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Stability of the European Option Value Function Under Jump-Diffusion Process

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Abstract. We consider a financial market where there are brusque variations in the price of an asset and an European option on this asset. In this setup the value as well as the hedging process functions are expressed in the form of infinite series. For finite expectation of the jumps proportions, we give sufficient condition on the payoff function which leads to the convergence of the infinite series. We also obtain the upper bound for the value function, hedging portfolio process as well as for the hedging process in the Black-Sholes setup. Moreover, we use probabilistic approach to investigate the variation of the value function.

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1. INTRODUCTION

A financial market is said to be complete if there is a replicating portfolio which ensures to price an option in it. In this setup, the stock price is the continuous function of time and there exits a unique probability measure under which the discounted stock price becomes a martingale. Considering complete financial markets, Black and Scholes [4] obtained explicit expression for European type options on the stock which do not pay dividends. Merton [25] extended the above mentioned work in a variety of very significant ways. Bensoussan [2] provided an axiomatic framework to provide the concept of portfolio and pricing risky operations for which there is no market. Bergman, Grundy and Wiener [3] investigated general properties of the option value function using an analysis of the parabolic partial differential equations satisfied by the option value function. They found that if the stock's volatility is the function of time and current price of the stock and the interest rate process is non-stochastic then the price function of the contingent claim is convex with respect to the price of the stock. Hussain and Shashiashvili [12] used this convexity property to show that having at hand any uniform approximation to unknown value function of the American style option it is possible to construct a discrete time hedging strategy the portfolio value process of which uniformly approximates the corresponding continuous time portfolio value process. El Karoui, Jeanblanc-Picque and Shreve [8] obtained similar results using Girsanov theorem and the theory of stochastic flows while Najafi [27] and Samimi [30] have have considered the pricing problem using CIR and Heston-Hull-White stochastic volatility models respectively. Rehman, Hussain and Wasim [29] considered the local volatility and investigated the continuity of the corresponding American option value function with respect to the variation of local volatility.

Incomplete markets are financial markets where perfect hedging of options is not possible, it means that there is no replicating portfolio which ensures to price an option in it. This difficulty comes from the fact that, for finite expiry time of the option, there are infinitely many equivalent probability measures (see, for example, Lamberton and Lapeyre [18]) under which the discounted stock price is martingale. Merton [26] did fundamental work in this setting. He considered the situation in which one plus the jump size is log-normal distributed. Ellio and Kopp [9] considered price process influenced by Poisson process and obtained the value function of a sum of European call options. Yan et al. [10] price the Cliquet options when stock prices follow a general jump-diffusion model with coefficients are explicitly functions of time. Hussain and Rehman [13] studied regularity properties of the American option value function under jump-diffusion model. Yongfeng Wu [34] studied the distribution of the jump-diffusion CIR model (JCIR) and discussed its applications in credit risk. Mercurio and Runggaldier [24] approximated the value function of the European call option when the coefficients explicitly depend on time and the price satisfy a jump-diffusion model. Bergman, Grundy and Wiener [3] gave an example that jumps in the stock price can lead to a non-convex European call price. Allanus et al. [33] analyze the pricing of European option when the riskfree interest rate follows a jump process. More recent works on option valuation and hedging in models with jumps can be found in Mehrdoust [1, 21, 22, 23], Hipp [11], Schweizer [31], Madan et al. [19], Brockhaus et al. [5], Madan and Malne [20], Das and Foresi [7], Jailet et al. [14], Karatzas and Shreve [15, 16], Lamberton and Villeneuve [17], Shreve [32] etc.

In an incomplete financial market the stock price has unexpected jumps due to the occurring of uncertain phenomena (for example, some natural disaster, release of unexpected funds to the market, political changes etc). In this case, it is clear that the price function and hedging process can be expressed in the form of an infinite series. These series are difficult to analyze. We impose sufficient condition on the payoff function (the particular case of which are both the European put as well as the call option) which leads to the convergence of these series and also ensures the convexity of the European price function. Moreover, we obtain several equivalent forms in terms of stochastic integrals (the detail study of such integrals can be found in [6] and the references therein) of the option value function and study its variational equations.

Our results can be used in the investigation of the discrete time hedging error estimate of the corresponding option, the optimal exercise boundary and its analysis and useful information to practitioners on the financial markets.

