r-2)\cdot(2r-2)\cdot(r-2) = k^n(rn-2)!_r \quad (1.11)$$
$$\vdots$$
$$(rn-(r-1),k)!_r = k^n n(n-r)\cdots[2r-(r-1)]\cdot[r-(r-1)] = k^n[rn-(r-1)]!_r. \quad (1.12)$$

Note: that for $r=1$, we get the definition of $(n,k)!$ and if $r=k=1$, then classical definition of factorial function.

Lemma (2.5): If r, n and m are any natural numbers and $\Gamma(z)$ denote the classical gamma function, then

(1) $n!_r \times (n-1)!_r \times (n-2)!_r \cdots [n-(r-1)]!_r = n!$
(2) $(rn)!_r = r^n \frac{\Gamma(n+\frac{r}{r})}{\Gamma(\frac{r}{r})} = r^n\Gamma(n+1) = r^n n!$
(3) $(rn-m)!_r = r^n \frac{\Gamma(n+\frac{r-m}{r})}{\Gamma(\frac{r-m}{r})}$
(4) $(-rn)!_r = \infty \quad ; \quad 0!_r = 1$
(5) $[-(rn+1)]!_r \cdot [-(rn+2)]!_r \cdots [-(rn+r-1)]!_r$
$= \frac{r}{(rn-1)!_r \times (rn-2)!_r \cdots [rn-(r-1)]!_r}$.

Proposition (2.6): For $n,\in \mathbb{N}$, $k>0$, classical gamma function $\Gamma(x)$ and gamma k-function $\Gamma_k(x)$, the expressions involving (n,k)-multiple factorial hold as

$$(a) \quad (n,k)!_r \times (n-1,k)!_r \times (n-2,k)!_r \times \cdots \times [n-(r-1),k]!_r = (n,k)!,$$
$$(b) \quad (rn,k)!_r = k^n(rn)! = k^n r^n n! = k^n r^n \Gamma(n+1) = r^n\Gamma_k(n+1)k$$

and

$$(c) \quad (rn-m,k)!_r = k^n r^n \frac{\Gamma(n+\frac{r-m}{r})}{\Gamma(\frac{r-m}{r})} = r^n \frac{\Gamma_k(nk+\frac{(r-m)k}{r})}{\Gamma_k(\frac{(r-m)k}{r})}, m = 1,\cdots,(r-1)\cdot$$

Proof: We give the proof of the above proposition by using the definition of $(n,k)!_r$ along with the lemma (2.5). Multiplying the equations (1.9) $\cdots$ (1.12), we have

$$(rn,k)!_r \times (rn-1,k)!_r \cdots (rn-(r-1),k)!_r = k^{rn}(rn)!_r(rn-1)!_r \cdots [rn-(r-1)]!_r\cdot$$

Replacing rn by n, we infer

$$(n,k)!_r \times (n-1,k)!_r \times \cdots \times [n-(r-1),k]!_r = k^n n!_r(n-1)!_r \cdots [n-(r-1)]!_r$$

and the lemma (2.5(1)) along with the relation (1.5) gives

$$(n,k)!_r \times (n-1,k)!_r \times \cdots \times [n-(r-1),k]!_r = k^n n! = (n,k)!\cdot$$

Part (b) is obvious from the definition of $(n,k)!_r$, the gamma function and the relation $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$. For the part (c), the equation (1.12) and lemma (2.5(3)) implies that

$$(rn-m,k)!_r = k^n(rn-m)!_r = k^n r^n \frac{\Gamma(n+\frac{r-m}{r})}{\Gamma(\frac{r-m}{r})}, m = 1,2,\cdots,(r-1)\cdot$$

Applying the relation $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$, we can easily obtain the required result in the form of gamma k-function.

