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A Numerical Treatment for the Stability of Josephson Vortices in BEC

Muhammad Irfan Qadir
Department of Mathematics,
University of Engineering and Technology, Lahore, Pakistan.
Email: mirfan@uet.edu.pk

Nisha Shiraz
Department of Mathematics,
University of Engineering and Technology, Lahore, Pakistan.
Email: nishashirax@gmail.com

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Abstract. We consider a system of a pair of quasi one dimensional Bose-Einstein condensates which are coupled with each other. The waveguides of the two condensates are assumed to be parallel. The system can be characterized by two coupled nonlinear Schrodinger equations. This system admits different topological structures such as vortices. We present a numerical treatment for the stability of Josephson vortices and dark solitons in a coupled system of Bose-Einstein condensate.

AMS (MOS) Subject Classification Codes: 03.75.Lm; 03.75.Mn; 74.50.+r
Key Words: Soliton, Bose-Einstein condensate, Bose-Josephson junction, Josephson vortex.

1. Introduction

One of the captivating phase of matter that was first predicted theoretically by Bose and Einstein in 1924 is Bose-Einstein condensate (BEC) [10, 17, 18]. For achieving this phase, atoms are cooled to a temperature close to absolute zero so that they coalesce into a single quantum state and can be described on a macroscopic scale. This phase of matter was realized experimentally in 1995 at Joint Institute for Laboratory Astrophysics (JILA) by E. Cornell and C. Weimann for the atoms of rubidium gas [2]. In the same year, W. Ketterle of Massachusetts Institute of Technology (MIT) attained a BEC of sodium atoms in the laboratory [6].

The experimental formation of atomic BEC leads to investigate the existence and stability of different nonlinear structures such as vortices [7, 24] and solitons [5, 9, 14, 26]. The subject of solitons has been an interesting area of study. A soliton is referred to as a wave that does not alter its shape when it is propagating with a specific velocity. In a nonlinear dispersive media, the solitons arise due to equalizing the effects of dispersion. There are different kinds of solitons but commonly studied solitons are bright and dark solitons. The terms dark and bright are coined from optics where they manifest dark and bright spots, respectively, in optical fibres.

Dark solitons are the nonlinear excitations which have a sudden dip in their intensity at the centre with a continuous wave background. One dimensional dark solitons are stable objects [8], yet in higher dimensions, they have snake instability [16]. Nonetheless, in condensates with spherical geometry, the instability induces the dark solitons to slip into stable vortices. Atomic vortices in BEC have been investigated experimentally as well as theoretically in [7].

A superconducting Josephson junction is a junction that consists of a pair of superconductors separated by a flimsy non-conducting material. A Bose-Josephson junction is similar to a superconducting Josephson junction that consists of two BECs separated by a barrier. The idea of Bose-Josephson junction was introduced by Smerzi [25] in 1997. Short Bose-Josephson junctions were studied in detail in [11, 19, 25]. Long Bose-Josephson junctions that admit different topological structures were investigated in [4]. In [4], the demonstration of the circulation of supercurrent in a quasi one dimensional parallel coupled waveguides was presented. This circulation of supercurrent was named as Josephson vortices. It was found that the phenomenon of the existence of these vortices is analogues to that of Josephson vortices in superconducting long Josephson junction. The system of quasi one dimensional BEC under the influence of a variety of magnetic potentials could have potential applications in atomic interferometry [3].

The interconversion of Josephson vortices and dark soliton was presented in [12, 13]. The authors discussed the stability of dark solitons and Josephson vortices by employing the energies of these structures. Following [12], we present a mathematical justification of the stability of Josephson vortices and dark solitons while varying the value of coupling strength in an effectively one dimensional parallel coupled BEC.

2. Mathematical Model and Methods

We consider the atoms of two quasi one dimensional cigar shaped coupled BEC with repulsive interaction. When the temperature is near to absolute zero, the thermal oscillations up to the first order can be ignored and the dynamics of the condensates can be characterized by the following system of nonlinear coupled Schrodinger equations

$$i\dot{\xi}_1 = -\frac{1}{2}\xi_{1xx} + \tau|\xi_1|^2\xi_1 - \omega\xi_1 + \eta\xi_2, \tag{2. 1}$$

$$i\dot{\xi}_2 = -\frac{1}{2}\xi_{2xx} + \tau|\xi_2|^2\xi_2 - \omega\xi_2 + \eta\xi_1, \tag{2. 2}$$

