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A Novel Labeling Algorithm on Several Classes of Graphs

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Abstract. A positive integer n is called super totient if the residues of n which are prime to n can be partitioned into two disjoint subsets of equal sums. Let G be a given graph with V, the set of vertices and E is the set of its edges. An injective function g defined on V into subset of integers will be termed as super totient labeling of the graph G, if the function $g^* : E \to \mathbb{N}$ defined by $g^*(xy) = g(x)g(y)$ assigns a super totient number for all edges $xy \in E,$ where $x, y \in V$. A graph admits this labeling is called a super totient graph. In the current manuscript, the authors investigate a novel labeling algorithm, called super totient labeling, for several classes of graphs such as friendship graphs, wheel graphs, complete graphs and complete bipartite graphs.

AMS (MOS) Mathematics subject classification (2000): 05C25, 11E04, 20G15
Key Words: Super totient number, Complete graph, Complete bipartite graph, Wheel graph, Friendship graph.

1. Introduction

The assignments of integers using some appropriate mathematical rule to vertices (or edges) of a given graph is called a vertex (or edge) labeling. Indeed, it is possible to define a simultaneous labeling for both vertices and edges of a graph as well. Thus it has become an autonomous and compelling interest of number theorist. Even after finding such functions, it is important to find classes of well known graphs which admit your number theoretic functions as graph labeling. Unadventurousl, most of the real world problems can be fixed and viewed by intrigues their graphs with labeling.

Formally, the subject of graph labeling has been introduced by A. Rosa [8] in 1967. Rosa defined an injective function on n edges from set of vertices of a graph to a subset of $\{1, 2, ..., n\}$. Later on Golomb [1], called such labeling as graceful labeling. Many interesting games and puzzles have been solved by means of graph labeling. A similar work regarding puzzles and packing has been presented very elegantly in [7]. A detailed survey of previous labeling over many well known classes of graphs has been studied by Gallian in [6]. In [2], K.P.S. Bhaskara Rao and Yuejian Peng explored many interesting results on Zumkeller numbers. B.J. Balamurugan, K. Thirusangu and D.G. Thomas introduce the concept of Zumkeller labeling and proposed Zumkeller labeling algorithms for complete bipartite graphs and wheel graphs [4].

In this paper, we propose novel labeling algorithms by means of super totient numbers. We call this labeling as super totient labeling. We give algorithms and prove that the well known classes of graphs such as complete graphs, complete bipartite graphs, wheel graphs and friendship graph admits the super totient labeling. Notations used in this paper are standard and we follow [2] to [4]. We state the following theorems of [5], without proof for use in the sequel.

Theorem 1.1. [5] *If p is a prime and k is any positive integer, then $\varphi(p^k) = p^k - p^{k-1}$.*

Theorem 1.2. [5] *For $n > 1$, the sum of positive integers less than n and relatively prime to n is $\frac{n\varphi(n)}{2}$.*

Before giving our proposed algorithm, we introduce the notion of super totient numbers and state few results of super totient numbers with straightforward proofs so as to make this paper self contained.

Definition 1.3. *A positive integer n is called super totient if the residues of n which are prime to n can be partitioned into two disjoint subsets of equal sums. The integers 5, 8, 10, 12, 14 and 15 are few examples of super totient numbers.*

Example 1.4. *Take $n = 14$, then the positive residues of 14 which are prime to 14 are 1, 3, 5, 9, 11, and 13. We can partition these residues into two disjoint subsets of equal sums such as: $A = \{1, 9, 11\}$ and $B = \{3, 5, 13\}$ with $\sum A = \sum B = 21$. Thus 14 is a super totient number.*

The following Lemma is very crucial and of vital importance. We shall be using this lemma throughout the paper.

Lemma 1.5. *Let m be a positive integer, if $4|\varphi(m)$ then m is super totient.*

Proof. We note that, $(t_i, m) = 1$ if and only if $(m - t_i, m) = 1$.