2. Notations and Assumptions

We consider an European option on a stock where there are unexpected huge jumps in its price. On the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we define a standard Brownian process $W = (W_t)_{0\leq t\leq T}$, a sequence $(U_j)_{j\geq 1}$ of independent and identically distributed random variables taking values in open interval $(-1, \infty)$ with finite expectation and Poisson process $N = (N_t)_{0\leq t\leq T}$ with intensity value λ. Assume the time horizon T is bounded and that the σ-algebras generated by $(W_t)_{0\leq t\leq T}$, $(U_j)_{j\geq 1}$ and $(N_t)_{0\leq t\leq T}$ respectively are independent. Denote by $(\mathcal{F}_t)_{0\leq t\leq T}$ the **P**-completion of the natural filtration of (W_t), $(U_j)I_{j\leq N_t}, j \geq 1$, and (N_t), $0 \leq t \leq T$. We also assume that, in each finite time interval, there are finite number of jumps in the asset price and the price jumps in the independent and identically distributed proportions.

On filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})_{0\leq t\leq T}$, let us consider a financial market on two assets $M_t, 0 \leq t \leq T$, the price of the unit of a money market account at given time t, and $S_t, 0 \leq t \leq T$-value of a share of the stock at this time, where the price jumps in the proportions $U_1, U_2, ...$ at random times $\tau_1, \tau_2, \ldots$. Let us suppose that the time τ_j's correspond to the jump times of Poisson process.

The asset M_t evolves according to the ordinary differential equation

$$dM_t = r(t)M_t dt,\ M_0 = 1,\ \ 0 \leq t \leq T,$$

where the interest rate process $r(t)$ is a non-negative, bounded and deterministic function of time.

The stock price S_t obeys the following stochastic differential equations

$$dS_t = S_{t-}\left(b(t)dt + \sigma(t)dW_t + d\left(\sum_{j=1}^{N_t} U_j\right)\right),\ \ 0 \leq t \leq T, \tag{2. 1}$$

where $(b(t), \mathcal{F}_t)_{0\leq t\leq T}$ is some progressively measurable process, S_{t-} denotes the left-hand limit of the stock price at t, the volatility $\sigma(t)$ is also deterministic functions of time. Moreover the following conditions are satisfied:

$$0 < \sigma(t) \leq \bar{\sigma},\ |b(t)| \leq \bar{\sigma},$$

for all $t \in [0, T]$ and where $\bar{\sigma}$ is some positive constant.

From the differential equation (2. 1), the dynamics of S_t can be expressed as:

$$S_t = S_0 \left(\prod_{j=1}^{N_t} (1+U_j) \right) \exp \left[\int_0^t \left(b(u) - \frac{\sigma^2(u)}{2} \right) du + \int_0^t \sigma(u) dW_u \right] \qquad (2.\ 2)$$

with the convention $\prod_{j=1}^{0} = 1$.

The discounted stock price $\tilde{S}_t = e^{-\int_0^t r(u)du} S_t$ is a martingale (see, for example, Lamberton and Lapeyre [18]) if and only if

$$\int_0^t b(u)du = \int_0^t (r(u) - \lambda E(U_1))du. \qquad (2.\ 3)$$

Under condition (2. 3), equation (2. 1) takes the form

$$dS_t = S_{t-} \left((r(t) - \lambda E(U_1))dt + \sigma(t)dW_t + d\left(\sum_{j=1}^{N_t} U_j \right) \right),\ 0 \le t \le T,$$

and expression (2. 2) becomes

$$\begin{aligned} S_t &= S_0 \left(\prod_{j=1}^{N_t} (1+U_j) \right) \\ &\times \exp \left[\int_0^t \left(r(u) - \lambda E(U_1) - \frac{\sigma^2(u)}{2} \right) du + \int_0^t \sigma(u) dW_u \right]. \end{aligned} \qquad (2.\ 4)$$

Moreover $\tilde{S}_t$ evolves as

$$d\tilde{S}_t = r(t)(\tilde{S}_{t-} - \tilde{S}_t)dt + \tilde{S}_{t-} \left(-\lambda E(U_1)dt + \sigma(t)dW_t + d\left(\sum_{j=1}^{N_t} U_j \right) \right),$$

with

$$\tilde{S}_t = S_0 \left(\prod_{j=1}^{N_t} (1+U_j) \right) \exp \left[- \int_0^t \left(\lambda E(U_1) + \frac{\sigma^2(u)}{2} \right) du + \int_0^t \sigma(u) dW_u \right].$$