where t and x denote the time and space variables respectively. The dot denotes the derivative with respect to time and ξ_1 and ξ_2 are the wave functions of the two condensates of BEC atoms. The coupled systems of BEC have been investigated in connection with various effects. For example, the interaction between the momentum and spin of a quantum particle i.e. spin-orbit coupling was studied in [27] and its dynamics was discussed in [15]. A comprehensive study of gap solitons with experimentally realizable spin-orbit coupling was presented in [28]. Excitation spectra for dark solitons in spin-orbit coupled BEC was investigated in [1]. The existence and stability of Josephson vortices and dark solitons in the presence of a harmonic trap was studied in [20, 21, 22] while the stability analysis of multi-Josephson vortices was presented in [23].

In our coupled system of eq. (2. 1) and eq. (2. 2), we have ignored the dissipative terms and assumed η as the coupling parameter, τ the nonlinearity coefficient and ω the chemical potential which is assumed to be same in both condensates. The sign of nonlinearity coefficient can be either positive or negative depending upon the interaction between the atoms of BEC. Here, we consider the interaction to be positive i.e. $\tau > 0$. The steady state solitons family for the system of eq. (2. 1) and eq. (2. 2) in an infinite medium is [12, 13]

$$\xi_1 = \sqrt{1+\eta}\tanh(ax) + ib\mathrm{sech}(ax), \tag{2. 3}$$

$$\xi_2 = \sqrt{1+\eta}\tanh(ax) - ib\mathrm{sech}(ax). \tag{2. 4}$$

The dark soliton in which $\xi_1 = \xi_2$ corresponds to $b = 0$ and $a = \sqrt{1+\eta}$. The Josephson vortices that satisfy both reflection and time reversal symmetries correspond to $b = \sqrt{1-3\eta}$ and $a = 2\sqrt{\eta}$. To obtain the time independent solution numerically, we may substitute $\dot{\xi}_1 = 0 = \dot{\xi}_2$, so that eq. (2. 1) and eq. (2. 2) becomes

$$-\frac{1}{2}\xi_{1xx} + \tau|\xi_1|^2\xi_1 - \omega\xi_1 + \eta\xi_2 = 0, \tag{2. 5}$$

$$-\frac{1}{2}\xi_{2xx} + \tau|\xi_2|^2\xi_2 - \omega\xi_2 + \eta\xi_1 = 0. \tag{2. 6}$$

As ξ_1 and ξ_2 are complex, we can write them in terms of real and imaginary parts such as $\xi_1 = \alpha_1 + i\alpha_2$ and $\xi_2 = \alpha_3 + i\alpha_4$. Using these values in eq. (2. 5) and eq. (2. 6) and then equating the real and imaginary parts, we get

$$\frac{1}{2}\alpha_{1xx} - \tau(\alpha_1^3 + \alpha_1\alpha_2^2) + \omega\alpha_1 - \eta\alpha_3 = 0, \tag{2. 7}$$

$$\frac{1}{2}\alpha_{2xx} - \tau(\alpha_1^2\alpha_2 + \alpha_2^3) + \omega\alpha_2 - \eta\alpha_4 = 0, \tag{2. 8}$$

$$\frac{1}{2}\alpha_{3xx} - \tau(\alpha_3^3 + \alpha_3\alpha_4^2) + \omega\alpha_3 - \eta\alpha_1 = 0, \tag{2. 9}$$

$$\frac{1}{2}\alpha_{4xx} - \tau(\alpha_3^2\alpha_4 + \alpha_4^3) + \omega\alpha_4 - \eta\alpha_2 = 0. \tag{2. 10}$$

Applying central difference approximations to above equations, we get

$$\frac{1}{2}\left[\frac{\alpha_{1,i-1} - 2\alpha_{1,i} + \alpha_{1,i+1}}{(\Delta x)^2}\right] - \tau(\alpha_{1,i}^3 + \alpha_{1,i}\alpha_{2,i}^2) + \omega\alpha_{1,i} - \eta\alpha_{3,i} = 0, \tag{2. 11}$$

$$\frac{1}{2}\left[\frac{\alpha_{2,i-1} - 2\alpha_{2,i} + \alpha_{2,i+1}}{(\Delta x)^2}\right] - \tau(\alpha_{1,i}^2\alpha_{2,i} + \alpha_{2,i}^3) + \omega\alpha_{2,i} - \eta\alpha_{4,i} = 0, \tag{2. 12}$$