Let $k = \varphi(m)/4$, we can partition the set of coprime residues

$$1 = t_1 < t_2 < t_3 < \cdots < t_{\varphi(m)} < m$$

in the following two disjoint sets:

$$\begin{aligned} A = \quad & \{\, t_1,\ t_2,\ t_3 \cdots,\ t_k\} \cup \{m - t_1,\ m - t_2,\ m - t_3 \cdots,\ m - t_k\} \\ B = \quad & \{t_{2k+1},\ t_{2k+2}, \cdots,\ t_{3k}\} \cup \{m - t_{2k+1},\ m - t_{2k+2}, \cdots,\ m - t_{3k}\} \end{aligned}$$

Then it is clear that

$$\sum_{a\in A} a = \sum_{i=1}^{k} t_i + (m - t_i) = mk = \sum_{j=2k+1}^{3k} t_j + (m - t_j) = \sum_{b\in B} b$$

□

The proof of the following theorem can be viewed by means of Lemma 1.5.

Theorem 1.6.

1. A prime number p is super totient if and only if $p \equiv 1(mod\ 4)$.

2. If a positive integer m has at leat two odd prime divisors then m is a super totient number.

3. If n is super totient number and m be any positive integer, then mn is a super totient number.

Definition 1.7. *Let G be a given graph with V, the set of vertices and E is the set of its edges. An injective function $g : V \to \mathbb{N}$ is termed as super totient labeling of the graph G, if the induced function $g^* : E \to \mathbb{N}$ given by $g^*(xy) = g(x)g(y)$ assigns a super totient number for each edge $xy \in E$, where, $x, y \in V$.*

Note that there will be no edge between vertices 2 and 3 as given in Fig.1 since 6 is not super totient number.

Definition 1.8. *We name a graph as super totient if it admits a super totient labeling.*

Example 1.9. *The super totient labeling of a graph is given in Fig.1.*

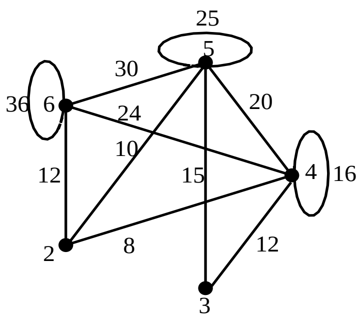

Fig.1

We propose algorithms for super totient labeling over different classes of graphs namely friendship graphs, wheel graphs, complete graphs and complete bipartite graphs. As consequences of these algorithms, we deduce that all cycles and all paths also admit super totient labeling. Finally, we prove the existence of super totient graphs for all these classes. Fig.1 depicts super totient labeling on a graph. It is note that loops can admit this labeling as well. For example, the vertices 4, 5 and 6 have loops and the edges must be labeled by 16, 25 and 36 respectively and all these are totient numbers. That is, 16, 25 and 36 are totient numbers. The motivation behind our research work was the concept of Zumkeller labeling and Zumkeller graphs given in [6]. Thus it was interesting and challenging to define a novel labeling algorithms which can be proved via number theory.

We note that the super totient numbers on the edges of a graph are obtained by multiplying the labels of the vertices of that edge. Thus, if we remove some vertices or edges then the remaining subgraph certainly admits super totient numbers for the remaining edges. Hence, we must arrive at the following proposition.

Proposition 1.10. *A non-totally disconnected subgraph of a super totient graph is a super totient graph.*

2. Super Totient Friendship Graphs

Definition 2.1. *For any integer n, a planar graph having $2n+1$vertices and $3n$ edges is termed as friendship (or n-fan) graph. It is denoted by F_n. This is traced by connecting n copies of a cyclic graph of order 3 such that these cyclic graphs have one common vertex. For $n=5$, the friendship graph F_5 has 11 vertices and 15 edges.*
The Fig.2, depicts the friendship graph F_5.

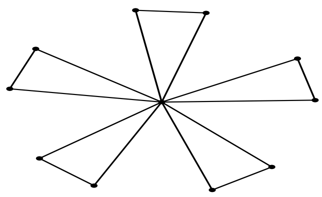

Fig.2

If a given friendship graph admits super totient labeling then we call this as a super totient friendship graph. In the following theorem, we prove existence of super totient labeling over a friendship graph.

