Unsteady Rotational Flow of a Second Grade Fluid with Non-Integer Caputo Time Fractional Derivative

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Abstract. This article presents some starting solutions corresponding to unsteady rotational flow of a second grade fluid with the non-integer Caputo time fractional derivatives through an infinite long cylinder. The fluid in the infinitely long cylinder is initially at rest and at $t = 0^+$, the fluid starts to rotate due to applied shear stress. By using modified Bessel equation and Laplace transformation, solutions for velocity field and shear stress are obtained. The solutions are presented in transformed domain in terms of modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$ and satisfy all imposed initial and boundary conditions. In this paper, inverse Laplace transformation has been calculated numerically by using Stehfest's algorithm. The hybrid technique presented in this paper has less computational effort and time cost as compared to other schemes in literature. Finally, the behavior of different parameters is graphically examined.

1. INTRODUCTION

The fluid flow in a translating or rotating cylinder is a healthy discussion for both practical and theoretical field. Large application of fluids in our daily life such as drilling operation, food industry, polymers industry, chemical industry and bio engineering made fluid mechanics most fascinating field for researchers.

Heat transfer in magnetohydrodynamic second grade fluid with porous impacts using Caputo-Fabrizio fractional derivatives is presented in [2]. The starting motions of second grade fluids with the help of linear constitutive relationship is discussed by Bandelli and Rajagopal [7], Han [12] also Liu and Hang in [18]. The starting solutions for the motions of second-grade fluid in a cylindrical domain is discussed by Ting [28] and Srivastava [25]. The first exact solution for velocity field in an infinite oscillating plate is examined by Stokes [27]. The fractional derivatives within the context of viscoelasticity were firstly described by Germant [11]. Casarella and Laura [8] presented the solutions of the Second grade fluid for torsional and longitudinal oscillation. Bagley and Torvik [6] studied the theory of viscoelasticity using constitutive relations having fractional derivatives. Later on, Rajagopal

[22] found solutions of a second grade fluid for torsional and longitudinal oscillations of an infinite rod. Some exact solutions for the motion of a second grade fluid are discussed in [11-19]. Exact solutions of a Maxwell fluid and a second grade fluid for torsional and longitudinal oscillations in a cylinder are described by Fetecau [10] and Vieru *et al* [31] respectively. Jamil and Fetecau [14] established exact analytic solutions for helical flows of a second grade fluid between two infinite coaxial cylinders. Cruz and Lins [9] provided the general solution of unsteady flow generated by an oscillating wall.

Flow of a second grade fluid with fractional derivatives due to a quadratic time dependent shear stress is discussed by Raza *et al* [23]. A note on the unsteady torsional sinusoidal flow of fractional viscoelastic fluid in an annular cylinder studied by Mahmood *et al* [19]. Awan *et al* [5] established the exact solution of fractional viscoelastic fluid between two coaxial circular cylinder. Abdulhameed *et al* [1] studied the exact solutions for unsteady flow of second grade fluid generated by oscillating wall with transpiration. Oscillating flows of fractionalized Second grade fluid in an infinite circular cylinder discussed by Jamil *et al* [15]. Effects of chemical reaction on the unsteady flow of an incompressible fluid over a vertical oscillating plate is presented in [4].

In this monograph, we have found the semi analytical solutions of second grade fluid having fractional derivatives in an infinite long cylinder. Semi analytical solutions are found by using Laplace transformation and modified Bessel equation. The solutions are in transformed form of modified Bessel function $I_0(\cdot)$ and $I_1(\cdot)$. To found the inverse Laplace transform of these type of functions analytically or using contour integration is almost impossible. Therefore, the inverse Laplace transformation has been achieved through numerical package by using Mathcad software. The validation of the numerical results for inverse Laplace transform is performed by presenting Table 1, which show comparison of our Mathcad results with two other numerical algorithms.