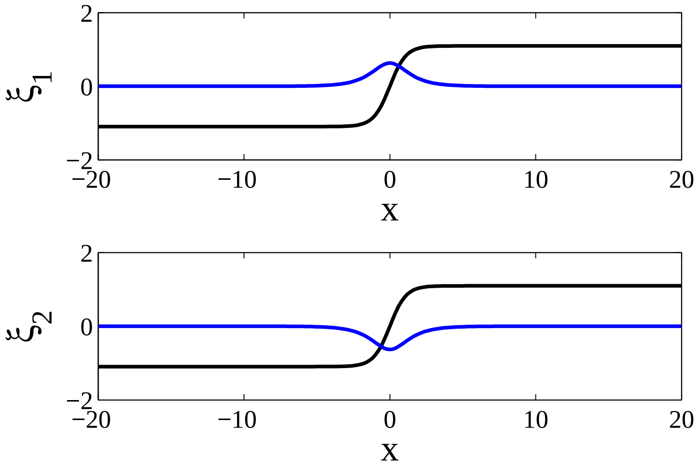

FIGURE 1. A Josephson vortices solution obtained numerically for $\tau = 1$, $\omega = 1$ and $\eta = 0.2$. The black curves represent the real parts and the blue curves are the imaginary part of the solution.

$$\frac{1}{2}\left[\frac{\alpha_{3,i-1} - 2\alpha_{3,i} + \alpha_{3,i+1}}{(\Delta x)^2}\right] - \tau(\alpha_{3,i}^3 + \alpha_{3,i}\alpha_{4,i}^2) + \omega\alpha_{3,i} - \eta\alpha_{1,i} = 0, \tag{2. 13}$$

$$\frac{1}{2}\left[\frac{\alpha_{4,i-1} - 2\alpha_{4,i} + \alpha_{4,i+1}}{(\Delta x)^2}\right] - \tau(\alpha_{3,i}^2\alpha_{4,i} + \alpha_{4,i}^3) + \omega\alpha_{4,i} - \eta\alpha_{2,i} = 0. \tag{2. 14}$$

where $i = 1, 2, 3, ...N$. The eq. (2. 11), eq. (2. 12), eq. (2. 13) and eq. (2. 14) represent a system of algebraic equations which is nonlinear. There are several methods available for getting the solutions of nonlinear systems. Using any suitable method, one can obtain the solution. We solved the above system numerically using Neumann conditions. The Josephson vortices and the dark soliton solutions are shown in Fig. 1 and Fig. 2 respectively.

3. STABILITY

Let us now investigate the stability of the dark solitons and the vortices solutions. Let $\hat{\xi}_1$ and $\hat{\xi}_2$ denote the time independent solutions of eq. (2. 1) and eq. (2. 2) and $p_1(x,t)$, $p_2(x,t)$ be the perturbation functions, respectively. These perturbations are assumed to be as small as their squares and higher powers may be ignored. So, we can write $\xi_1(x,t) = \hat{\xi}_1 + p_1(x,t)$, $\xi_2(x,t) = \hat{\xi}_2 + p_2(x,t)$. Inserting these values in eq. (2. 1) and eq. (2. 2) and linearizing the resulting equations, we obtain

$$i\dot{p}_1 = -\frac{1}{2}p_{1xx} + 2\tau p_1|\hat{\xi}_1|^2 + \tau\bar{p}_1(\hat{\xi}_1)^2 - \omega p_1 + \eta p_2, \tag{3. 15}$$

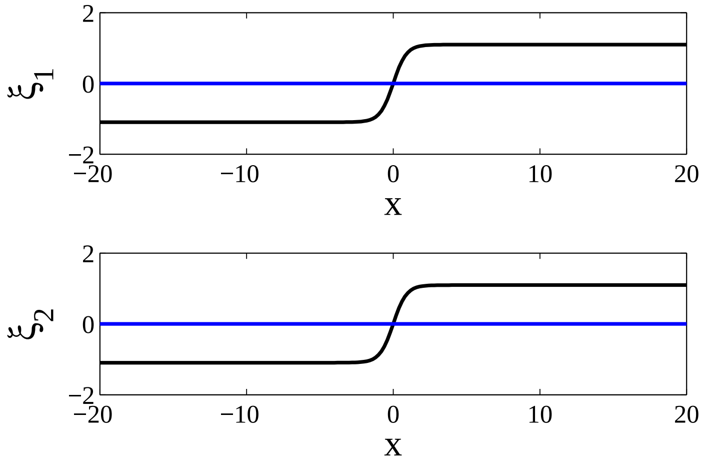

FIGURE 2. A coupled dark soliton solution obtained numerically for $\tau = 1$, $\omega = 1$ and $\eta = 0.2$. The black curves represent the real parts and the blue lines are the imaginary part of the solution.