Theorem 2.2. *The friendship graph F_n admits super totient labeling. That is, for each n, F_n are super totient friendship graphs.*

Proof. Let $p,\ q$ and r be distinct odd primes. Choose v_0 as the vertex which is adjacent to vertices $v_1, v_2, v_3, \cdots, v_n$ to construct the $n-$fans of a given friendship graph F_n. Then by definition of F_n, $E = \{e_i = v_i v_{i+1},\ i = 1,3,5,\cdots,2n-1\} \cup \{e_i^{'} = v_0 v_i,\ i = 1,2,3,\cdots,2n\}$ is the set of edges. We define an injective function g on the vertex set V such as

$$g(v_i) = \begin{cases} r, & \text{if } i = 0 \\ p^{\frac{i+1}{2}}, & \text{if } i = 1,3,5,\cdots,2n-1 \end{cases}$$

and

$$g(v_{i+1}) = q^{\frac{i+1}{2}},\ i = 1,3,5,\cdots,2n-1$$

Let g^* be an induced function to g defined on the vertex set V by

$$\begin{aligned} g^*(e_i) &= g^*(v_i v_{i+1}) = g(v_i)g(v_{i+1}),\ i = 1,3,5,\cdots,2n-1 \\ g^*(e_i^{'}) &= g^*(v_0 v_i) = g(v_0)g(v_i),\ i = 1,2,3,\cdots,2n \end{aligned}$$

Then by definition of g, we obtain,

$$g^*(e_i) = g^*(v_i v_{i+1}) = g(v_i)g(v_{i+1}) = p^{\frac{i+1}{2}} q^{\frac{i+1}{2}} \quad (2.\ 1)$$

$$g^*(e_i^{'}) = g^*(v_0 v_i) \ = g(v_0)g(v_i) \ = rp^{\frac{i+1}{2}} \quad (2.\ 2)$$

$$g^*(e_i^{'}) = g^*(v_0 v_{i+1}) = g(v_0)g(v_{i+1}) = rq^{\frac{i+1}{2}} \quad (2.\ 3)$$

Note that if p and q are distinct odd primes, then $\varphi(pq) = \varphi(p)\varphi(q)$ and $\varphi(p)$ is an even number. Hence, by Lemma 1.5, equations (2.1)-(2.3) yield that the friendship graph F_n is a super totient friendship graph. □

Example 2.3. *For $n = 6$, the friendship graph F_6 is a super totient-graph, its super totient labeling with $v_0 = 11,\ p = 3,\ q = 7$ is given in Fig.3 .*

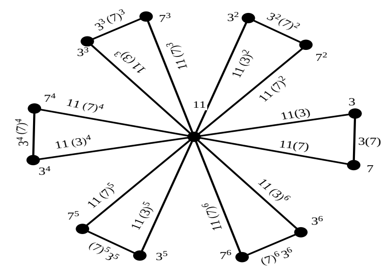

Fig.3

Algorithm 1
(super totient labeling of friendship graph F_n)

This algorithm computes integers for vertices of the friendship graph F_n to label the edges with super totient numbers.

Step 1. (Input) F_n, a friendship graph over $2n + 1$ vertices.
V : Set of vertices of F_n.
E: set of edges of F_n and $E = \{e_i = v_i v_{i+1},\ i = 1, 3, 5, \cdots, 2n-1\} \cup \{e_i^{'} = v_0 v_i,\ i = 1, 2, 3, \cdots, 2n\}$
p: where p is an odd prime.
q: where q is an odd prime.
q: where r is an odd prime.
$p \neq q \neq r$.
g : g is a one-one function on V with $g(v_0) = r$.
g^* : g^* is an induced function by g on E.
Set v_0 as the central vertex of the wheel F_n with $g(u_0) = r$.

Step 2.
$\{$
for
$\quad\quad \{$
$i = 1,\ 3,\ 5,\ ...,\ 2n-1$ **do**
$\quad\quad\quad \{$
$\quad\quad\quad\quad g(v_i) = p^{\frac{i+1}{2}}$
$\quad\quad\quad\quad g(v_{i+1}) = q^{\frac{i+1}{2}}$

}
}
}

Step 3. Output (super totient friendship graph.

3. Super Totient Wheel Graphs

Definition 3.1. *A graph over n vertices is said to be a wheel graph if a fixed vertex u_0 is adjacent to all remaining vertices $u_2, u_2, u_3, ..., u_{n-1}$ where the vertices $u_2, u_2, u_3, ..., u_{n-1}$ form a cycle of length $n-1$. This graph is denoted by W_n and has $2n-2$ edges.*
In the following theorem, we prove that all wheel graphs admit the super totient labeling.