2. Formulation of the Problem

The velocity field **F** for the movement of fluid [32] is considered as

$$\mathbf{F} = \mathbf{F}(\sigma, t) = f(\sigma, t)\mathbf{e}_\theta, \tag{2.1}$$

where $\mathbf{e}_\theta$ is the unit vector of cylindrical coordinates (σ, θ, z) in θ-direction. Moreover, when the fluid starts to move, we have

$$\mathbf{F}(\sigma, 0) = \mathbf{0}. \tag{2.2}$$

Consider the governing equations of ordinary second grade fluid [3]

$$\xi(\sigma, t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \sigma} - \frac{1}{\sigma}\right)f(\sigma, t), \tag{2.3}$$

$$\rho \frac{\partial f(\sigma, t)}{\partial t} = \left(\frac{\partial}{\partial \sigma} + \frac{2}{\sigma}\right)\xi(\sigma, t). \tag{2.4}$$

Eliminating $\xi(\sigma, t)$ from Eqs. (3) and (4), we have

$$\frac{\partial f(\sigma, t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial}{\partial \sigma} - \frac{1}{\sigma^2}\right)f(\sigma, t), \tag{2.5}$$

where $\xi(\sigma, t) = T_{\sigma\theta}(\sigma, t)$ is the only nontrivial shear stress, μ is the dynamics viscosity, $\nu = \frac{\mu}{\rho}$ is the kinematics viscosity, ρ is its constant density, α_1 is a normal stress module

and $\alpha = \frac{\alpha_1}{\rho}$. The flow of second grade fluid with fractional derivatives is governed by the following model

$$\xi(\sigma,t) = \mu\left(1 + \lambda\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)f(\sigma,t), \qquad (2.\ 6)$$

$$\frac{\partial f(\sigma,t)}{\partial t} = \nu\left(1 + \lambda\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial}{\partial\sigma} - \frac{1}{\sigma^2}\right)f(\sigma,t), \qquad (2.\ 7)$$

where $\frac{\alpha_1}{\mu} = \lambda$. The Caputo fractional differential operator [12] is defined as

$$D_t^\varrho f(t) = \begin{cases} \frac{1}{\Gamma(1-\varrho)}\int\limits_0^t \frac{df(\zeta)}{dt}\frac{1}{(t-\zeta)^\varrho}\,d\zeta,\ 0 \leq \varrho < 1; \\ \frac{df(t)}{dt},\ \varrho = 1, \end{cases} \qquad (2.\ 8)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

By using the Caputo fractional differential operator D_t^β, the equations related to the second grade fluid having fractional derivatives can be written as

$$\xi(\sigma,t) = \mu\left(1 + \lambda^\beta D_t^\beta\right)\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)f(\sigma,t), \qquad (2.\ 9)$$

$$\frac{\partial f(\sigma,t)}{\partial t} = \nu\left(1 + \lambda^\beta D_t^\beta\right)\left(\frac{\partial^2}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial}{\partial\sigma} - \frac{1}{\sigma^2}\right)f(\sigma,t). \qquad (2.\ 10)$$

Consider an infinitely long cylinder of radius R through which a second grade fluid is moving. At $t = 0$, the fluid is at rest, at $t = 0^+$, the cylinder starts to move due to shear stress. As a result of applied shear stress, the fluid is gradually moved. The appropriate initial and boundary conditions are

$$f(\sigma,0) = 0;\ \ \sigma \in [0,R], \qquad (2.\ 11)$$

$$\xi(R,t) = \mu\left(1 + \lambda^\beta D_t^\beta\right)\left(\frac{\partial}{\partial r} - \frac{1}{\sigma}\right)f(\sigma,t)\bigg|_{\sigma=R} = G\sin\omega t; \qquad (2.\ 12)$$

where G is a constant and ω is angular frequency. Eqs. (9) and (10) involving fractional derivatives are solved by using the tool of Laplace transformation and modified Bessel equation.