Assume an investor starts trading with a non-random initial portfolio value $\Pi(0)$. He holds $\triangle(t), 0 \le t \le T$, number of stocks of the underlying asset and invests $\Pi(t) - \triangle(t)S_t$ on a bank account a time t. Then the portfolio value process $\Pi(t)$ evolves as

$$d\Pi(t) = r(t) \left[\Pi(t) - \triangle(t)S_t \right] dt + \triangle(t)dS_t, 0 \le t \le T,$$

with

$$\Pi(t) = e^{\int_0^t r(v)dv} \left[\Pi(0) + \int_0^t \triangle(v) d\left(e^{-\int_0^v r(u)du} S_v \right) \right], 0 \le t \le T. \qquad (2.\ 5)$$

Since the portfolio process $\triangle(t), 0 \le t \le T$, is bounded (as it will be clear from the later expressions) the above expression (2. 5) shows that the discounted portfolio value process is a martingale if and only if the condition (2. 3) does hold.

A non-path dependent European contingent claim is defined through an adapted stochastic process $g(S_T)$ at expiration time T, where $g(y)$ is called the payoff function of the claim if exercised at time T. The payoff of an European call (respectively put) option on an asset is defined as $g(y) = (y - K)^+$ (respectively $g(y) = (K - y)^+$), where the non-negative constant K is called the exercise price. Throughout the work, we suppose that the payoff function $g(y)$ is convex and is only the function of the price of the risky asset at time $t, 0 \le t \le T$. This function is continuous and has one sided derivatives (see, for example, Niculescu and Persson [28]) on the open interval $(0, \infty)$. We also assume

$$|g'(y\pm)| \le c, \; y \in (0, \infty), \tag{2. 6}$$

where c is some non-negative constant.

The typical examples of our model are put and call options where the payoff function satisfies $|g'(y)| \le 1$.

From Lamberton and Lapeyre [18] the explicit expression for the European option at time t is given as

$$v(t, S_t) = \mathbb{E}\left(e^{-\int_t^T r(v)dv} g(S_T)|\mathcal{F}_t\right),$$

using (2. 4) we can write

$$v(t,x) = \mathbb{E}\left(e^{-\int_t^T r(v)dv} g\left(xe^{\int_t^T \left(r(v)-\lambda E(U_1)-\frac{\sigma^2(v)}{2}\right)dv+\int_t^T \sigma(v)dW_v+\sum_{j=N_t+1}^{N_T}\ln(1+U_j)}\right)\right), \tag{2. 7}$$

where $S_t = x > 0$.

3. Main Results in the Classical Black-Scholes Setup

In this section, we investigate the evolution of the value function in the Black-Scholes setup and see how the value function and hedging process are bounded from above and further we investigate how these bounds depend on the proportions of the jump size.

In the classical Black-Scholes setup, let us denote by

$$v(t, S_t) \equiv B(T - t, S_t; r, \sigma, \lambda, E(U_1)), \; 0 \le t < T,$$

where

$$\begin{aligned} & B(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\ = \; & \mathbb{E}\left(e^{-r\mathbf{t}} g\left(xe^{\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}+\sigma W_{\mathbf{t}}+\sum_{j=1}^{N_{\mathbf{t}}}\ln(1+U_j)}\right)\right), \end{aligned} \tag{3. 8}$$

where $\mathbf{t} = T - t$, $x > 0$ and where we have used the fact that the law of the Poisson process $N_t - N_s$ is identical to the law of N_{t-s}, for all t, s, $0 \le s, t \le T$.

Further (3. 8) can be expressed as

$$
\begin{aligned}
& B(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
= \quad & \frac{e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}} \\
\times \quad & E\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} g\left(xe^{u}\prod_{j=1}^{n}(1+U_j)\right)\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!}\, e^{\frac{-1}{2\sigma^2\mathbf{t}}\;\; u-\;\; r-\lambda E(U_1)-\frac{\sigma^2}{2}\;\;\mathbf{t}\;\;^{2}} du \\
= \quad & \frac{e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}} E\sum_{n=0}^{\infty}\int_{0}^{\infty} g\left(z\prod_{j=1}^{n}(1+U_j)\right)\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\;\;\ln z-\ln x-\;\; r-\lambda E(U_1)-\frac{\sigma^2}{2}\;\;\mathbf{t}\;\;^{2}}\frac{dz}{z},
\end{aligned}
\tag{3. 9}
$$

where we have used the change of variable $z = xe^{u}$.