Theorem (2.7): For $n,r \in \mathbb{N}$, $k>0$ and the rth order factorial $!_r$, prove that

$$[-(rn+1),k]!r \times [-(rn+2),k]!r \times \cdots \times [-(rn+r-1),k]!r$$

$$= (-1)^{(1-r)n}\frac{r}{(rn-1,k)!r\times(rn-2,k)!r\times\cdots\times[rn-(r-1),k]!r}.$$

Proof. From the equations (1.9)$\cdots$ (1.12) and the proposition (2.6(a)), we observe that

$$(rn-1,k)!_r\times(rn-2,k)!_r\times\cdots\times[rn-(r-1),k]!_r$$

$$= (rn-1,k)!_r\times(rn-2,k)!_r\times\cdots\times[rn-(r-1),k]!_r\times\frac{(rn,k)!_r}{(rn,k)!r} = \frac{(rn,k)!}{(rn,k)!_r}$$

$$\Rightarrow (rn-1,k)!_r\cdot(rn-2,k)!_r\cdots(rn-(r-1),k)!_r = \frac{k^{rn}(rn)!}{k^n r^n n!} = \frac{k^{(r-1)n}\Gamma(1+rn)}{r^n\Gamma(1+n)}.$$

Replacing n by $-n$, we have

$$\Big(-(rn+1),k\Big)!_r\times\Big(-(rn+2),k\Big)!_r\times\cdots\times\Big(-(rn+r-1),k\Big)!_r$$

$$= \frac{k^{-(r-1)n}\Gamma(1-rn)}{r^{-n}\Gamma(1-n)} = \frac{r^n\Gamma\Big(-(rn-1)\Big)}{k^{(r-1)n}\Gamma\Big(-(n-1)\Big)}.$$

By the singular point formula, $\frac{\Gamma\left(-(rn-1)\right)}{\Gamma\left(-(n-1)\right)} = (-1)^{(n-1)-(rn-1)}\frac{(n-1)!}{(rn-1)!}$ $= (-1)^{(1-r)n}\frac{(n-1)!}{(rn-1)!}$, the above equation becomes

$$\Big(-(rn+1),k\Big)!_r\times\Big(-(rn+2),k\Big)!_r\times\cdots\times\Big(-(rn+r-1),k\Big)!_r$$

$$= (-1)^{(1-r)n}\frac{r^n(n-1)!}{k^{(r-1)n}(rn-1)!} = \frac{(-1)^{(1-r)n}r^n rn(n-1)!}{k^{(r-1)n}rn(rn-1)!} = \frac{(-1)^{(1-r)n}r\cdot r^n n!}{k^{(r-1)n}(rn)!}.$$

To convert into (n,k)-factorials, we multiply the numerator and denominator by k^n on R.H.S. and proceed as

$$\frac{(-1)^{(1-r)n}r\cdot r^n n!}{k^{(r-1)n}(rn)!} = \frac{(-1)^{(1-r)n}r\cdot r^n k^n n!}{k^{rn}(rn)!} = \frac{(-1)^{(1-r)n}r\cdot r^n(n,k)!}{(rn,k)!}$$

By the proposition (2.6(a)) and the relation (1.9), we have the R.H.S. as

$$\frac{(-1)^{(1-r)n}r\cdot r^n(n,k)!}{(rn,k)!_r\times(rn-1,k)!_r\times(rn-2,k)!_r\times\cdots\times[rn-(r-1),k]!_r}$$

$$= \frac{(-1)^{(1-r)n}r\cdot r^n(n,k)!}{r^n(n,k)!\times(rn-1,k)!_r\times(rn-2,k)!_r\times\cdots\times[rn-(r-1),k]!_r}$$

$$= \frac{(-1)^{(1-r)n}r}{(rn-1,k)!_r\times(rn-2,k)!_r\times\cdots\times[rn-(r-1),k]!_r}$$

3. Applications of higher order factorials in k-symbol

In this section, we give the applications of $(n,k)!_r$ in the expansion of some elementary functions.