3. CALCULATION OF THE VELOCITY FIELD

Taking the Laplace transformation of Eqs. (10) and (12), we have

$$\frac{\partial^2 \overline{f}(\sigma,p)}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial\overline{f}(\sigma,p)}{\partial\sigma} - \frac{1}{\sigma^2}\overline{f}(\sigma,p) - \frac{p}{\nu(1 + \lambda^\beta p^\beta)}\overline{f}(\sigma,p) = 0, \qquad (3.\ 13)$$

$$\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)\overline{f}(\sigma,p)\bigg|_{\sigma=R} = \frac{G\omega}{\mu(1 + \lambda^\beta p^\beta)(p^2 + \omega^2)}, \qquad (3.\ 14)$$

where p is the Laplace transform parameter. We can write Eqs. (13) and (14) as

$$\frac{\partial^2 \overline{f}(\sigma,p)}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial\overline{f}(\sigma,p)}{\partial\sigma} - \frac{\overline{f}(\sigma,p)}{\sigma^2} - A(p)\overline{f}(\sigma,p) = 0, \qquad (3.\ 15)$$

$$\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)\overline{f}(\sigma,p)\bigg|_{\sigma=R} = B(p), \qquad (3.\ 16)$$

where

$$A(p) = \frac{p}{\nu(1 + \lambda^\beta p^\beta)} \text{ and } B(p) = \frac{f\omega}{\mu(1 + \lambda^\beta p^\beta)(p^2 + \omega^2)}.$$

By using variable transformation $m = \sigma\sqrt{A(p)}$ in Eq. (15), we get

$$m^2 \frac{d^2\overline{f}}{dm^2} + m\frac{d\overline{f}}{dm} - (1 + m^2)\overline{f} = 0, \tag{3. 17}$$

which is the modified Bessel equation and the general solution of this equation is given by

$$\overline{f}(m, p) = C_1 I_1(m) + C_2 K_1(m). \tag{3. 18}$$

Here C_1 and C_2 are constants, $I_1(m)$ and $K_1(m)$ are the modified Bessel functions of the first and second kind respectively. In order to have a finite solution at $m = 0(\sigma = 0)$, C_2 must be zero. Then Eq. (18) becomes

$$\overline{f}(\sigma, p) = C_1 I_1(\sigma\sqrt{A(p)}). \tag{3. 19}$$

taking derivative of Eq. (19) with respect to σ, we get

$$\frac{\partial\overline{f}(\sigma, p)}{\partial\sigma} = C_1\sqrt{A(p)}I_1'(\sigma\sqrt{A(p)}), \tag{3. 20}$$

by using derivative formula of modified funtion

$$\frac{\partial\overline{f}(\sigma, p)}{\partial\sigma} = C_1\sqrt{A(p)}\left[I_0(\sigma\sqrt{A(p)}) - \frac{1}{\sigma\sqrt{A(p)}}I_1(\sigma\sqrt{A(p)})\right]. \tag{3. 21}$$

The dimensionless boundary condition can be written as

$$\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)\overline{f}(\sigma, p)\bigg|_{\sigma=R} = B(p). \tag{3. 22}$$

By solving Eqs. (21) and (22), we obtain the following value of C_1

$$C_1 = \frac{B(p)R}{\left(R\sqrt{A(p)}I_0(R\sqrt{A(p)}) - 2I_1(R\sqrt{A(p)})\right)}, \tag{3. 23}$$

substituting the value of C_1 in Eq. (19), we get

$$\overline{f}(\sigma, p) = \frac{G\omega R I_1(\sigma\sqrt{A(p)})}{m(p)}. \tag{3. 24}$$

where

$$m(p) = \mu(p^2 + \omega^2)(1 + \lambda^\beta p^\beta)\left((R\sqrt{A(p)}I_0(R\sqrt{A(p)}) - 2I_1(R\sqrt{A(p)})\right).$$

The expression of Eq. (24) is in form of modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$ of first kind of order zero and one respectively. For the solution of velocity field, it is almost impossible to calculate inverse Laplace transform analytically. Therefore, to overcome this problem, we have used some numerical package to obtain the inverse Laplace transform. Here, we have found the inverse Laplace transform numerically through Mathcad software.