Theorem 3.2. *For each integer n, the wheel graphs W_n are super totient wheel graphs. That is, every wheel graph admits super totient labeling.*

Proof. Let u_0 be the vertex of a given wheel graph which is connected with all other vertices (say) $u_1, u_2, u_3, \cdots, u_n$. Define the edge set E of W_n as follows:
$E = \{e_i' = u_0u_i,\ i = 1, 2, ..., n\} \cup \{e_i = u_iu_{i+1},\ i = 1, 2, ..., n-1\} \cup \{u_nu_1 = e_n\}$.
We discuss the cases separately according as n is even and odd. Consider the wheel graph W_n when n is even. Let $p,\ q$ and r be distinct odd primes . To label vertices, we define an injective function g on the vertex set V such as:

$$g(u_i) = \begin{cases} r, & \text{if } i = 0 \\ p^{\frac{i+1}{2}}, & \text{if } i \equiv 1(mod\,2),\ 1 \leq i \leq n \\ q^{\frac{i}{2}}, & \text{if } i \equiv 0(mod\,2),\ 1 \leq i \leq n \end{cases}$$

Now, we define an induced function g^* to g as follows,

$$\begin{aligned} g^*(e_i) &= g^*(u_iu_{i+1}) = g(u_i)g(u_{i+1})\ ,\ i = 1, 2, ...,\ n-1 \\ g^*(e_i') &= g^*(u_0u_i) = g(u_0)g(u_i)\ ,\ i = 1, 2, 3...,\ n \end{aligned}$$

Then by definition of g, we obtain,

$$g^*(e_i) = g^*(u_iu_{i+1}) = g(u_i)g(u_{i+1}) = p^{\frac{i+1}{2}}q^{\frac{i+1}{2}}, i \equiv 1(mod\,2) \tag{3. 4}$$
$$g^*(e_i) = g^*(u_iu_{i+1}) = g(u_{i+1})g(u_i) = q^{\frac{i}{2}}p^{\frac{i+2}{2}}, i \equiv 0(mod\,2) \tag{3. 5}$$
$$g^*(e_i') = g^*(u_0u_{i+1}) = g(u_0)g(u_{i+1}) = rp^{\frac{i+1}{2}}, i \equiv 1(mod\,2) \tag{3. 6}$$
$$g^*(e_i') = g^*(u_0u_{i+1}) = g(u_0)g(u_{i+1}) = rq^{\frac{i}{2}}, i \equiv 0(mod\,2) \tag{3. 7}$$

Note that if p and q are distinct odd primes, then $\varphi(pq) = \varphi(p)\varphi(q)$ and $\varphi(p)$ is an even number. Hence, by Lemma 1.5, equations (3.4)-(3.7) yield that the wheel graph W_n is a super totient wheel graph.
Now if n is an odd number then, we take distinct odd primes p, q, r and s. Again, we label vertices as the values of the function g defined on the vertex set V such that,

$$g(u_i) = \begin{cases} r, & \text{if } i = 0 \\ p^{\frac{i+1}{2}}, & \text{if } i \equiv 1(mod\,2),\ 1 \leq i \leq n \\ q^{\frac{i}{2}}, & \text{if } i \equiv 0(mod\,2),\ 1 \leq i \leq n \\ s, & \text{if } i = n, \end{cases}$$

Also we define an induced function g^* to g as follows,

$$
\begin{aligned}
g^*(e_i) &= g^*(u_iu_{i+1}) = g(u_i)g(u_{i+1})\ ,\ i = 1, 2, ..., \ n-2 && (3.\ 8)\\
g^*(e_{n-1}) &= g^*(u_{n-1}u_n) = g(u_{n-1})g(u_n) && (3.\ 9)\\
g^*(e_i^{'}) &= g^*(u_0u_{i+1}) = g(u_0)g(u_i)\ ,\ i = 1, 2, 3..., \ n && (3.\ 10)\\
g^*(e_n) &= g^*(u_nu_1) = g(u_n)g(u_1) && (3.\ 11)
\end{aligned}
$$

Equations (3.8) and (3.10) follow the previous case, so we only to prove that the equations (3.9) and (3.11) assign super totient numbers.

$$
\begin{aligned}
g^*(e_{n-1}) &= g^*(u_{n-1}u_n) = g(u_{n-1})g(u_n) = q^{\frac{i}{2}}s && (3.\ 12)\\
g^*(e_n) &= g^*(u_nu_1) = g(u_n)g(u_1) = sp^{\frac{i+1}{2}} && (3.\ 13)
\end{aligned}
$$

Since $p,\ q,\ r$ and s are distinct odd primes, so again by Lemma 1.1, equations (3.12)-(3.13) assign super totient numbers. Consequently, the wheel graph W_n is a super totient wheel graph. □

It is well known that cycle C_n can be obtained by deleting the vertex u_0 of the wheel graph W_n. Thus the cycle graphs are the non-totally disconnected subgraph of wheels graphs. Similarly, by deleting some more vertices we can obtain path graphs P_n as the non-totally disconnected subgraph of wheels graphs. The following corollaries are the simple consequences of Theorem 3.2 and proposition 1.10.