Proposition (3.1): If n and r are the natural numbers and a is an integer such that $|a| < n$, then the Pochhammer's symbol in terms of $(n,k)!_r$ is expressed as

$$\left(\frac{a}{r}\right)_n = \frac{1}{(rk)^n}[a+r(n-1),k]!_r \quad \text{when } a>0 \tag{1.13}$$

and

$$\left(\frac{a}{r}\right)_n = \frac{1}{(rk)^n}a[a+r(n-1),k]!_r \quad \text{when } a<0 \tag{1.14}$$

Proof. For $a > 0$, by the definition of Pochhammer's symbol given in relation(1.4), we see that

$$\left(\frac{a}{r}\right)_n = (\frac{a}{r})(\frac{a}{r}+1)(\frac{a}{r}+2)\cdots(\frac{a}{r}+(n-1)) = \frac{a(a+r)(a+2r)\cdots(a+r(n-1))}{r^n}. \tag{1.15}$$

When $r = 2,3,4,\cdots$, from the equation (1.15) along with the property $(nk,k)! = k^n(nk)!$ of newly defined factorial in k symbol, we observe that

$$\left(\frac{a}{2}\right)_n = \frac{a(a+2)\cdots(a+2(n-1))}{2^n} = \frac{[a+2(n-1)]!!}{(2)^n} = \frac{[a+2(n-1),k]!!}{(2k)^n},$$

$$\left(\frac{a}{3}\right)_n = \frac{a(a+3)\cdots(a+3(n-1))}{3^n} = \frac{[a+3(n-1)]!!!}{(3)^n} = \frac{[a+3(n-1),k]!!!}{(3k)^n},$$

$$\vdots$$

$$\left(\frac{a}{r}\right)_n = \frac{a(a+r)\cdots(a+r(n-1))}{r^n} = \frac{[a+r(n-1)]!_r}{(r)^n} = \frac{[a+r(n-1),k]!_r}{(rk)^n}.$$

When $a < 0$, $a!_r = 1$, $(a+1\cdot r) > 0, (a+2r) > 0, \cdots, a+r(n-1) > 0$, then

$$a = a\cdot a!_r = \frac{1}{k^n}ak^n\cdot a!_r = \frac{1}{k^n}a\cdot(a,k)!_r \quad \text{, when } n=1$$

$$a(a+r) = a(a+r)!_r = \frac{1}{k^n}k^n a(a+r)!_r = \frac{1}{k^n}a(a+r,k)!_r \quad \text{, when } n=2$$

$$a(a+r)(a+2r) = a(a+2r)!_r = \frac{1}{k^n}k^n a(a+2r)!_r = \frac{1}{k^n}a(a+2r,k)!_r \quad \text{, when } n=3$$

and by induction, we get

$$\frac{a(a+r)(a+2r)\cdots[a+(n-1)r]}{r^n} = \frac{a\cdot[a+(n-1)r]!_r}{r^n}$$

$$= \frac{1}{k^n}\frac{a\cdot k^n[a+(n-1)r]!_r}{r^n} = \frac{a[a+r(n-1),k]!_r}{(rk)^n}$$

Now, we express the hypergeometric function of which parameters are rational numbers smaller than unity in terms of the newly defined higher order (n,k)- factorials.

Theorem (3.2): If r is a natural number greater than one and a is an integer such that $|a| < r$, then the hypergeometric function ${}_2F_1$, in terms of $(n,k)!_r$ is expressed as

$$ {}_2F_1\left(\frac{a}{r}, b, c; x\right) = 1 + \sum_{n=1}^{\infty} \frac{[a + r(n-1), k]!_r (b)_n x^n}{(c)_n (rn, k)!_r} \quad \text{when } a > 0 \tag{1.16} $$

and

$$ = 1 + a \sum_{n=1}^{\infty} \frac{[a + r(n-1), k]!_r (b)_n x^n}{(c)_n (rn, k)!_r} \quad \text{when } a < 0 \tag{1.17} $$

Proof. By the definition of classical hypergeometric function, we have

$$ {}_2F_1\left(\frac{a}{r}, b; c; x\right) = \sum_{n=0}^{\infty} \frac{(\frac{a}{r})_n (b)_n}{(c)_n} \frac{x^n}{n!}. $$

Using the relation (1.13), we have

$$ {}_2F_1\left(\frac{a}{r}, b; c; x\right) = 1 + \sum_{n=1}^{\infty} \frac{[a + r(n-1), k]!_r (b)_n}{(rk)^n (c)_n} \frac{x^n}{n!} \ a > 0 $$

and the fact $r^n k^n n! = (rn, k)!_r$, we get the required result (1.16). Similarly, we can have the result (1.17) for $a < 0$.