4. CALCULATION OF THE SHEAR STRESS

Taking Laplace transform of Eq. (9), we have

$$\overline{\xi}(\sigma, p) = \mu(1 + \lambda^\beta p^\beta) \left(\frac{\partial \overline{f}(\sigma, p)}{\partial \sigma} - \frac{\overline{f}(\sigma, p)}{\sigma} \right), \qquad (4.\,25)$$

differentiating Eq. (24) with respect to σ, we have

$$\frac{\partial \overline{f}(\sigma, p)}{\partial \sigma} = \frac{G\omega R\sqrt{A(p)}I_1'(\sigma\sqrt{A(p)})}{m(p)}. \qquad (4.\,26)$$

The suitable form of above expression is given by

$$\frac{\partial \overline{f}(\sigma, p)}{\partial \sigma} = \frac{G\omega R\left[\sigma\sqrt{A(p)}I_0(\sigma\sqrt{A(p)}) - I_1(\sigma\sqrt{A(p)})\right]}{\sigma m(p)}, \qquad (4.\,27)$$

Substituting Eqs. (24) and (27) into Eq. (25), we obtain

$$\overline{\xi}(\sigma, p) = \mu(1 + \lambda^\beta p^\beta)\frac{G\omega R\left(\sigma\sqrt{A(p)}I_0(\sigma\sqrt{A(p)}) - 2I_1(\sigma\sqrt{A(p)})\right)}{\sigma m(p)}. \qquad (4.\,28)$$

After simplification, we have

$$\overline{\xi}(\sigma, p) = \frac{G\omega R\left((\sigma\sqrt{A(p)}I_0(\sigma\sqrt{A(p)}) - 2I_1(\sigma\sqrt{A(p)})\right)}{\sigma(p^2 + \omega^2)\left((R\sqrt{A(p)}I_0(R\sqrt{A(p)}) - 2I_1(R\sqrt{A(p)}))\right)}. \qquad (4.\,29)$$

After taking inverse Laplace transform of Eq. (29), we obtain the solution of shear stress. The expression of Eq. (29) is of the same type of Eq. (24). Here, again we have found the inverse Laplace transform numerically through Mathcad.

5. RESULTS AND DISCUSSION

In this article, semi analytical solutions of shear stress and velocity field of a second grade fluid with fractional derivatives moving in an infinitely long circular cylinder of radius R are procured. The tool of Laplace transformation and modified Bessel equation are used for determining these solutions. The results are obtained in transformed form in terms of modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$. The solutions satisfy all the initial and boundary conditions. The hybrid technique used by us proves itself more suitable in validity than the previous methods. The time cost and computational efforts of our technique are also very economical as compared to other existing methods.

The validation of obtained results of velocity is presented in Table 1 and 2. The results achieved with Mathcad program are compared with two numerical algorithms, called Stehfest's algorithm [30] and Tzou's algorithm [29]. The Stehfest's algorithm for the inverse Laplace transform is defined as

$$u(r, t) = \frac{ln2}{t} \sum_{k=1}^{N} V_k \overline{u}\left(r, \frac{ln2}{t} \right),$$

$$V_k = (-1)^{k+\frac{N}{2}} \sum_{j=\left[\frac{k+1}{2}\right]}^{min(k, \frac{N}{2})} \frac{j^{\frac{N}{2}}(2j)!}{(\frac{N}{2} - j)!j!(j - 1)!(k - j)!(2j - k)!}, \qquad (5.\,30)$$

where N is the index of the expansion and be an even number.