$$i\dot{p}_2 = -\frac{1}{2}p_{2xx} + 2\tau p_2|\hat{\xi}_2|^2 + \tau\bar{p}_2(\hat{\xi}_2)^2 - \omega p_2 + \eta p_1, \tag{3. 16}$$

where bar is used for the complex conjugate. Taking conjugate of the above equations, we can write

$$-i\overline{\dot{p}_1} = -\frac{1}{2}\bar{p}_{1xx} + 2\tau\bar{p}_1|\hat{\xi}_1|^2 + \tau p_1\overline{(\hat{\xi}_1)^2} - \omega\bar{p}_1 + \eta\bar{p}_2, \tag{3. 17}$$

$$-i\overline{\dot{p}_2} = -\frac{1}{2}\bar{p}_{2xx} + 2\tau\bar{p}_2|\hat{\xi}_2|^2 + \tau p_2\overline{(\hat{\xi}_2)^2} - \omega\bar{p}_2 + \eta\bar{p}_1. \tag{3. 18}$$

For simplification, we substitute $p_1 = \alpha$, $p_2 = \beta$, $\bar{p}_1 = \delta$ and $\bar{p}_2 = \Gamma$ in the above equations to obtain

$$i\dot{\alpha} = -\frac{1}{2}\alpha_{xx} + 2\tau\alpha|\hat{\xi}_1|^2 + \tau\delta(\hat{\xi}_1)^2 - \omega\alpha + \eta\beta, \tag{3. 19}$$

$$i\dot{\beta} = -\frac{1}{2}\beta_{xx} + 2\tau\beta|\hat{\xi}_2|^2 + \tau\Gamma(\hat{\xi}_2)^2 - \omega\beta + \eta\alpha_1, \tag{3. 20}$$

$$-i\dot{\bar{\delta}} = -\frac{1}{2}\delta_{xx} + 2\tau\delta|\hat{\xi}_1|^2 + \tau\alpha\overline{(\hat{\xi}_1)^2} - \omega\delta + \eta\Gamma, \tag{3. 21}$$

$$-i\dot{\bar{\Gamma}} = -\frac{1}{2}\Gamma_{xx} + 2\tau\Gamma|\hat{\xi}_2|^2 + \tau\beta\overline{(\hat{\xi}_2)^2} - \omega\Gamma + \eta\delta. \tag{3. 22}$$

It is easy to see that the above system of equations after discretization process can be expressed as an eigenvalue problem $AX = \lambda X$ with the eigenvalues λ. The solution will be stable if every eigenvalue is real. Nonetheless, if any one of the eigenvalue is complex, the solution will become unstable.

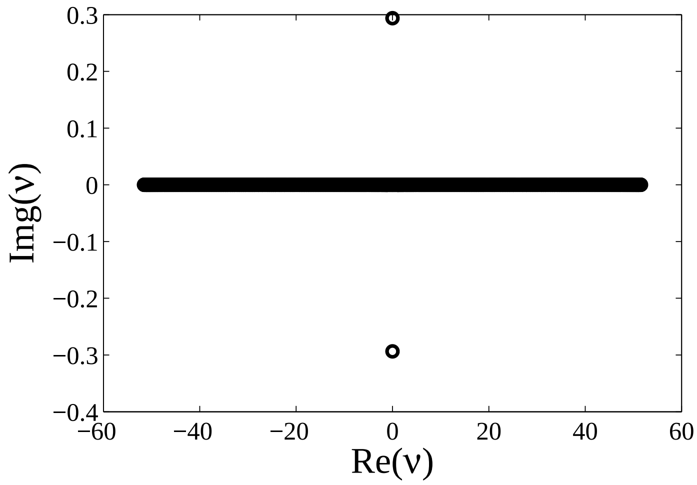

FIGURE 3. The graph of the real versus the imaginary parts of the eigenvalues for the dark solitons with $\tau = 1$, $\omega = 1$ and $\eta = 0.2$.