In general, using condition (2. 6) and the fact that U_j and W_t are independent, we can bound as

$$
\begin{aligned}
& B(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
\leq \quad & \frac{e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} E\left[cxe^{u}\prod_{j=1}^{n}(1+U_j)+b\right]\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\;\; u-\;\; r-\lambda E(U_1)-\frac{\sigma^2}{2}\;\;\mathbf{t}\;\;^{2}} du \\
= \quad & e^{-r\mathbf{t}}\left(cxe^{\lambda\mathbf{t}E(U_1)}+\frac{b}{2\sqrt{2}}\right),
\end{aligned}
\tag{3. 10}
$$

where b is an arbitrary constant satisfying $b + 2\sqrt{2}cxe^{\lambda\mathbf{t}E(U_1)} \geq 0$.

The right side of (3. 10) is finite only if $E(U_1) < \infty$.

In particular, the value function of the European call, where $g(y) = (y - K)^{+}$, can be expressed as

$$
\begin{aligned}
& B^{c}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
= \quad & \frac{e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}} E\sum_{n=0}^{\infty}\int_{0}^{\infty}\left(z\prod_{j=1}^{n}(1+U_j)-K\right)^{+}\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\;\;\ln z-\ln x-\;\; r-\lambda E(U_1)-\frac{\sigma^2}{2}\;\;\mathbf{t}\;\;^{2}}\frac{dz}{z} \\
= \quad & x - e^{-r\mathbf{t}}E\sum_{n=0}^{\infty}\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!}\prod_{j=1}^{n}(1+U_j)xe^{(r-\lambda E(U_1))\mathbf{t}} \\
\times \quad & \mathcal{N}\left(\frac{\ln\frac{K}{x}-\sum_{j=1}^{n}\ln(1+U_j)-\left(r-\lambda E(U_1)+\frac{\sigma^2}{2}\right)\mathbf{t}}{\sigma\sqrt{\mathbf{t}}}\right) - Ke^{-r\mathbf{t}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad Ke^{-r\mathbf{t}} E \sum_{n=0}^{\infty} \frac{e^{-\lambda \mathbf{t}}(\lambda \mathbf{t})^n}{n!} \\
&\times \quad \mathcal{N}\left(\frac{\ln \frac{K}{x} - \sum_{j=1}^{n} \ln(1+U_j) - \left(r - \lambda E(U_1) - \frac{\sigma^2}{2}\right)\mathbf{t}}{\sigma\sqrt{\mathbf{t}}} \right).
\end{aligned}
$$

The latter expression gives the jump-free bound

$$
B^c(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \quad \leq \quad x + Ke^{-r\mathbf{t}}. \tag{3. 11}
$$

While the value function of the European put option, with payoff $g(y) = (K-y)^+$, can be expressed in the form of an infinite series as

$$
\begin{aligned}
& B^p(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
= \quad & e^{-r\mathbf{t}} E \sum_{n=0}^{\infty} \frac{e^{-\lambda \mathbf{t}}(\lambda \mathbf{t})^n}{n!} \Bigg[-\prod_{j=1}^{n}(1+U_j) x e^{(r-\lambda E(U_1))\mathbf{t}} \\
\times \quad & \mathcal{N}\left(\frac{\ln \frac{K}{x} - \sum_{j=1}^{n} \ln(1+U_j) - \left(r - \lambda E(U_1) + \frac{\sigma^2}{2}\right)\mathbf{t}}{\sigma\sqrt{\mathbf{t}}} \right) \\
+ \quad & K \cdot \mathcal{N}\left(\frac{\ln \frac{K}{x} - \sum_{j=1}^{n} \ln(1+U_j) - \left(r - \lambda E(U_1) - \frac{\sigma^2}{2}\right)\mathbf{t}}{\sigma\sqrt{\mathbf{t}}} \right) \Bigg],
\end{aligned}
$$

and the jump free bounded is given as

$$
B^p(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \quad \leq \quad e^{-r\mathbf{t}} K. \tag{3. 12}
$$

Comparing (3. 11) and (3. 12) we observe that the value of the European call can take larger value than that of European put.