The Tzou's algorithm is based on the Riemann-sum approximation. According to this method, the inverse Laplace is defined as

$$u(r,t) = \frac{e^{4.7}}{t} \left[\frac{1}{2}\overline{u}\left(r, \frac{4.7}{t}\right) + Re\left(\sum_{k=1}^{N_1} (-1)^k \overline{u}\left(r, \frac{4.7+k\pi i}{t}\right)\right)\right], \qquad (5.31)$$

where $Re(\cdot)$ is the real part, i is the imaginary unit and N_1 is a natural number. The values obtained with Mathcad and Eqs. (30) and (31) are given in the following table.

Table 1: Validation of inverse Laplace numerical algorithms for shear stress

σ	$\xi(\sigma,t)$ (Stehfest's)[26]	$\xi(\sigma,t)$ (Tzou's)[30]	$\xi(\sigma,t)$ (Talbot's)[29]
0	0	0	0
0.05	0.075649	0.075743	0.075649
0.1	0.160779	0.160972	0.160880
0.15	0.265980	0.266283	0.265540
0.2	0.404168	0.404459	0.404278
0.25	0.592016	0.592596	0.592326
0.3	0.851722	0.852483	0.851521
0.35	1.213259	1.214241	1.213319
0.4	1.717268	1.718522	1.717378
0.45	2.418775	2.420365	2.418625
0.5	3.391958	3.393960	3.391818

Table 2: Validation of inverse Laplace numerical algorithms for velocity field

σ	$f(\sigma,t)$ (Stehfest's)[26]	$f(\sigma,t)$ (Tzou's) [30]	$f(\sigma,t)$ (Talbot's) [29]
0	0.000461	0.000461	0.000463
0.05	0.116477	0.116435	0.116428
0.1	0.386015	0.385879	0.384660
0.15	0.834121	0.833834	0.836715
0.2	1.494883	1.494383	1.499594
0.25	2.417277	2.416496	2.413299
0.3	3.667521	3.666382	3.667833
0.35	5.332112	5.330529	5.333199
0.4	7.521576	7.519448	7.529401
0.45	10.37494	10.37216	10.376442
0.5	14.06491	14.06144	14.068432

It is observed from the Table 1 and 2 that the results obtained with different numerical algorithms have a good agreement between them.

In the end, the reaction of different physical parameters is presented by graphs. Finally, we have plotted some graphs for shear stress $\xi(\sigma,t)$ and velocity field $f(\sigma,t)$ of the fluid by using Eqs. (24) and (29) respectively, to see the effect of various material parameters on our results. All the graphs have been plotted against the values of σ. Figs. 1(a) and 1(b) depicted for different values of time, the shear stress $\xi(\sigma,t)$ and velocity field $f(\sigma,t)$ both are directly proportional to t. Figs. 2(a) and 2(b) show that the shear stress and the velocity field are increasing function to β. From Figs. 3(a) and 3(b) figures, we conclude

that the shear stress and the velocity field are decreasing function to R. Figs. 4(a) and 4(b) represent shear stress and velocity field are increasing function to λ. Figs. 5(a) and 5(b) represent shear stress and velocity field are increases by enhancement of kinematic viscosity. Figs 6(a) depict velocity field is decrease as value of dynamic viscosity enhanced.

6. CONCLUSIONS

In this monograph, the motion of a second grade fluid having fractional derivatives is studied. The flow domain is the inside of a cylinder of radius R. The flow is produced due to longitudinal stress-force given on the cylinder boundary. Since the Laplace transforms of the solutions (see Eqs. (24) and (29)) are very complex therefore, we have obtained the inverse Laplace transforms by mean of numerical procedures. Firstly, we employed a Mathcad numerical code later to provide a validation of obtained results, we have used other two numerical algorithms, called Stehfest's algorithm and Tzou's algorithm.

- It is concluded from the Table 1 and 2 that we found a good comparison between results of the three numerical methods.
- The motion of the fluid near the cylinder walls is faster than that in the inner region.
- The shear stress has similar behavior with velocity; therefore, it is increasing when the fractional parameter enhances.
- An increase in the parameters t, ν and β causes greater stress and velocity function.
- An increase in the parameters R, μ, λ causes weaker stress and velocity function.