First, we consider the dark soliton solution and find the eigenvalues of the coefficient matrix A. These eigenvalues are shown in Fig. 3. It is evident from this figure that only two eigenvalues are complex and all the remaining eigenvalues are real. This shows that the dark soliton is unstable for the parameter values shown where the value of coupling factor is 0.2. We found the eigenvalues for different values of the coupling parameter and noticed that the dark soliton solution remains unstable for $\eta < 1/3$. In Fig. 4, the graph between coupling parameter and the maximum value of the imaginary parts of the eigenvalues has been shown by black curve. This curve shows that the dark soliton solution is unstable for $\eta < 1/3$. At $\eta = 1/3$, the solution becomes stable.

Next, we consider the Josephson vortices solution and find the eigenvalues of A for different values of coupling parameter. It is found that all eigenvalues remain real for every value of coupling parameter. Since all eigenvalues were found to be real and lying horizontally, we did not show that graph. However, in Fig. 4 the graph for the stability of Josephson vortices solution has been shown while varying the coupling parameter by red line. This shows that the solution exists and remains stable for $0 < \eta < 1/3$. At $\eta = 1/3$, solution changes into dark soliton.

Finally, the full system of eq. (2. 1) and eq. (2. 2) has been solved numerically to see the evolution of the unstable solutions at later stages which has been shown in Fig. 5. A typical evolution of the dark soliton has been depicted ensuing the instability of dark solitons. Due to the chaotic behavior, the radiations emerge close to $t = 30$ and validates the results obtained above as well as in [12].

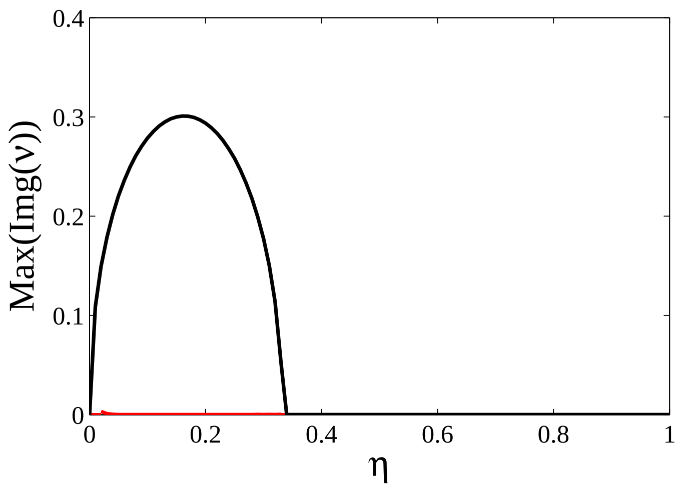

FIGURE 4. The graph of the stability curves for dark solitons and Josephson vortices solutions. The black curve is for dark solitons and the red curve is for Josephson vortices.

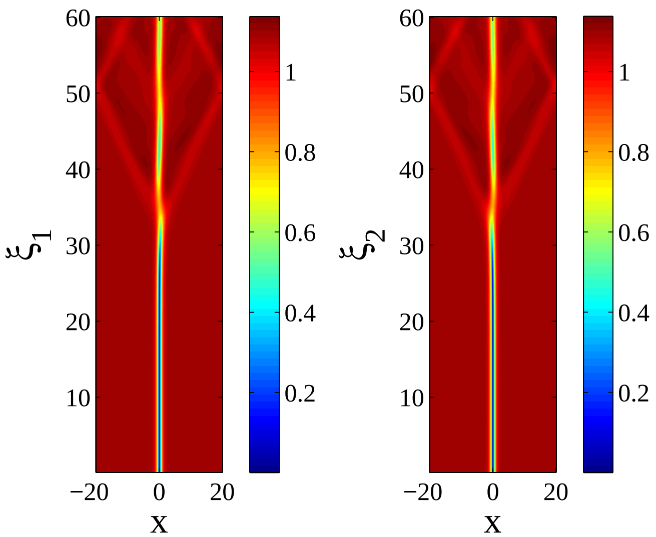

FIGURE 5. The time evolution of the unstable dark soliton solution obtained numerically in Fig. 2. The left part corresponds to ξ_1 and the right part to ξ_2.

4. Conclusions

In this paper, we have considered a system of two quasi parallel coupled one dimensional waveguides of Bose-Einstein condensates. The dynamics of the system has been characterized by two coupled nonlinear Schrodinger equations. The system exhibits different topological spatial configurations. Typically, we have obtained coupled dark soliton and Josephson vortices solutions numerically. A mathematical justification for the stability of both solutions was presented. It was found that the Josephson vortices solution remains stable in its whole domain of existence. Nevertheless, the dark soliton exists and remains unstable for $\eta < 1/3$ but becomes stable for $\eta \geq 1/3$.

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