Since the exponential function containing x in the second expression of the value function in (3. 9) is continuous for all $(x, z) \in (0, \infty) \times (0, \infty)$ and has also continuous first order derivative with respect to x, therefore we can interchange the derivative and the improper integral and can write from (3. 9) as

$$
\begin{aligned}
& \frac{\partial B}{\partial x}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
= \quad & \frac{e^{-r\mathbf{t}}}{\sigma^3 x \mathbf{t}\sqrt{2\pi \mathbf{t}}} E \sum_{n=0}^{\infty} \int_0^{\infty} g\left(z \prod_{j=1}^{n}(1+U_j) \right) \frac{e^{-\lambda \mathbf{t}}(\lambda \mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2 \mathbf{t}} \left(\ln z - \ln x - \left(r - \lambda E(U_1) - \frac{\sigma^2}{2} \right) \mathbf{t} \right)^2} \\
\times \quad & \left(\ln z - \ln x - \left(r - \lambda E(U_1) - \frac{\sigma^2}{2} \right) \mathbf{t} \right) \frac{dz}{z}.
\end{aligned} \tag{3. 13}
$$

The expression (3. 13) guarantees us to write from (3. 9) an equivalent form for the partial derivative with respect to x as

$$
\begin{aligned}
&\frac{\partial B}{\partial x}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
= \quad & \frac{e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}} E \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} g'\left(xe^u \prod_{j=1}^{n}(1+U_j)\right) e^u \prod_{j=1}^{n}(1+U_j) \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\left(u-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2} du \\
= \quad & \frac{e^{-r\mathbf{t}}}{\sigma x\sqrt{2\pi\mathbf{t}}} E \sum_{n=0}^{\infty} \int_{0}^{\infty} g'\left(z \prod_{j=1}^{n}(1+U_j)\right) \prod_{j=1}^{n}(1+U_j) \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\left(\ln z-\ln x-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2} dz. \qquad (3.\ 14)
\end{aligned}
$$

Further from (3. 13), using (2. 6) we can estimate the absolute value

$$
\begin{aligned}
&\left|\frac{\partial B}{\partial x}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1))\right| \\
\leq \quad & \frac{c\, e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^u (1+E(U_1))^n \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!}\, e^{\frac{-1}{2\sigma^2\mathbf{t}}\left(u-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2} du \\
\leq \quad & c\, e^{\lambda(1+E(U_1))\mathbf{t}}. \qquad (3.\ 15)
\end{aligned}
$$

It is clear from the latter expression that the partial derivative $\frac{\partial B}{\partial x}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1))$ is bounded by a constant.

Using the same argument as above we can also write

$$
\begin{aligned}
&\frac{\partial B}{\partial \mathbf{t}}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
= \quad & -\left(r+\frac{1}{2\mathbf{t}}\right) \frac{e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}} E \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} g\left(xe^u \prod_{j=1}^{n}(1+U_j)\right) \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\left(u-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2} du \\
+ \quad & \frac{e^{-r\mathbf{t}}}{2\sigma^3\mathbf{t}^2\sqrt{2\pi\mathbf{t}}} E \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} g\left(xe^u \prod_{j=1}^{n}(1+U_j)\right) \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\left(u-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2} \left(u-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2 du \\
+ \quad & \frac{\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)e^{-r\mathbf{t}}}{\sigma^3\mathbf{t}\sqrt{2\pi\mathbf{t}}} E \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} g\left(xe^u \prod_{j=1}^{n}(1+U_j)\right) \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad e^{\frac{-1}{2\sigma^2\mathbf{t}}\quad u-\quad r-\lambda E(U_1)-\frac{\sigma^2}{2}\quad \mathbf{t}\quad ^2} du \left(u - \left(r - \lambda E(U_1) - \frac{\sigma^2}{2}\right)\mathbf{t}\right) du \\
&+ \quad \frac{\lambda e^{-r\mathbf{t}}}{\sigma\sqrt{2\pi\mathbf{t}}} \int_{-1}^{\infty} \left[E \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \left(g \left(xe^u \prod_{j=1}^{n} (1+U_j)(1+z) \right) \right.\right. \\
&- \quad \left.\left. g \left(xe^u \prod_{j=1}^{n} (1+U_j) \right) \right) e^{\frac{-1}{2\sigma^2\mathbf{t}}\quad u-\quad r-\lambda E(U_1)-\frac{\sigma^2}{2}\quad \mathbf{t}\quad ^2} du \right] \varrho(dz), \qquad (3.\ 16)
\end{aligned}
$$