Corollary (3.3): If r is a natural number greater than one and a, b, and c are the integers such that $|a|, |b|, |c| < r$, then we have

$$ {}_2F_1\left(\frac{a}{r}, \frac{b}{r}; \frac{c}{r}; x\right) = 1 + \sum_{n=1}^{\infty} \frac{[a + r(n-1), k]!_r [b + r(n-1), k]!_r x^n}{[c + r(n-1), k]!_r (rn, k)!_r}, \text{when } a, b, c > 0, $$

$$ = 1 + a \sum_{n=1}^{\infty} \frac{[a + r(n-1), k]!_r [b + r(n-1), k]!_r x^n}{[c + r(n-1), k]!_r (rn, k)!_r} \quad \text{only } a < 0, $$

$$ = 1 + \frac{1}{c} \sum_{n=1}^{\infty} \frac{[a + r(n-1), k]!_r [b + r(n-1), k]!_r x^n}{[c + r(n-1), k]!_r (rn, k)!_r} \quad \text{only } c < 0, $$

Remarks: From the above discussion, we observe that a symbol whose sign is negative among a, b or c serves as a coefficient of $\sum$.

Examples (3.4): Here, e give some examples involving the above $(n,k)!_r$ for some elementary functions.

(i). To find the expansion of $(1-x)^{-\frac{1}{3}}$ by using the $(n,k)!$.

Consider the hypergeometric function defined in the theorem (3.2) for positive a, by taking $a = \frac{1}{3}$ and $b = c$ as

$$ {}_2F_1\left(\frac{1}{3}, c; c; x\right) = 1 + \frac{(1,k)!_3}{(3,k)!_3} x + \frac{(4,k)!_3}{(6,k)!_3} x^2 + \frac{(7,k)!_3}{(9,k)!_3} x^3 + \frac{(10,k)!_3}{(12,k)!_3} x^4 + \cdots $$

$$ = 1 + \frac{(1,k)!!!}{(3,k)!!!} x + \frac{(4,k)!!!}{(6,k)!!!} x^2 + \frac{(7,k)!!!}{(9,k)!!!} x^3 + \frac{(10,k)!!!}{(12,k)!!!} x^4 + \cdots = (1-x)^{-\frac{1}{3}}. $$

(ii). To find the expansion of $(1-x)^{\frac{1}{3}}$ by using the $(n, k)!$.
Consider the hypergeometric function defined in the theorem (3.2) for negative a, by taking $a = -\frac{1}{3}$ and $b = c$ as

$$
\begin{aligned}
&{}_2F_1\Big(\frac{-1}{3}, c; c; x\Big) \\
&= 1 + (-1)\Big[\frac{(-1,k)!!!}{(3,k)!!!}x + \frac{(2,k)!!!}{(6,k)!!!}x^2 + \frac{(5,k)!!!}{(9,k)!!!}x^3 + \frac{(8,k)!!!}{(12,k)!!!}x^4 + \cdots\Big] \\
&= 1 - \frac{(-1,k)!!!}{(3,k)!!!}x - \frac{(2,k)!!!}{(6,k)!!!}x^2 - \frac{(5,k)!!!}{(9,k)!!!}x^3 - \frac{(8,k)!!!}{(12,k)!!!}x^4 + \cdots = (1-x)^{\frac{1}{3}}.
\end{aligned}
$$

(iii). To find the expansion of $\frac{Sin^{-1}x}{x}$ by using the $(n, k)!$.
Consider the hypergeometric function defined in the corollary (3.3) for $a = \frac{1}{2}$, $b = \frac{1}{2}$ and $c = \frac{3}{2}$ and x for x^2 as

$$
\begin{aligned}
{}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\Big) &= 1 + \frac{(1,k)!!}{(2,k)!!}\frac{x^2}{3} + \frac{(3,k)!!}{(4,k)!!}\frac{x^4}{5} + \frac{(5,k)!!}{(6,k)!!}\frac{x^6}{7} + \frac{(7,k)!!}{(8,k)!!}\frac{x^8}{9} + \cdots \\
&= \frac{Sin^{-1}x}{x}
\end{aligned}
$$

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