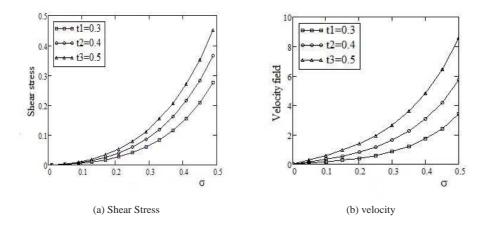(a) Shear Stress   (b) velocity

Figure 1: Shear stress and velocity field graphs for $R = 0.5$, $f = 90$, $\nu = 0.035754$, $\mu = 0.5$, $\beta = 0.5$, $\lambda = 1$ and various values of t.
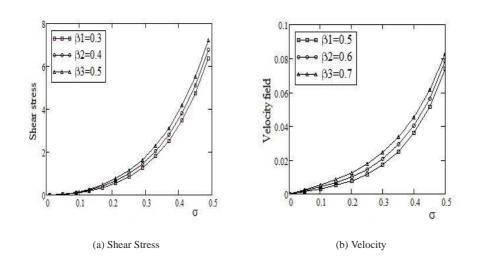
(a) Shear Stress   (b) Velocity

Figure 2: Shear stress and velocity field graphs for $R = 0.5$, $f = 90$, $\nu = 0.035754$, $\mu = 0.5$, $t = 0.3$, $\lambda = 1$ and various values of β.

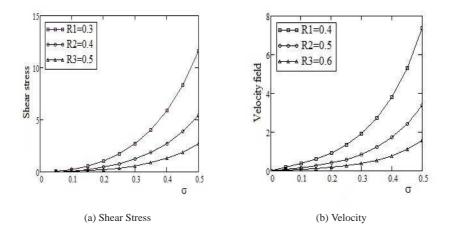(a) Shear Stress   (b) Velocity

Figure 3: Shear stress and velocity field graphs for $t = 0.3$, $\nu = 0.035754$, $\mu = 0.5$, $f = 90$, $\beta = 0.5$, $\lambda = 1$ and various values of R.
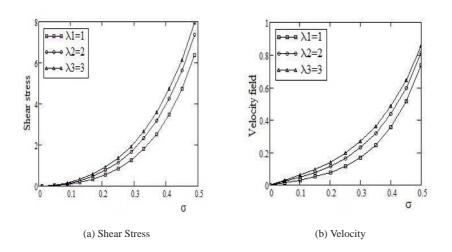
(a) Shear Stress

(b) Velocity

Figure 4: Shear stress and velocity field graphs for $f = 90$, $\nu = 0.035754$, $\mu = 0.5$, $\beta = 0.5$, $t = 0.3$, $R = 0.5$ and various values of λ.
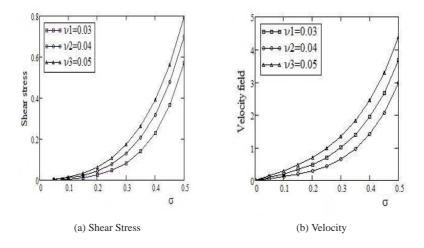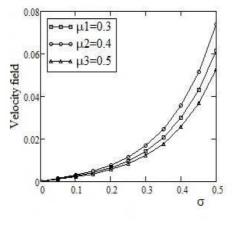

(a) Shear Stress

(b) Velocity

Figure 5: Shear stress and velocity field graphs for $f = 90$, $\lambda = 1$, $\mu = 0.5$, $\beta = 0.5$, $t = 0.3$, $R = 0.5$ and various values of ν.

(a) Velocity

Figure 6: Velocity field graph for $f = 90$, $\lambda = 1$, $\nu = 0.035754$, $\beta = 0.5$, $t = 0.3$, $R = 0.5$ and various values of μ.

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