where $\varrho(dz)$ is a probability measure which gives the distribution of the jumps $U_j, j = 1, 2, ...$ in the log-price of $S_{\mathbf{t}}, 0 < \mathbf{t} \leq T$.

From expressions (3. 13) and (3. 16) we observe that, under the condition (2. 6), $B(\cdot, \cdot; r, \sigma, \lambda, E(U_1))$ is in class $C^{1,2}([0,T) \times (0, \infty))$. Moreover, since $g'(y)$ is an increasing function, the first expression of the partial derivative in (3. 14) guarantees the convexity of the value function with respect to x.

From Lamberton and Lapeyre [18] an admissible hedging strategy $\Delta(t)$ minimizing the risk at expiration time T can be expressed as

$$
\begin{aligned}
\Delta(t) \quad &= \quad \frac{1}{\sigma^2 + \lambda \int z^2 d\varrho(z)} \left(\sigma^2 \frac{\partial B}{\partial x}(T-t, x; r, \sigma, \lambda, E(U_1)) \right. \\
&+ \quad \lambda \int \frac{z}{x} \big(B(T-t, x; r, \sigma, \lambda, E(U_1)) \\
&- \quad \left. B(T-t, x(1+z); r, \sigma, \lambda, E(U_1)) \big)\, d\varrho(z) \right),
\end{aligned}
$$

where $\varrho(z)$ is the law of the random variables $U_j, j = 1, 2, ...$

Using (3. 15) we can obtain the bound for the admissible strategy for $x > 0$ and all t, $0 \leq t < T$.

The portfolio value process $\Pi(t), 0 \leq t \leq T$, can be obtained by inserting the value of $\Delta(t), 0 \leq t \leq T$, in the expression (2. 5).

Moreover (3. 13) gives

$$
\begin{aligned}
& x^2 \frac{\partial^2 B}{\partial x^2}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\
= \quad & \frac{-e^{-r\mathbf{t}}}{\sigma^3 \mathbf{t}\sqrt{2\pi\mathbf{t}}} E \sum_{n=0}^{\infty} \int_0^{\infty} g \left(z \prod_{j=1}^{n} (1+U_j) \right) \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\quad \ln z - \ln x -\quad r-\lambda E(U_1)-\frac{\sigma^2}{2}\quad \mathbf{t}\quad ^2} \\
\times \quad & \left(\ln z - \ln x - \left(r - \lambda E(U_1) - \frac{\sigma^2}{2} \right) \mathbf{t} \right) \frac{dz}{z} \\
+ \quad & \frac{e^{-r\mathbf{t}}}{\sigma^5 \mathbf{t}^2 \sqrt{2\pi\mathbf{t}}} E \sum_{n=0}^{\infty} \int_0^{\infty} g \left(z \prod_{j=1}^{n} (1+U_j) \right) \frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad e^{\frac{-1}{2\sigma^2\mathbf{t}}\ \ \ln z-\ln x-\ \ r-\lambda E(U_1)-\frac{\sigma^2}{2}\ \ \mathbf{t}\ \ ^2} \\
&\times \quad \left(\ln z-\ln x-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2 \frac{dz}{z} \\
&- \quad \frac{e^{-r\mathbf{t}}}{\sigma^3\mathbf{t}\sqrt{2\pi\mathbf{t}}} E\sum_{n=0}^{\infty}\int_0^{\infty} g\left(z\prod_{j=1}^{n}(1+U_j)\right)\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
&\times \quad e^{\frac{-1}{2\sigma^2\mathbf{t}}\ \ \ln z-\ln x-\ \ r-\lambda E(U_1)-\frac{\sigma^2}{2}\ \ \mathbf{t}\ \ ^2}\frac{dz}{z}. \qquad (3.\ 17)
\end{aligned}
$$