Corollary 3.3. *A cycle C_n admits a super totient labeling.*

Corollary 3.4. *A path P_n with n vertices is a super totient graph.*

Example 3.5. *For $n = 8$ the wheel graph W_8 is a super totient graph, its super totient labeling with $r = 11$, $p = 3$ and $q = 7$ and for $n = 9$, the wheel graph W_9 is a super totient graph, its super totient labeling with $r = 11$, $p = 3,\ q = 7$ and $s = 19$ is given in Fig.4 .*

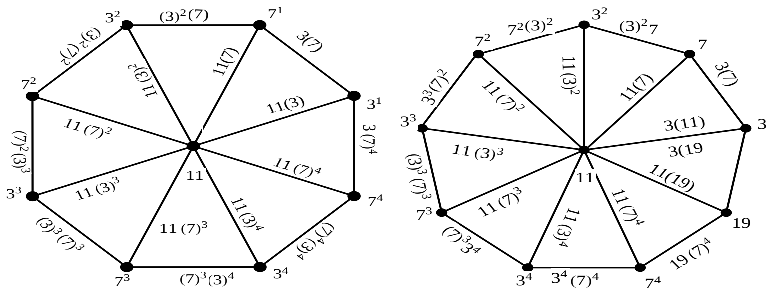

Fig.4

In the following algorithm, we summarize Theorem 3.2.

Algorithm 2
This algorithm finds the set of vertices of a given wheel graph W_n such that each edge must be labeled by a super totient numbers by means by Theorem 3.2.
Step 1. (Input)

W_n, a wheel graph over n vertices;
V : Set of vertices of W_n;
E : Set of edges of W_n and $E = \{e_i^{'} = u_0 u_i,\ i = 1, 2, ..., n\} \cup \{e_i = u_i u_{i+1},\ i = 1, 2, ..., n-1\} \cup \{u_n u_1 = e_n\}$;
g : g is a one-one function on V;
p : p is an odd prime;
q : q is an odd prime;
r : r is an odd prime;
s : s is an odd prime;
$p \neq q \neq r \neq s$
Set v_0 as the central vertex of the wheel W_n with $g(u_0) = r$;
Step 2. **do** $\{$
if n is even **then**
$\{$
for $i = 1, 3, ..., n-1$ **do**
$\{$
$g(u_i) = p^{\frac{i+1}{2}}$
$g(u_{i+1}) = q^{\frac{i+1}{2}}$
$\}$
$\}$
else
$\{$
for $i = 1, 3, ..., n-2$ **do**
$\{$
$g(u_i) = p^{\frac{i+1}{2}}$
$g(u_{i+1}) = q^{\frac{i+1}{2}}$
$\}$
$\}$
if $i = n$ **then** $g(u_n) = s$
$\}$
Step 3. Output (super totient wheel graph).

4. Super Totient Complete Graphs

In this section, we prove the existence of super totient labeling over complete graphs, that is, existence of super totient complete graphs and propose an algorithm to understand the notion of super totient labeling over complete graphs.

Definition 4.1. *If any two distinct vertices of a simple graph are adjacent, then it is called a complete graph. These are denoted by K_n with n vertices and $\frac{n(n-1)}{2}$ edges.*
The complete graph K_5 is shown in Fig.5.