And

$$
\begin{aligned}
&\frac{\partial B}{\partial\sigma}(\mathbf{t},x;r,\sigma,\lambda,E(U_1)) \\
= \quad & \frac{-e^{-r\mathbf{t}}}{\sigma^2\sqrt{2\pi\mathbf{t}}} E\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} g\left(xe^u\prod_{j=1}^{n}(1+U_j)\right)\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\ \ u-\ \ r-\lambda E(U_1)-\frac{\sigma^2}{2}\ \ \mathbf{t}\ \ ^2} du \\
+ \quad & \frac{e^{-r\mathbf{t}}}{\sigma^4\mathbf{t}\sqrt{2\pi\mathbf{t}}} E\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} g\left(xe^u\prod_{j=1}^{n}(1+U_j)\right)\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\ \ u-\ \ r-\lambda E(U_1)-\frac{\sigma^2}{2}\ \ \mathbf{t}\ \ ^2}\left(u-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right)^2 du \\
- \quad & \frac{e^{-r\mathbf{t}}}{\sigma^2\sqrt{2\pi\mathbf{t}}} E\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} g\left(xe^u\prod_{j=1}^{n}(1+U_j)\right)\frac{e^{-\lambda\mathbf{t}}(\lambda\mathbf{t})^n}{n!} \\
\times \quad & e^{\frac{-1}{2\sigma^2\mathbf{t}}\ \ u-\ \ r-\lambda E(U_1)-\frac{\sigma^2}{2}\ \ \mathbf{t}\ \ ^2} du\left(u-\left(r-\lambda E(U_1)-\frac{\sigma^2}{2}\right)\mathbf{t}\right) du. \qquad (3.\ 18)
\end{aligned}
$$

Combining equations (3. 17) and (3. 18) we find

$$
x^2\frac{\partial^2 B}{\partial x^2}(\mathbf{t},x;r,\sigma,\lambda,E(U_1)) = \frac{1}{\sigma\mathbf{t}}\frac{\partial B}{\partial\sigma}(\mathbf{t},x;r,\sigma,\lambda,E(U_1)),\ 0<\mathbf{t}\le T, \qquad (3.\ 19)
$$

and using (3. 9), (3. 13), (3. 16) and (3. 18) we obtain

$$
\begin{aligned}
&\frac{\partial B}{\partial\mathbf{t}}(\mathbf{t},x;r,\sigma,\lambda,E(U_1)) \\
= \quad & -rB(\mathbf{t},x;r,\sigma,\lambda,E(U_1)) + x\left(r-\lambda E(U_1)\right)\frac{\partial B}{\partial x}(\mathbf{t},x;r,\sigma,\lambda,E(U_1)) \\
+ \quad & \frac{\sigma}{2\mathbf{t}}\frac{\partial B}{\partial\sigma}(\mathbf{t},x;r,\sigma,\lambda,E(U_1)) \\
+ \quad & \lambda\int_{-1}^{\infty}(B(\mathbf{t},x(1+z);r,\sigma,\lambda,E(U_1)) - B(\mathbf{t},x;r,\sigma,\lambda,E(U_1)))\varrho(dz). \\
& \qquad (3.\ 20)
\end{aligned}
$$

Combination of (3. 19) and (3. 20) also give us

$$\begin{aligned} & \frac{\partial B}{\partial \mathbf{t}}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\ = \quad & -rB(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) + x\,(r - \lambda E(U_1))\,\frac{\partial B}{\partial x}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\ + \quad & \frac{\sigma^2 x^2}{2}\frac{\partial^2 B}{\partial x^2}(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)) \\ + \quad & \lambda \int_{-1}^{\infty} (B(\mathbf{t}, x(1+z); r, \sigma, \lambda, E(U_1)) - B(\mathbf{t}, x; r, \sigma, \lambda, E(U_1)))\varrho(dz). \end{aligned} \tag{3. 21}$$

From (3. 21) we see that if the stock price path is continuous, that is if $\lambda = 0$, then this is simply the Black-Scholes' equation satisfied by the value function of the option.

4. Conclusion

We consider a risky stock in an incomplete financial market where there are unexpected huge jumps in the prices of assets. In this case there is no perfect valuation as well as hedging. However, if the corresponding payoff is convex with respect to the current stock price then the value function is also convex and satisfy several variational equalities which strongly depends on the distribution of the jumps.

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