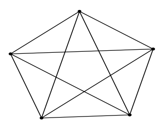

Fig. 5

Theorem 4.2. *The complete graph K_n admits super totient labeling. That is, for each n, K_n are super totient complete graphs.*

Proof. Let $\{v_1, v_2, v_3, \cdots, v_n\}$ be the set of vertices and $\bigcup_{i<k,\ i,k=1}^{n} \{e_{ik} = v_i v_k\}$ be the set of edges of the complete graph K_n. Denote these sets by V and E respectively. Let $\mathbb{N}$ be the set of positive integers.
Define functions $g : V \to N$ such that $g(v_i) = 2^i,\ 1 \leq i \leq n$ and an induced function $g^* : E \to \mathrm{N}$ such that $g^*(e_{ik}) = g(v_i)g(v_k),\ 1 \leq k \leq n,\ i < k$.
We need to show that the numbers on the edges are the super totient numbers. For this, we see that

$$\begin{aligned} g^*(e_{ik}) = g^*(v_i v_k) &= g(v_i)g(v_k) \\ &= 2^i 2^k \\ &= 2^{i+k},\ i + k \geq 3 \end{aligned}$$

Thus by Lemma 1.5, $g^*(e_{ik})$ is a super totient number. □

Algorithm 3
This algorithm finds the set of vertices of a given complete graph K_n such that each edge must be labeled by a super totient number.
Step 1. (Input)
K_n, a complete graph over n vertices;
V : Set of vertices of K_n;
E : Set of edges of K_n and $E = \bigcup_{i<k,\ i,k=1}^{n} \{e_{ik}\}$ be the set of edges of the complete graph K_n.
Step 2.
{
for $i = 1$ **to** n **do**
{
{
$g(v_i) = 2^i$, where g is an injective function defined over V in K_n.
}
}
}
Step 3. Output (super totient complete graph).

Example 4.3. *For $n = 6$ the complete graph K_6 is a super totient graph, is given in Fig.6.*

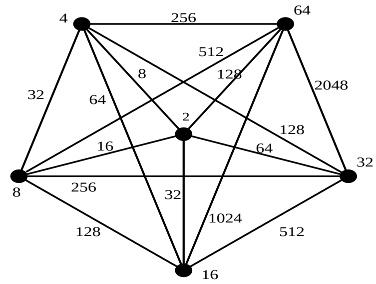

Fig.6

5. SUPER TOTIENT COMPLETE BIPARTITE GRAPHS

Definition 5.1. *A simple graph with $m+n$ vertices is called bipartite if the vertex set can be partitioned into two subsets A and B containing m, and n, vertices respectively such that the graph contains no edge between any pair of vertices form A and from B itself. That is, the edges can be built only for those pair for which one vertex is taken from A and the other is from the vertex set B. In addition, if each vertex of A is adjacent to every vertex of B, the bipartite graph is called a complete bipartite graph. It is denoted by $K_{m,n}$. In this case, the complete bipartite graph has mn edges.*
The complete bipartite graph $K_{3,3}$ is shown in Fig.7.

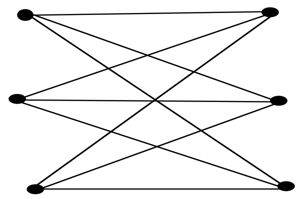

Fig.7

In the following theorem, we prove the existence of super totient complete bipartite graphs.

Theorem 5.2. *The complete bipartite graph $K_{m,n}$ is a super totient complete bipartite graph. That is, all complete bipartite graphs admit super totient labeling.*

Proof. Let A and B be the two partitioned subsets containing m and n elements respectively for the vertex set V of a complete bipartite graph $K_{m,n}$. Take $A=\{v_1,v_2,v_3,\cdots,v_m\}$ and $B=\{\acute{v_1},\acute{v_2},\acute{v_3},\cdots,\acute{v_n}\}$ where $V=A\cup B$. Define $E=\{e_{ij}=v_i\acute{v_j}\mid 1\le i\le m,\ 1\le j\le n\}$. Then E is the edge set of $K_{m,n}$. Let p and q be two distinct odd primes and define a function $g:V\to\mathbb{N}$ such as,

$$g(v)=\begin{cases} p^i, & \text{if } v=v_i\in A\\ q^i, & \text{if } v=v_i^{'}\in B\end{cases}$$

and an induced function $g^*:E\to N$ defined as

$$\begin{aligned} g^*(e_{ij}) &= g^*(v_i\acute{v_j})\\ &= g(v_i)g(\acute{v_j}) \text{ for, } 1\le i\le m, \text{ and } 1\le j\le n\end{aligned}$$

But then,

$$g^*(e_{ij}) = g^*(v_i \acute{v_j}) = g(v_i)g(\acute{v_j}) = p^i q^j \ \forall\ i, j$$

Since p and q are distinct odd primes so, by Theorem 1.6(2), $g^*(e_{ij})$ is a super totient number. Hence, $K_{m,n}$ is a super totient complete bipartite graph. □

Example 5.3. *The complete bipartite graph $K_{5,5}$ is a super totient complete bipartite graph. Fig.8 depicts its super totient labeling when $p = 3,\ q = 7.$.*

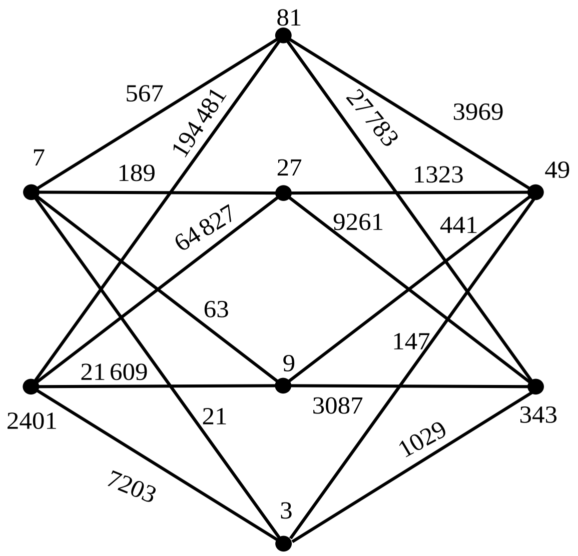

Fig.8

Since every bipartite graph is a non-totally disconnected subgraph of a complete bipartite graph and complete bipartite graphs are super totient graphs, thus we simply arrive at the following corollary.

Corollary 5.4. *Every bipartite graph is a super totient bipartite graph. That is, every bipartite graph admits super totient labeling.*

In the following algorithm, we present super totient labeling for a given complete bipartite graph $K_{m,n}$. This algorithm assigns labels to edges assuring that each edge must be labeled by a super totient number.

Algorithm 4

Step 1. (Input)

$K_{m,m}$, a complete bipartite graph over $m + n$ vertices;

V : Set of vertices of $K_{m,n}$.

$A = \{v_1, v_2, v_3, \cdots, v_m\}$

$B = \{\acute{v_1}, \acute{v_2}, \acute{v_3}, \cdots, \acute{v_n}\}$ where, $V = A \cup B$ and $A \cap B = \emptyset$

E : Set of edges of $K_{m,n}$ where, $E = \{e_{ij} = v_i\acute{v_j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

g : g is a one-one function on V.

g^* : g^* is an induced function by g on E.

p, where p is an odd prime.

q, where q is an odd prime.
$p \neq q$.
Step 2.
{
for i:=1**to**m**do**
$g(v_i) = p^i$
for $j := 1$ **to** n **do**
$g(\acute{v_j}) = q^j$
if $i \neq j$ **then**

{
{
for $i := 1$ **to** m **do**
{
for $j := 1$ **to** n **do**
$g^*(v_i\acute{v_j}) = g(u_i)g(\acute{v_j})$
}
}
else
$g^*(v_i\acute{v_i}) = g(u_i)g(\acute{v_i})$
}
}
Step 3. Output (super totient complete bipartite graph).

The explicit constructions of super totient labelings for different families of graphs have been discussed separately. Therefore, these families of graphs are actually super totient graphs. It is proved in Proposition 1.1, that every non-totally disconnected subgraph of a super totient graph is again a super totient graph. This means that, the property of being super totient is preserved via graph restriction. Moreover, in Theorem 4.1, it is shown that every complete graph admits super totient labeling and hence every complete graph is a super totient graph. It is also well-known that every graph with n vertices can be viewed as a subgraph of a complete graph K_n. Therefore, we conclude from Proposition 1.1 and Theorem 4.1, that every graph with at least one edge ($i.e$, a non-totally disconnected graph) admits a super totient labeling as well. For instance, if we start with the labeling of K_6 in Fig.6 and delete the edges $\{\{4,64\},\{2,64\},\{2,4\},\{8,16\},\{16,32\},\{8,32\}\}$, then we immediately recover a super totient labeling for $K_{3,3}$ (which of course is different from the one given in Fig.8). Thus, the above discussion leads to the following theorem.

Theorem 5.5. *Every graph on n vertices having at least one edge is a super totient graph. That is, every graph on n vertices having at least one edge admits super totient labeling.*

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